Theory of Stochastic Processes
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DISCRETE ANALOGUE OF THE KRYLOV–VERETENNIKOV EXPANSION

We consider a difference analogue of the stochastic flow with interaction in $\mathbb{R}$. The discrete-time flow is given by a difference equation with random perturbation which is defined by a sequence of stationary Gaussian processes. We obtain the Itô–Wiener expansion for a solution to the stochastic difference equation which can be regarded as a discrete analogue of the Krylov–Veretennikov representation for a solution to the stochastic differential equation.

1. Introduction

In the present paper, we study a discrete-time system of interacting particles which is generated by a sequence of independent stationary Gaussian processes $\{\xi_n(u), u \in \mathbb{R}\}_{n\geq 1}$ with covariance function Γ, $\Gamma(0) = 1$, via the recurrence equation

$$\begin{cases} x_{n+1}(u) = x_n(u) + \xi_{n+1}(x_n(u)) \\ x_0(u) = u, \ u \in \mathbb{R}. \end{cases} \tag{1}$$

Here, $\{x_n(u), n \geq 0\}$ describes the motion of a particle which starts from a point $u \in \mathbb{R}$. The interaction is understood as a probabilistic dependence between the random variables $\{x_n(u), u \in \mathbb{R}\}$.

Similar schemes were investigated in connection with the synchronization of a system of oscillators evolving in the presence of a random force [1]. For example, the phase dynamics equation for an oscillator with a random perturbation has the form

$$\frac{d\psi(t)}{dt} = -\nu + \varepsilon(t)q(\psi(t)), \tag{2}$$

where $\psi(t) = \varphi(t) - \omega t$ is a difference between the phase of oscillations φ and the phase of an external force; $\nu = \omega - \omega_0$ is the detuning frequency; ω is the frequency of an external force, ω_0 is the internal frequency; and $\varepsilon(t)$ is the amplitude of an external force. The main question for model (2) is as follows: "when does the noise destroy or enhance the synchronization of oscillators?"

In the case where $\nu = 0$ and with the discretization on the time, Eq. (2) yields

$$\psi_{n+1} = \psi_n + \eta_{n+1} q(\psi_n). \tag{3}$$

Consider model (1) with $\{\xi_n(u), u \in \mathbb{R}\}_{n\geq 1}$ of the following kind:

$$\xi_n(u) = \eta'_n \cos u + \eta''_n \sin u,$$

where $\{\eta'_n\}_{n\geq 1}$, $\{\eta''_n\}_{n\geq 1}$ are independent sequences of independent standard Gaussian random variables. In this case, we obtain the equation similar to (3):

$$x_{n+1}(u) = x_n(u) + \eta'_{n+1} \cos x_n(u) + \eta''_{n+1} \sin x_n(u). \tag{4}$$

The distance between points at the two-point motion of system (4) can be viewed as the difference between the phases of oscillators.

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Our model can be also regarded as a general iterative scheme:

$$X_n = T_n \circ T_{n-1} \circ \ldots \circ T_1 X_0,$$

where $\{T_n\}_{n\geq 1}$ is a sequence of independent and identically distributed random maps of $\mathbb{R}^k$ into itself, and X_0 is an initial state. Such models are considered by many authors (see, e.g., [2, 3, 4]). In general, the main questions that are studied for the random sequence $\{X_n\}_{n\geq 1}$ are the following ones:

i) existence and properties of stationary random measures of the system ;

ii) convergence of the sequence $\{X_n\}_{n\geq 1}$ (almost sure, in probability, of the distribution) .

Since $f(x_n(u))$ is a function of Gaussian processes, there exists the Itô–Wiener series expansion for a solution to system (1). Our main goal is to obtain the explicit form of such series expansion. The individual terms of the expansion can be used to analyze the asymptotic behavior of our model. Note that, for the solution to a stochastic differential equation, the Itô–Wiener series expansion was obtained in [6]. So, the obtained series expansion for a solution to system (1) can be regarded as a discrete analogue of the Krylov–Veretennikov representation.

2. Some properties of one- and two-point motion

In this section, we consider one- and two-point motions of system (1) and observe some properties.

Lemma 1. *Consider the sequences* $\{x_n(u)\}_{n\geq 0}$ *and* $\{x_n(u) - x_n(v)\}_{n\geq 0}$ *for any fixed points* $u,\ v \in \mathbb{R}$.

i) Let $\{\zeta_n\}_{n\geq 1}$ *be a sequence of independent standard Gaussian random variables, and let* $\{z_n\}_{n\geq 0}$ *be a sequence such that*

$$z_0 = u,\ z_{n+1} = z_n + \zeta_{n+1}. \tag{5}$$

Then the sequences $\{x_n(u)\}_{n\geq 1}$ *and* $\{z_n\}_{n\geq 1}$ *are identically distributed.*

ii) Let $\{\nu_n\}_{n\geq 1}$ *be a sequence of independent standard Gaussian random variables, and let* $\{y_n\}_{n\geq 0}$ *be a sequence such that*

$$y_0 = u - v,\ y_{n+1} = y_n + \sqrt{2 - 2\Gamma(y_n)}\nu_{n+1}. \tag{6}$$

Then the sequences $\{x_n(u) - x_n(v)\}_{n\geq 1}$ *and* $\{y_n\}_{n\geq 1}$ *are identically distributed.*

Proof. The proof is trivial and is omitted. □

From Lemma 1, (i) we can see that the one-point motions of system (1) are Gaussian symmetric random walks. So, our model can be regarded as a discrete analogue of the Harris flows with a Brownian one-point motion [7]. Note that the asymptotic behavior for Gaussian symmetric random walks is well known:

$$\mathbb{P}\{\overline{\lim_{n\to\infty}} \frac{x_n(0)}{\sqrt{2n \ln\ln n}} = 1\} = 1.$$

From Lemma 1, (ii) we can see that the joint motion of two points is specified by the function Γ. Let us give the example with a different behavior of two-point motions in system (1).

Example 1. For any two fixed points $u_1,\ u_2\ \in \mathbb{R}$, we denote $y_n = x_n(u_1) - x_n(u_2)$. Consider two covariance functions $\Gamma_1 \equiv 1$ and

$$\Gamma_2(u) = \begin{cases} 1 - |u|, & |u| \leq 1, \\ 0, & |u| > 1. \end{cases}$$

In the first case, we get $y_{n+1} = y_n$, so the distance between two any points does not change in time. In the second case, we denote $\tau = \inf\{n : y_n \in [-1, 1]\}$. Then $\{y_n, n < \tau\}$ is a symmetric random walk with Gaussian increments.

In [5], the following stochastic difference equation is considered:

$$x_{n+1} = x_n\big(1 - hf(x_n) + \sqrt{h}g(x_n)\xi_{n+1}\big). \tag{7}$$

Here, ξ_n are independent random variables, and $x_0 \in \mathbb{R}$. The authors obtained [5] some results on the asymptotic stability and the instability of the trivial solution $x_n \equiv 0$ which allow us to formulate the following proposition.

Lemma 2. *If the covariance function Γ satisfies the condition*

$$1 > \Gamma(x) \geq 1 - \frac{K^2x^2}{2} \tag{8}$$

for any $x \in \mathbb{R} \setminus \{0\}$ and some constant $K > 0$, then, for any fixed points $u,\ v \in \mathbb{R}$,

$$\lim_{n\to\infty} (x_n(u) - x_n(v)) = 0 \ \ a.s.$$

Proof. Let the sequence $\{y_n\}_{n\geq 1}$ satisfy (6). We prove that $\lim\limits_{n\to\infty} y_n = 0$ $a.s.$ From (6), we get

$$y_{n+1} = y_n\big(1 + \sqrt{h}g(y_n)\nu_{n+1}\big),$$

where $g(x) = \frac{\sqrt{2-2\Gamma(x)}}{\sqrt{h}x}$. Theorem 6 in [5] implies that if a function g is bounded and $g(x) \neq 0,\ x \in \mathbb{R} \setminus \{0\}$, then $\lim\limits_{n\to\infty} y_n = 0$ $a.s.$ From condition (8), it is follows that the function g is bounded and $g(x) \neq 0,\ x \in \mathbb{R} \setminus \{0\}$, so $\lim\limits_{n\to\infty} y_n = 0$ $a.s.$ □

3. Itô–Wiener expansion

In this section, we introduce basic definitions and notations related to the Itô–Wiener expansion of Gaussian functionals and the Krylov–Veretennikov representation of solutions to a stochastic differential equation driven by the Wiener process (see [8, 9, 10, 11]).

Let $(\Omega, \mathcal{F}, P)$ be a probability space, and let H be a real separable Hilbert space with inner product $(\cdot, \cdot)$ and norm $\|\cdot\|$. By ξ, we denote a generalized Gaussian random element in H which has the zero mean and the identical correlation operator. This means that ξ is a linear map which maps elements of H into the set of Gaussian random variables and has the property

$$\forall\, \varphi \in H : \quad \mathbb{E}(\xi, \varphi) = 0,\ \mathbb{E}(\xi, \varphi)^2 = \|\varphi\|^2.$$

By ξ, we also denote the white noise in H.

Let $L_2(\Omega, \mathcal{F}, P)$ be the set of all square-integrable random variables on $(\Omega, \mathcal{F}, P)$. Let $H_k, k \geq 1$, be the space of k-linear symmetric Hilbert–Schmidt forms on H. We define the inner product and the norm in H_k in the usual way:

$$\forall\, T_k, S_k \in H_k : \quad (T_k, S_k)_k := \sum_{i_1,\ldots,i_k=1}^{\infty} T_k(e_{i_1}, \ldots, e_{i_k}) S_k(e_{i_1}, \ldots, e_{i_k}),$$

$$\|T_k\|_k^2 = (T_k, T_k)_k,$$

where $\{e_i, i \geq 1\}$ is an orthonormal basis for H. For any $Q_k \in H_k$, let us consider the random variable $Q_k(\xi, \ldots, \xi)$ defined by

$$Q_k(\xi, \ldots, \xi) := \sum_{i_1,\ldots,i_k=1}^{\infty} q_{i_1,\ldots,i_k} : (e_{i_1}, \xi) \cdot \ldots \cdot (e_{i_k}, \xi) :,$$

where $q_{i_1,\dots,i_k} = Q_k(e_{i_1},\dots,e_{i_k})$, and $: \ \cdot \ :$ is the Wick product [11]. Note that

$$\mathbb{E}Q_k(\xi,\dots,\xi)^2 = k!\|Q_k\|_k^2. \tag{9}$$

Suppose that the random variable η is $\sigma(\xi)$-measurable and in $L_2(\Omega,\mathcal{F},P)$. It is well known [9] that there exists a unique sequence of forms $Q_k \in H_k$ such that

$$\eta = \sum_{k=0}^{\infty} Q_k(\xi,\dots,\xi), \tag{10}$$

where the series converges in the square mean. Representation (10) is called the Itô–Wiener expansion.

Example 2. Consider $H = L_2([0,1])$. We assume that a generalized Gaussian random element ξ is given by

$$(f,\xi) = \int_0^1 f(t)dw(t),$$

where $\{w(t), t \in [0,1]\}$ is a Wiener process. Then, for any Hilbert–Schmidt form A_k, there exists the unique function a_k that is invariant under any permutation of the arguments such that, for any $x_1,\dots,x_k \in L_2([0,1])$,

$$A_k(x_1,\dots,x_k) = \int_0^1 \overset{k}{\dots} \int_0^1 a_k(t_1,\dots,t_k)x_1(t_1)\dots x_k(t_k)dt_1\dots dt_k.$$

Let η be a $\sigma(w)$-measurable random variable, and let $\eta \in L_2(\Omega,\mathcal{F},P)$. Then the Itô–Wiener expansion is as follows:

$$\eta = \mathbb{E}\eta + \int_0^1 f_1(t)dw(t) + \sum_{n=2}^{\infty}\int_0^1\int_0^{t_n}\dots\int_0^{t_2} f_n(t_1,\dots,t_n)dw(t_1)\dots dw(t_n),$$

where the functions $f_n \in L_2([0,1]^n)$ are invariant under any permutation of the arguments.

The Itô–Wiener expansion for the solution to the a stochastic differential equation was obtained in [6]. Let $\{w_t, t \in [0,1]\}$ be a one-dimensional Wiener process which is defined on a complete probability space $(\Omega,\mathcal{F},P)$. Consider the Cauchy problem for the one-dimensional stochastic differential equation

$$\begin{cases} dx(t) = \sigma(x(t))dw(t) + b(x(t))dt, \quad t \ge 0 \\ x(0) = x, \end{cases} \tag{11}$$

where $x \in \mathbb{R}$, $\sigma(\cdot)$, $b(\cdot)$ are measurable bounded functions. Suppose that

$$\exists\, \mu > 0 \ \ \forall\, x \in \mathbb{R}: \quad |\sigma(x)| \ge \mu.$$

It is known that, under the given conditions, this stochastic equation has a weak solution [6]. The Krylov–Veretennikov representation can be written with the use of the fundamental solution to a parabolic partial differential equation associated with the stochastic differential equation. Denote $a(x) = \frac{1}{2}\sigma^2(x)$ and consider

$$\begin{cases} \frac{\partial}{\partial s}u(s,x) + a(x)\frac{\partial^2}{\partial x^2}u(s,x) + b(x)\frac{\partial}{\partial x}u(s,u) = 0, \ s \in [0,t), \\ u(t,x) = \varphi(x), \ \varphi \in C_0^\infty(\mathbb{R}), \ t \in \mathbb{R}. \end{cases} \tag{12}$$

Let T_{t-s}, $s < t$, be a set of operators that define a solution to (12) (which is solved backward in time). It is known that $T_t\varphi(x) = \mathbb{E}\varphi(x(t))$, where $x(t)$ is a solution to (11).

Denote $Q_{t-s}\varphi(x) = \sigma(x)\frac{\partial}{\partial x}T_{t-s}\varphi(x)$. Then the Itô–Wiener expansion for $\varphi(x(t))$ has the form

$$\varphi(x_t) = T_t\varphi(x_0) + \sum_{i=1}^{\infty} \int \cdots \int_{t^i<\ldots<t^1<t} T_{t^i}Q_{t^{i-1}-t^i}\ldots Q_{t-t^1}\varphi(x_0)dw_{t^i}\ldots dw_{t^1}.$$

4. Main result

Let us return to our system of interacting particles (1). Assume that a covariance function Γ has the form

$$\Gamma(u) = \int_{\mathbb{R}} \psi(u-v)\psi(v)dv,$$

where ψ is a symmetric function on $\mathbb{R}$ such that

$$\int_{\mathbb{R}} \psi^2(u)du = 1.$$

Let us find an analogue of the Krylov–Veretennikov representation for $\varphi(x_n(u))$.

First of all, let us define the white noise $\dot{\xi}$ which produces the sequence $\{\xi_n(u), u \in \mathbb{R}\}_{n\geq 1}$. For this purpose, we define the white noise $\dot{\xi}_n$ on $L_2(\mathbb{R})$ which produces the process $\{\xi_n(u), u \in \mathbb{R}\}$. Under the assumption on the covariance function Γ, it can be proved that there exists a Wiener process $w_n(t), t \in \mathbb{R}$, such that

$$\xi_n(u) = \int_{\mathbb{R}} \psi(u-v)dw_n(v).$$

Define the white noise $\dot{\xi}_n$ on the space $L_2(\mathbb{R})$ by

$$(f, \dot{\xi}_n) = \int_{\mathbb{R}} f(v)dw_n(v).$$

We now define a separable Hilbert space

$$\mathcal{H} = \{F : F = (f_1, f_2, \ldots),\ f_k \in L_2(\mathbb{R}),\ \sum_{k=1}^{\infty} \|f_k\|^2_{L_2} < +\infty\},$$

with the inner product on $\mathcal{H}$:

$$(F, G) = \sum_{k=1}^{\infty} (f_k, g_k)_{L_2}.$$

We define the white noise $\dot{\xi}$ in $\mathcal{H}$ as follows:

$$(F, \dot{\xi}) = \sum_{n=1}^{\infty} (f_n, \dot{\xi}_n).$$

Observe that

$$\forall\ n \geq 1\ \forall\ u \in \mathbb{R}\ \ \xi_n(u) = (\Psi_n(\cdot - u), \dot{\xi}),$$

where $\Psi_n(u) = (\underbrace{0, \ldots, 0}_{n-1}, \psi(u), 0, \ldots) \in \mathcal{H}$.

Consider $\Phi = \cap_{n=1}^{\infty} L_2(\mathbb{R}, N(0,n))$, where

$$L_2(\mathbb{R}, N(0,n)) = \{f : \mathbb{R} \to \mathbb{R} |\ \frac{1}{\sqrt{2\pi n}} \int_{\mathbb{R}} f^2(x)e^{-\frac{x^2}{2n}}dx < \infty\}.$$

We define the norm in $L_2(\mathbb{R}, N(0,n))$ by

$$\|f\|^2_{N(0,n)} = \frac{1}{\sqrt{2\pi n}} \int_{\mathbb{R}} f^2(x)e^{-\frac{x^2}{2n}}dx.$$

We now define the distance ρ in the space Φ. For any $\varphi_1, \varphi_2 \in \Phi$, we take

$$\rho(\varphi_1, \varphi_2) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\|\varphi_1 - \varphi_2\|_{N(0,n)}}{1 + \|\varphi_1 - \varphi_2\|_{N(0,n)}}.$$

Let us verify that, for any $\varphi \in \Phi$ and $r \in \mathbb{R}$, the random variable $\varphi(r + \xi_1(r)) \in L_2(\Omega, \mathcal{F}, P)$.

For any $\varphi \in \Phi$, $r \in \mathbb{R}$ and $n \geq 2$, we have

$$\mathbb{E}\varphi^2(r + \xi_1(r)) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \varphi^2(x) e^{-\frac{(r-x)^2}{2}} dx =$$

$$= \sqrt{n} \frac{1}{\sqrt{2\pi n}} \int_{\mathbb{R}} \varphi^2(x) e^{-\frac{x^2}{2n}} e^{-x^2\left(\frac{n-1}{2n}\right) + xr - \frac{r^2}{2}} dx \leq$$

$$\leq \sqrt{n} \sup_{x \in \mathbb{R}} e^{-x^2 \frac{n-1}{2n} + xr - \frac{r^2}{2}} \frac{1}{\sqrt{2\pi n}} \int_{\mathbb{R}} \varphi^2(x) e^{-\frac{x^2}{2n}} dx = \sqrt{n} e^{\frac{r^2}{2(n-1)}} \|\varphi\|^2_{N(0,n)} < +\infty, \tag{13}$$

where we used

$$\sup_{x \in \mathbb{R}} e^{-x^2 \frac{n-1}{2n} + xr - \frac{r^2}{2}} = e^{-x^2 \frac{n-1}{2n} + xr - \frac{r^2}{2}} \Big|_{x = r\frac{n}{n-1}} = e^{\frac{r^2}{2(n-1)}}.$$

Thus, $\varphi(r+\xi_1(r)) \in L_2(\Omega, \mathcal{F}, P)$ for any $\varphi \in \Phi$ and $r \in \mathbb{R}$. We now denote the Itô–Wiener expansion for $\varphi(r + \xi_1(r))$ by

$$\varphi(r + \xi_1(r)) = \sum_{k=0}^{\infty} Q_k \varphi(r, \dot{\xi}_1, \ldots, \dot{\xi}_1). \tag{14}$$

We note also that, for any $k \in \mathbb{N}$, relation (9) yields

$$\mathbb{E}\varphi(r + \xi_1(r))^2 = \sum_{j=0}^{\infty} \mathbb{E}Q_j \varphi(r, \dot{\xi}_1, \ldots, \dot{\xi}_1)^2 \geq \mathbb{E}Q_k \varphi(r, \dot{\xi}_1, \ldots, \dot{\xi}_1)^2 = k! \|Q_k\|_k^2. \tag{15}$$

Lemma 3. *For any $k \geq 0$ and $r \in \mathbb{R}$, the function $\varphi \mapsto Q_k \varphi(r, \cdot, \ldots, \cdot)$ is a linear continuous mapping from Φ onto $L_2(\mathbb{R}^k)$.*

Proof. The linearity is obvious. The continuity follows from

$$\|Q_k \varphi_1(r, \cdot, \ldots, \cdot) - Q_k \varphi_2(r, \cdot, \ldots, \cdot)\|_k^2 \leq \frac{1}{k!} \mathbb{E}((\varphi_1(r + \xi_1(r)) - \varphi_2(r + \xi_1(r)))^2 \leq$$

$$\leq \frac{1}{k!} \sqrt{n} e^{\frac{r^2}{2(n-1)}} \|\varphi_1 - \varphi_2\|^2_{N(0,n)},$$

where we used (15) and (13). The lemma is proved. □

We now define an action of the operators Q_k on the functions with values in a Hilbert space. Let H be a Hilbert space. By $\Phi(\mathbb{R}, H)$, we denote the set of all measurable mappings f from $\mathbb{R}$ onto H such that $\|f\|_H \in \Phi$. We define the distance in $\Phi(\mathbb{R}, H)$ by

$$d(\varphi', \varphi'') := \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\big\| \|\varphi'(\cdot) - \varphi''(\cdot)\|_H \big\|_{N(0,n)}}{1 + \big\| \|\varphi'(\cdot) - \varphi''(\cdot)\|_H \big\|_{N(0,n)}}.$$

Let $\{e_j\}_{j=1}^{\infty}$ be an orthonormal basis for the Hilbert space H. Then, for any $x \in \mathbb{R}$, we have

$$\varphi(x) = \sum_{j=1}^{\infty} \varphi_j(x) e_j.$$

Note that

$$\big\| \|\varphi(\cdot)\|_H \big\|^2_{N(0,n)} = \frac{1}{\sqrt{2\pi n}} \int_{\mathbb{R}} \|\varphi(x)\|_H^2 e^{-\frac{x^2}{2n}} dx = \tag{16}$$

$$= \frac{1}{\sqrt{2\pi n}} \sum_{j=1}^{\infty} \int_{\mathbb{R}} \varphi_j^2(x) e^{-\frac{x^2}{2n}} dx = \sum_{j=1}^{\infty} \|\varphi_j(\cdot)\|^2_{N(0,n)}.$$

It follows from $\varphi_j(\cdot) \in \Phi,\ j \geq 1$, that, for any $r \in \mathbb{R}$,

$$\varphi_j(r + \xi_1(r)) = \sum_{k=0}^{\infty} Q_k \varphi_j(r, \dot{\xi}_1, \dots, \dot{\xi}_1).$$

We now define an action of the operator Q_k on $\varphi \in \Phi(\mathbb{R}, H)$ by

$$Q_k \varphi(r, \cdot, \dots, \cdot) := \sum_{j=1}^{\infty} Q_k \varphi_j(r, \cdot, \dots, \cdot) e_j.$$

From (15) and (13), it follows that

$$\sum_{j=1}^{\infty} Q_k \varphi_j(r, x_1, \dots, x_k)^2 \leq |x_1|^2 \dots |x_k|^2 \sum_{j=1}^{\infty} \|Q_k \varphi_j(r, \cdot, \dots, \cdot)\|_k^2 \leq$$

$$\leq |x_1|^2 \dots |x_k|^2 \sum_{j=1}^{\infty} \frac{1}{k!} \mathbb{E} \varphi_j^2(r + \xi_1(r)) \leq const \sum_{j=1}^{\infty} \|\varphi_j\|^2_{N(0,n)} < +\infty.$$

So, the action of the operator Q_k on $\varphi \in \Phi(\mathbb{R}, H)$ is well-defined. It can be easily verified that the definition does not depend on the choice of the basis $\{e_j, j \geq 1\}$.

Thus, the operator Q_k maps the function $\varphi \in \Phi(\mathbb{R}, H)$ into the Hilbert–Schmidt form with values in the Hilbert space H. As in the one-dimensional case, for any $\varphi \in \Phi(\mathbb{R}, H)$, we obtain

$$\varphi(r + \xi_1(r)) = \sum_{k=0}^{\infty} \sum_{j=1}^{\infty} Q_k \varphi_j(r, \dot{\xi}_1, \dots, \dot{\xi}_1) e_j =: \sum_{k=0}^{\infty} Q_k \varphi(r, \dot{\xi}_1, \dots, \dot{\xi}_1),$$

where $Q_k \varphi(r, \dot{\xi}_1, \dots, \dot{\xi}_1)$ are random elements in H.

Lemma 4. *For any $k \geq 0$ and $r \in \mathbb{R}$, the function*

$$\Phi(\mathbb{R}, H) \ni \varphi \mapsto Q_k \varphi(r, \cdot, \dots, \cdot)$$

is a linear continuous mapping from $\Phi(\mathbb{R}, H)$ onto the space of multilinear symmetric Hilbert–Schmidt forms with values in the Hilbert space H.

Proof. The proof is similar to that of Lemma 2 and is omitted. □

Example 3. For the sequel, we need to define an action of Q_k on multilinear Hilbert–Schmidt forms. Let $H = L_2(\mathbb{R}^k)$. Consider $A_k \in \Phi(\mathbb{R}, H)$. Let $\{E_k^i(\cdot, \dots, \cdot)\}_{i=1}^{\infty}$ be an orthogonal basis for $L_2(\mathbb{R}^k)$. Then we have

$$A_k(x, \cdot, \dots, \cdot) = \sum_{i=1}^{\infty} a_i(x) E_k^i(\cdot, \dots, \cdot).$$

For any $r \in \mathbb{R}$ with the use of the general construction described above, we obtain

$$A_k(r + \xi_1(r), \underbrace{\cdot, \dots, \cdot}_{k}) = \sum_{j=0}^{\infty} Q_j A_k(r, \underbrace{\dot{\xi}_1, \dots, \dot{\xi}_1}_{j}, \underbrace{\cdot, \dots, \cdot}_{k}),$$

where we denoted

$$Q_j A_k(r, \underbrace{\dot{\xi}_1, \dots, \dot{\xi}_1}_{j}, \underbrace{\cdot, \dots, \cdot}_{k}) := \sum_{i=1}^{\infty} Q_j(a_i, r, \dot{\xi}_1, \dots, \dot{\xi}_1) E_k^i(\cdot, \dots, \cdot).$$

In general, the action of a random mapping on random elements is not well-defined. But in the case where a random mapping Q_j and a random element $A_k(r+\xi(r))$ are independent, their composition is well-defined [10].

We note that the white noise $\dot{\xi}_j$ can be regarded as that on the space

$$\mathcal{L}^j = \{F = \{\underbrace{0,\ldots,0}_{j-1}, f, 0, \ldots\}),\ f \in L_2(\mathbb{R})\}.$$

Assume that the mapping A_k is defined on the space $\mathcal{L}^2$.

Then the form $A_k(r+\xi_1(r), \underbrace{\cdot,\ldots,\cdot}_{k})$ is defined on the space $\mathcal{L}^2$ and measurable with respect to the white noise in $\mathcal{L}^1$. From the orthogonality of the spaces $\mathcal{L}^1$ and $\mathcal{L}^2$, it follows that

$$A_k(r+\xi_1(r), \underbrace{\dot{\xi}_2,\ldots,\dot{\xi}_2}_{k}) = \sum_{j=0}^{\infty} Q_j A_k(r, \underbrace{\dot{\xi}_1,\ldots,\dot{\xi}_1}_{j}, \underbrace{\dot{\xi}_2,\ldots,\dot{\xi}_2}_{k}).$$

Theorem 1. *Let $\{x_n(u), u \in \mathbb{R}\}_{n\geq 1}$ be a sequence that satisfies Eq. (1). For any $\varphi \in \Phi, n \geq 1,$ we have*

$$\varphi(x_n(u)) = \sum_{k=0}^{\infty} \sum_{\substack{l_1+\ldots+l_n=k \\ l_1,\ldots,l_n\geq 0}} Q_{l_n} Q_{l_{n-1}} \ldots Q_{l_1} \varphi(u, \underbrace{\dot{\xi}_n,\ldots,\dot{\xi}_n}_{l_n}, \ldots, \underbrace{\dot{\xi}_1,\ldots,\dot{\xi}_1}_{l_1}).$$

Proof. First of all, let us verify that the iterated action of the operators Q_j is well-defined. Note that $\{\xi_n(u)\}_{n\geq 1}$ are identically distributed. So, for $x_n(u)$ by (14), we have

$$\varphi(x_n(u)) = \varphi(x_{n-1}(u) + \xi_n(x_{n-1}(u))) = \sum_{k_1=0}^{\infty} Q_{k_1} \varphi(x_{n-1}(u), \dot{\xi}_n, \ldots, \dot{\xi}_n).$$

From the definition of an action of the operators Q_k on the functions with values in a Hilbert space, the action of the operator Q_{k_2} on Q_{k_1} is well-defined in the case where $\|Q_{k-1}\varphi(\cdot, \underbrace{\cdot,\ldots,\cdot}_{k_1})\|^2_{k_1} \in \Phi$. To verify this condition, we note that relations (15) and (13) imply that, for any n,

$$\|Q_{k_1}\varphi(r,\cdot,\ldots,\cdot)\|^2_{k_1} \leq \frac{1}{k_1!}\mathbb{E}\varphi^2(r+\xi_1(r)) \leq \frac{1}{k_1!}\sqrt{n}e^{\frac{r^2}{2(n-1)}}\|\varphi\|^2_{N(0,n)}.$$

Thus, for any $m \geq 1$, we put $n > m+1$ and obtain

$$\frac{1}{\sqrt{2\pi m}} \int_{\mathbb{R}} \|Q_k\varphi(r,\cdot,\ldots,\cdot)\|^2_{k_1} e^{-\frac{r^2}{2m}} dr \leq \frac{1}{k_1!}\sqrt{n}\|\varphi\|^2_{N(0,n)} \int_{\mathbb{R}} e^{\frac{r^2}{2(n-1)}} e^{-\frac{r^2}{2m}} dr < +\infty.$$

So, the action of the operator Q_{k_2} on Q_{k_1} is well-defined. Applying the construction from Example 3 to the mapping $A_k(x,\cdot,\ldots,\cdot) = Q_{k_1}\varphi(x,\cdot,\ldots,\cdot)$, we obtain

$$Q_{k_1}\varphi(x_{n-2}(u) + \xi_{n-1}(x_{n-2}(u)), \underbrace{\cdot,\ldots,\cdot}_{k_1}) =$$

$$= \sum_{k_2=0}^{\infty} Q_{k_2} Q_{k_1} \varphi(x_{n-1}(u), \underbrace{\dot{\xi}_{n-1},\ldots,\dot{\xi}_{n-1}}_{k_2}, \underbrace{\cdot,\ldots,\cdot}_{k_1}).$$

Note also that the Hilbert space

$$\mathcal{H} = \{F : F = (f_1, f_2, \ldots),\ f_k \in L_2(\mathbb{R}),\ \sum_{k=1}^{\infty} \|f_k\|^2_{L_2} < +\infty\},$$

splits into the direct sum

$$\mathcal{H} = \bigoplus_{n=1}^{\infty} \mathcal{L}^n,$$

where $\mathcal{L}^n$ is the subspace of $\mathcal{H}$:

$$\mathcal{L}^n = \{F \in \mathcal{H} :\ F = (\underbrace{0, \ldots, 0}_{n-1}, f, 0, \ldots)\},\ f \in L_2(\mathbb{R}).$$

In these terms, the form $Q_k\varphi(x_{n-1}(u) + \xi_n(x_{n-1}(u)), \underbrace{\cdot, \ldots, \cdot}_{k})$ is defined on the subspace $\mathcal{L}^{n+1}$ and measurable with respect to the white noise in $\mathcal{L}^n$. This implies that

$$Q_{k_1}\varphi(x_{n-1}(u) + \xi_n(x_{n-1}(u)), \dot{\xi}_{n+1}, \ldots, \dot{\xi}_{n+1}) =$$
$$= \sum_{k_2=0}^{\infty} Q_{k_2}Q_{k_1}\varphi(x_{n-1}(u), \underbrace{\dot{\xi}_n, \ldots, \dot{\xi}_n}_{k_2}, \underbrace{\dot{\xi}_{n+1}, \ldots, \dot{\xi}_{n+1}}_{k_1}).$$

We note also that

$$\|Q_{k_m}Q_{k_{m-1}} \ldots Q_{k_1}\varphi(u, \cdot, \ldots, \cdot)\|^2_{k_m+\ldots+k_1} \leq$$
$$\leq \frac{1}{k_m!}\sqrt{l}e^{\frac{u^2}{2(l-1)}} \big\| \|Q_{k_{m-1}} \ldots Q_{k_1}\varphi(\cdot, \cdot, \ldots, \cdot)\|_{k_{m-1}+\ldots+k_1}\big\|^2_{N(0,l)} \in \Phi.$$

Using similar arguments, we complete the proof.

□

It is obvious that the zero-order expansion term for $\varphi(x_n(u))$ is equal to $\mathbb{E}\varphi(x_n(u))$. It follows from Lemma 1 that

$$\mathbb{E}\varphi(x_n(u)) = \frac{1}{\sqrt{2\pi n}} \int_{\mathbb{R}} \varphi(u + v)e^{-\frac{v^2}{2n}} dv.$$

Let us give the explicit form of the first-order expansion term for $\varphi(x_n(u))$. When $n = 1$, we have

$$\varphi(u + \xi_1(u)) = \sum_{k=0}^{\infty} Q_k\varphi(u, \dot{\xi}_1, \ldots, \dot{\xi}_1).$$

On the other hand, $\varphi(u + \xi_1(u))$ as a function of the Gaussian random variable can be represented as a series in Hermite polynomials,

$$\varphi(u + \xi_1(u)) = \sum_{k=0}^{\infty} H_k(\xi_1(u)) \frac{1}{k!} \int_{\mathbb{R}} \varphi(u + v)H_k(v)p_1(v)dv,$$

where $p_n(x) = \frac{1}{\sqrt{2\pi n}}e^{-\frac{x^2}{2n}}$. Using this representation, we get

$$Q_0\varphi(u) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \varphi(u + v)e^{-\frac{v^2}{2}} dv$$

$$Q_1\varphi(u, \dot{\xi}_1) = H_1(\xi_1(u)) \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \varphi(u + v)H_1(v)e^{-\frac{v^2}{2}} dv =$$
$$= \xi_1(u) \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \varphi(u + v)ve^{-\frac{v^2}{2}} dv = \xi_1(u) \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \varphi'(u + v)e^{-\frac{v^2}{2}} dv =$$
$$= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \psi(u - y)dw_1(y) \int_{\mathbb{R}} \varphi'(u + v)e^{-\frac{v^2}{2}} dv =$$
$$= \int_{\mathbb{R}} \int_{\mathbb{R}} \varphi'(u + v)p_1(v)\psi(u - y)dvdw_1(y).$$

We recall that

$$\varphi(x_n(u)) = \sum_{k=0}^{\infty} \sum_{\substack{l_1,\dots,l_n \geq 0 \\ l_1+\dots+l_n=k}} Q_{l_1} Q_{l_2} \dots Q_{l_n} \varphi(u, \underbrace{\dot{\xi}_1, \dots, \dot{\xi}_1}_{l_1}, \dots, \underbrace{\dot{\xi}_n, \dots, \dot{\xi}_n}_{l_n}).$$

Thus, the first-order expansion term for $\varphi(x_n(u))$ has the form

$$\sum_{j=0}^{n-1} Q_0^j Q_1 Q_0^{n-j-1} \varphi(u, \dot{\xi}_{j+1}).$$

For $Q_0^k \varphi(u)$, we get the explicit form

$$Q_0^k \varphi(u) = \int_{\mathbb{R}} \varphi(u+v) p_k(v) dv.$$

Then the action of Q_0^{n-j-1} on the random variable $\varphi(x_n(u)) = \varphi(x_{n-1}(u) + \xi_n(x_{n-1}(u)))$ is as follows:

$$Q_0^{n-j-1} \varphi(x_{j+1}(u)) = \int_{\mathbb{R}} \varphi(x_{j+1}(u) + v) p_{j+1} dv.$$

By definition, we put $(T_k f)_u = Q_0^k(f, u)$ and $\theta_x f = f(\cdot + x)$. Then we obtain

$$Q_0^j Q_1 Q_0^{n-j-1} \varphi(u, \dot{\xi}_{j+1}) = T_j \int_{\mathbb{R}} \int_{\mathbb{R}} ((\theta_v T_{n-j-1} \varphi)'_v \theta_{-y} \psi)_u p_1(v) dv dw_{j+1}(y) =$$

$$= \int_{\mathbb{R}} \int_{\mathbb{R}} (T_j (\theta_v T_{n-j-1} \varphi)'_v \theta_{-y} \psi)_u p_1(v) dv dw_{j+1}(y).$$

Example 4. For the covariance function $\Gamma(u) = \cos u$, we will give the explicit form of the first expansion term. In this case, system (1) has the form

$$\begin{cases} x_{n+1}(u) = x_n(u) + \eta'_{n+1} \cos x_n(u) + \eta''_{n+1} \sin x_n(u) \\ x_0(u) = u, \end{cases}$$

where $\{\eta'_n\}_{n\geq 1}, \{\eta''_n\}_{n\geq 1}$ are independent sequences of independent standard Gaussian random variables. The first expansion term for $x_n(u)$ has the form

$$\sum_{k=1}^{n} (\eta'_k \cos u + \eta''_k \sin u) e^{-\frac{k-1}{2}}. \tag{17}$$

Indeed, for $n = 1$, we have

$$x_1(u) = u + \eta'_1 \cos u + \eta''_1 \sin u,$$

and relation (15) is obvious. Suppose that the first expansion term for $x_n(u)$ has the form (15). For $x_{n+1}(u)$, we obtain

$$x_{n+1}(u) = x_n(u) + \eta'_{n+1} \cos x_n(u) + \eta''_{n+1} \sin x_n(u).$$

We note that η'_{n+1} does not depend on $\sigma(\eta'_1, \dots, \eta''_n)$, and η''_{n+1} does not depend on $\sigma(\eta'_1, \dots, \eta''_n)$. So, the first expansion term for $x_{n+1}(u)$ has the form

$$\sum_{k=1}^{n} (\eta'_k \cos u + \eta''_k \sin u) e^{-\frac{k-1}{2}} + \eta'_{n+1} \mathbb{E} \cos x_n(u) + \eta''_{n+1} \mathbb{E} \sin x_n(u) =$$

$$= \sum_{k=1}^{n+1} (\eta'_k \cos u + \eta''_k \sin u) e^{-\frac{k-1}{2}},$$

where we used the relations

$$\mathbb{E} \cos x_n(u) = \int_{\mathbb{R}} \cos(u+v) p_n(v) dv = e^{-\frac{n}{2}} \cos u,$$

$$\mathbb{E}\sin x_n(u) = \int_{\mathbb{R}} \sin(u+v)p_n(v)dv = e^{-\frac{n}{2}}\sin u,$$

$p_n(x) = \frac{1}{\sqrt{2\pi n}}e^{-\frac{x^2}{2n}}$.

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