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CORRELOGRAM ESTIMATION OF RESPONSE FUNCTIONS OF LINEAR SYSTEMS IN SCHEME OF SOME INDEPENDENT SAMPLES

The problem of estimation of an unknown response function from $L_2(\mathbf{R})$ of a linear system is considered. The inputs are supposed to be stationary zero-mean Gaussian almost surely sample continuous processes. We take the integral-type sample input-output cross-correlograms as estimators of the response function and apply the scheme of some independent samples, when the pair of inputs and outputs are observed. The asymptotic normality of the distributions of centered cross-correlogram estimations in the space of continuous functions and the construction of the confidence bands for the limiting process are discussed.

1. Introduction

We consider a problem of estimation of an impulse response function $H = (H(\tau), \tau \in \mathbf{R})$, of a time-invariant linear system by observations of responses of the system on certain inputs using the cross-correlogram method. According to this approach, the sample correlograms between the inputs and outputs are taken as estimators for the unknown function H ([1], [5], [2], [7]-[10]).

To construct the cross-correlogram estimators, two schemes are usually applied; the first is the scheme of one sample (one realization is observed on the unbounded expansion of the observing set), and the second is the scheme of some independent samples (some independent realizations are observed on the fix observing set). Using the first scheme, the asymptotic properties of the cross-correlogram estimator for the response function by perturbations of the system by Gaussian stationary processes were investigated, for instance, in works ([5], [2], [9]). Using the second scheme, the asymptotic normality of the correlogram estimators for the correlation function of a Gaussian process or a homogeneous Gaussian field was considered, for example, in ([4], [6], [17]).

In this paper, we consider the estimators of the response real-valued function $H \in L_2(\mathbf{R})$ of the linear system that is perturbed by a family of stationary zero-mean Gaussian almost surely sample continuous processes, using the scheme of some independent samples, when the pair of inputs and outputs are observed. We remark that such a scheme for solving the problem was not considered earlier.

We investigate the conditions for asymptotic normality of the integral-type zero-mean cross-correlogram estimators for the response function in the space of continuous functions and construct the confidence functional bands for the limiting process.

2. Definitions and preliminaries

We introduce the following notations used throughout the work:
$\mathbf{R} = (-\infty, \infty)$;

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$L_p(\mathbf{R}), p \in [1, \infty)$, is the Banach space of Lebesgue p-integrable complex-valued functions $\phi = (\phi(x),\ x \in \mathbf{R})$ with the norm

$$\|\phi\|_p = \left(\int_{-\infty}^{\infty} |\phi(x)|^p \, dx \right)^{\frac{1}{p}};$$

$L_\infty(\mathbf{R})$ is the Banach space of complex-valued essentially bounded functions $\phi = (\phi(x),\ x \in \mathbf{R})$ with the norm $\|\phi\|_\infty = \sup_{x \in \mathbf{R}} |\phi(x)|$;

$C[a, b],\ [a, b] \subset \mathbf{R}$, is the separable Banach space of real-valued continuous functions $\phi = (\phi(x),\ x \in [a, b])$ with the norm $\|\phi\|_\infty = \sup_{x \in [a,b]} |\phi(x)|$.

Assume that $X_\Delta = (X_\Delta(t),\ t \in \mathbf{R})$, $\Delta > 0$, is a family of measurable real-valued separable stationary zero-mean Gaussian processes that disturb a linear time-invariant system. Let $f_\Delta = (f_\Delta(\lambda),\ \lambda \in \mathbf{R})$, $\Delta > 0$, be a family of spectral densities of the processes X_Δ, $\Delta > 0$, satisfying the following conditions:

$$f_\Delta(\lambda) = f_\Delta(-\lambda),\ \lambda \in \mathbf{R}; \tag{1a}$$

$$f_\Delta \in L_\infty(\mathbf{R}); \tag{1b}$$

$$f_\Delta \in L_1(\mathbf{R}). \tag{1c}$$

We also suppose that all X_Δ are almost surely (a.s.) sample continuous processes.

Remark 2.1. By the Dudley theorem [12], a separable Gaussian mean-square continuous centered process $U = (U(t), t \in \mathbf{R})$ is a.s. sample continuous on $\mathbf{R}$, if its Dudley integral is convergent for any $[a, b] \subset \mathbf{R}$. That is, for all $h > 0$ and any $[a, b] \subset \mathbf{R}$,

$$\int_0^h \mathcal{H}_{\sigma_U}^{\frac{1}{2}}([a, b], \varepsilon) d\varepsilon < \infty, \tag{DI}$$

where $\mathcal{H}_{\sigma_U}([a, b], \varepsilon)$ is a metric entropy of an interval $[a, b]$ with respect to the pseudometric driven by the mean-square deviation of the process U: $\sigma_U(t, s) = \big(\mathbf{E}|U(t) - U(s)|^2\big)^{\frac{1}{2}},\ t, s \in \mathbf{R}$.

Moreover, if U is a stationary process, then, by the Fernique theorem (see [13]), condition (DI) is not only sufficient but also is necessary for the a.s. continuity of the realizations of U.

The processes $X_\Delta, \Delta > 0$, are a.s. sample continuous on $\mathbf{R}$, if, for example, their spectral densities have the form

$$f_\Delta(\lambda) = \frac{1}{2\pi} \exp\left\{ -\frac{\lambda^2}{\Delta} \right\},\ \lambda \in \mathbf{R}.$$

The reaction of the system to an input signal X_Δ is represented by the integral

$$Y_\Delta(t) = \int_{-\infty}^{\infty} H(t - s) X_\Delta(s) \, ds,\ t \in \mathbf{R}, \tag{2}$$

which is interpreted as a mean-square Riemann integral.

From the definitions of the processes X_Δ and Y_Δ, it follows that these processes are jointly Gaussian and jointly stationary. Moreover, we suppose that the processes Y_Δ, $\Delta > 0$, are a.s. sample continuous on $\mathbf{R}$. Without loss of generality, we also suppose that all Y_Δ are separable processes.

Remark 2.2. Under certain restrictions on the response function H, the a.s. sample continuity of the process X_Δ yields the a.s. sample continuity of the process Y_Δ.

Consider the pseudometrics driven by the mean-square deviations of the processes X_Δ and Y_Δ:

$$\sigma_{X_\Delta}(t,s) = \Big[\mathbf{E}|X_\Delta(t)-X_\Delta(s)|^2\Big]^{\frac{1}{2}} = 2\Big[\int_{-\infty}^{\infty}\sin^2\Big(\frac{(t-s)\lambda}{2}\Big)f_\Delta(\lambda)d\lambda\Big]^{\frac{1}{2}},$$

$$\sigma_{Y_\Delta}(t,s) = \Big[\mathbf{E}|Y_\Delta(t)-Y_\Delta(s)|^2\Big]^{\frac{1}{2}} = 2\Big[\int_{-\infty}^{\infty}\Big|H^*(\lambda)\Big|^2\sin^2\Big(\frac{(t-s)\lambda}{2}\Big)f_\Delta(\lambda)d\lambda\Big]^{\frac{1}{2}},$$
$$t,s\in\mathbf{R},$$

where H^* is the Fourier–Plancherel transform of the function H in the space $L_2(\mathbf{R})$.

For example, if $\sup_{\lambda\in\mathbf{R}}|H^*(\lambda)|<\infty$, then the inequality

$$\sigma_{Y_\Delta}(t,s)\le\Big(\sup_{\lambda\in\mathbf{R}}|H^*(\lambda)|\Big)\sigma_{X_\Delta}(t,s),\quad t,s\in\mathbf{R},$$

holds true.

From the Marcus–Shepp comparison inequality for the Gaussian processes [18], which is applied to the separable zero-mean Gaussian processes X_Δ and Y_Δ, where $X_\Delta\in C(\mathbf{R})$, it follows that Y_Δ is an a.s. sample continuous process on $\mathbf{R}$.

Suppose that we observe n independent realizations of the pair of processes $(X_\Delta(t), t\in[0,T])$, $(Y_\Delta(t), t\in[0,T+T_1])$ (here, $T+T_1>0$), which will be denoted by $\big(X_\Delta^{(j)}, Y_\Delta^{(j)}\big)$, $j=1,\dots,n$.

The properties of X_Δ and Y_Δ imply that $X_\Delta^{(j)}$ and $Y_\Delta^{(j)}$, $j=1,...,n$, are jointly Gaussian and a.s. sample continuous processes.

The so-called cross-correlogram

$$\widehat{H}_{T,\Delta}^{(n)}(\tau)=\frac{1}{n}\sum_{j=1}^{n}\frac{1}{T}\int_0^T Y_\Delta^{(j)}(t+\tau)X_\Delta^{(j)}(t)\,dt,\ \tau\in[0,T_1], \tag{3}$$

will be used as an estimator for the response function H (here, T is the length of the averaging interval $[0,T]$).

Since $X_\Delta^{(j)}$ and $Y_\Delta^{(j)}$ are a.s. sample continuous processes, the process $\widehat{H}_{T,\Delta}^{(n)} = (\widehat{H}_{T,\Delta}^{(n)}(\tau)$,
$\tau\in[0,T_1])$ is also a.s. sample continuous. Moreover,

$$\mathbf{E}\widehat{H}_{T,\Delta}^{(n)}(\tau)=\int_{-\infty}^{\infty}H(s)K_{X_\Delta}(\tau-s)\,ds,\ \tau\in[0,T_1].$$

By the last relation, $\mathbf{E}\widehat{H}_{T,\Delta}^{(n)}$ does not depend on the quantity n and the length T of the averaging interval $[0,T]$.

Since, generally speaking, $\mathbf{E}\widehat{H}_{T,\Delta}^{(n)}(\tau)\ne H(\tau)$, estimator (3) is biased.

Consider the empirical stochastic process

$$\widehat{Z}_{T,\Delta}^{(n)}(\tau)=\sqrt{n}\Big(\widehat{H}_{T,\Delta}^{(n)}(\tau)-\mathbf{E}\widehat{H}_{T,\Delta}^{(n)}(\tau)\Big),\tau\in[0,T_1], \tag{4}$$

which is zero-mean and has the following correlation function for all $\tau_1,\tau_2\in[0,T_1]$:

$$C_{T,\Delta}^{(n)}(\tau_1,\tau_2)=\mathbf{E}\widehat{Z}_{T,\Delta}^{(n)}(\tau_1)\widehat{Z}_{T,\Delta}^{(n)}(\tau_2)=$$
$$=\frac{1}{nT^2}\sum_{j=1}^{n}\sum_{l=1}^{n}\int_0^T\int_0^T\Big[\mathbf{E}Y_\Delta^{(j)}(t+\tau_1)Y_\Delta^{(l)}(s+\tau_2)\times\mathbf{E}X_\Delta^{(l)}(s)X_\Delta^{(j)}(t)+$$
$$+\mathbf{E}Y_\Delta^{(j)}(t+\tau_1)X_\Delta^{(l)}(s)\times\mathbf{E}Y_\Delta^{(l)}(s+\tau_2)X_\Delta^{(j)}(t)\Big]\,dt\,ds=$$
$$=\frac{1}{T^2}\int_0^T\int_0^T\Big[K_{Y_\Delta}(t-s+\tau_1-\tau_2)\times K_{X_\Delta}(s-t)+K_{Y_\Delta,X_\Delta}(t-s+\tau_1)$$

$$\times K_{Y_\Delta,X_\Delta}(s-t+\tau_2)\Big]\,dt\,ds,$$

where K_{Y_Δ} is the correlation function of Y_Δ, K_{X_Δ} is the correlation function of X_Δ, and K_{Y_Δ,X_Δ} is the joint correlation function of Y_Δ and X_Δ.

Using the spectral representations of the correlation functions,

$$K_{X_\Delta}(t)=\int_{-\infty}^{\infty}e^{it\lambda}f_\Delta(\lambda)d\lambda,\quad t\in\mathbf{R};$$

$$K_{Y_\Delta}(t)=\int_{-\infty}^{\infty}e^{it\lambda}|H^*(\lambda)|^2f_\Delta(\lambda)d\lambda,\quad t\in\mathbf{R};$$

$$K_{Y_\Delta,X_\Delta}(t)=\int_{-\infty}^{\infty}e^{it\lambda}H^*(\lambda)f_\Delta(\lambda)d\lambda,\quad t\in\mathbf{R},$$

we see, consequently, that the function $C_{T,\Delta}^{(n)}=\Big(C_{T,\Delta}^{(n)}(\tau_1,\tau_2),\ \tau_1,\ \tau_2\in[0,T_1]\Big)$ does not depend on the quantity n and has the form

$$(5)\quad C_{T,\Delta}^{(n)}(\tau_1,\tau_2)=C_{T,\Delta}(\tau_1,\tau_2)=$$
$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Big[e^{i(\tau_1-\tau_2)\lambda_2}|H^*(\lambda_2)|^2+e^{i(\tau_1\lambda_1+\tau_2\lambda_2)}H^*(\lambda_1)H^*(\lambda_2)\Big]f_\Delta(\lambda_1)f_\Delta(\lambda_2)\times$$
$$\times\frac{1}{T^2}\left|\int_0^T e^{it(\lambda_2-\lambda_1)}\,dt\right|^2\,d\lambda_1d\lambda_2=$$
$$=\frac{2\pi}{T}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Big[e^{i(\tau_1-\tau_2)\lambda_2}|H^*(\lambda_2)|^2+e^{i(\tau_1\lambda_1+\tau_2\lambda_2)}H^*(\lambda_1)H^*(\lambda_2)\Big]\Phi_T(\lambda_2-\lambda_1)\times$$
$$\times f_\Delta(\lambda_1)f_\Delta(\lambda_2)\ d\lambda_1d\lambda_2,$$

where H^* is the Fourier–Plancherel transform of the function H in the space $L_2(\mathbf{R})$, and Φ_T is the Fejer kernel; that is,

$$\Phi_T(x)=\frac{1}{2\pi T}\Big(\frac{\sin(Tx/2)}{x/2}\Big)^2,\ x\in\mathbf{R}.$$

Further, we will discuss the properties of the process $\widehat{Z}_{T,\Delta}^{(n)}$ as $n\to\infty$ in the space of continuous functions, by supposing that the parameters T and Δ have fixed values. For this, we need some additional statements.

The first one is the strong law of large numbers in the separable Banach space $C[0,M]$, where $M>0$, (see [14]).

Statement 2.1. We assume that $\{(\xi_j(t),t\in[0,M]),j\ge1\}$, is a sequence of independent real-valued a.s. sample continuous stochastic processes that are the copies of a process $(\xi(t),t\in[0,M])$. If $\mathbf{E}\sup_{t\in[0,M]}|\xi(t)|<\infty$. Then

$$\mathsf{P}\left\{\lim_{n\to\infty}\sup_{t\in[0,M]}\Big|\frac{1}{n}\sum_{j=1}^{n}\xi_j(t)-\mathbf{E}\xi(t)\Big|=0\right\}=1.$$

Moreover, $\mathbf{E}\xi(\cdot)\in C[0,M]$.

The second statement is the following version of the central limit theorem for the space of continuous functions defined on the compact sets (see [15]).

Statement 2.2. Assume that $\{(\xi_j(t),t\in[0,M]),j\ge1\}$ is a sequence of independent real-valued a.s. sample continuous stochastic processes that are the copies of a process $(\xi(t),t\in[0,M])$ such that $\mathbf{E}\xi(t)=0$ and $\mathbf{E}\xi^2(t)<\infty,\ t\in[0,M]$. If there exists a function g, which is the modulus of continuity on some interval $[0,u_0], u_0>0$, and the inequality

$$|\xi(t)-\xi(s)|<Vg(|t-s|),\ |t-s|<u_0,$$

holds true a.s., where V and g are such that $\mathbf{E}V^2<\infty$ and

$$\int_0^{u_0}\frac{g(u)}{u|\ln u|^{\frac{1}{2}}}du<\infty,$$

then the distributions of the stochastic processes $\{\frac{1}{\sqrt{n}}\sum_{j=1}^{n}\xi_j(t),\ t\in[0,M]\}$ converge weakly as $n\to\infty$ in the space $C[0,M]$ to a Gaussian process with a zero mean and a correlation function $\mathbf{E}\xi(t_1)\xi(t_2),\ t_1,t_2\in[0,M]$.

3. Main results

In this section, we discuss the conditions for the asymptotic normality of the process $\widehat{Z}_{T,\Delta}^{(n)}$ as $n\to\infty$ in the space of continuous functions and construct the confidence functional bands for the limiting process.

3.1. **Asymptotic properties of the centered estimator.** Without loss of generality, we suppose that, on the same complete probability space $\{\Omega,\mathfrak{F},\mathsf{P}\}$, where the processes $X_\Delta,\ Y_\Delta,\ \widehat{H}_{T,\Delta}^{(n)}$, and $\widehat{Z}_{T,\Delta}^{(n)}$ are considered, a separable Gaussian process $\{Z_{T,\Delta}(\tau),\ \tau\in[0,T_1]\}$ with a zero mean and the correlation function $C_{T,\Delta}$ is defined.

In the following lemma, we state the sufficient conditions for the asymptotic normality of the process $\widehat{Z}_{T,\Delta}^{(n)}$ as $n\to\infty$ in the space of continuous functions $C[0,T_1]$.

Lemma 3.1. *If there exists a function g, which is the modulus of continuity on some interval $[0,u_0], u_0>0$, and such that the relation*

$$\eta=\sup_{\substack{t\neq s\\ |t-s|<u_0\\ t,s\in[0,T+T_1]}}\frac{|Y_\Delta(t)-Y_\Delta(s)|}{g(|t-s|)}<\infty, \tag{6}$$

holds true a.s., and the condition

$$\int_0^{u_0}\frac{g(u)}{u|\ln u|^{\frac{1}{2}}}du<\infty, \tag{7}$$

is satisfied, then the distributions of the processes $\{\widehat{Z}_{T,\Delta}^{(n)}(\tau),\ \tau\in[0,T_1]\}$ converge weakly as $n\to\infty$ in the space $C[0,T_1]$ to the distributions of the Gaussian process $\{Z_{T,\Delta}(\tau),\ \tau\in[0,T_1]\}$ with a zero mean and the correlation function $C_{T,\Delta}$.

Proof of Lemma 3.1. Let us verify the conditions of Statement 2.2. We write (4) as

$$\widehat{Z}_{T,\Delta}^{(n)}(\tau)=\sqrt{n}\Big[\widehat{H}_{T,\Delta}^{(n)}(\tau)-\mathbf{E}\widehat{H}_{T,\Delta}^{(n)}(\tau)\Big]=$$
$$=\frac{1}{\sqrt{n}}\sum_{j=1}^{n}\frac{1}{T}\int_0^T\Big[Y_\Delta^{(j)}(t+\tau)X_\Delta^{(j)}(t)-\mathbf{E}Y_\Delta(t+\tau)X_\Delta(t)\Big]dt$$

and note that, for all $j=1,...,n$,

$$\xi^{(j)}(\tau)=\frac{1}{T}\int_0^T\Big[Y_\Delta^{(j)}(t+\tau)X_\Delta^{(j)}(t)-\mathbf{E}Y_\Delta(t+\tau)X_\Delta(t)\Big]dt,\quad \tau\in[0,T_1].$$

We remark that $\Big\{(\xi^{(j)}(\tau),\ \tau\in[0,T_1]),\ j\ge 1\Big\}$ is a sequence of independent a.s. sample continuous zero-mean stochastic processes that are the copies of the process

$$\xi(\tau)=\frac{1}{T}\int_0^T\Big[Y_\Delta(t+\tau)X_\Delta(t)-\mathbf{E}Y_\Delta(t+\tau)X_\Delta(t)\Big]dt,\quad \tau\in[0,T_1].$$

It is clear that

$$\mathbf{E}\xi(\tau)=0,\quad \tau\in[0,T_1];$$
$$\mathbf{E}\xi^2(\tau)=C_{T,\Delta}(\tau,\tau)=$$
$$=\frac{1}{T^2}\int_0^T\int_0^T\Big[K_{Y_\Delta}(t-s)K_{X_\Delta}(s-t)+K_{Y_\Delta,X_\Delta}(t-s+\tau)K_{Y_\Delta,X_\Delta}(s-t+\tau)\Big]dtds<\infty.$$

Since Y_Δ is the Gaussian sample continuous process and g is the continuous function, it follows from (6) that, for any $\beta>0$,

$$\mathbf{E}|\eta|^\beta=\left(\sup_{\substack{t\ne s\\ |t-s|<u_0\\ t,s\in[0,T+T_1]}}\frac{|Y_\Delta(t)-Y_\Delta(s)|}{g(|t-s|)}\right)^\beta<\infty. \tag{8}$$

As $|\tau_1-\tau_2|<u_0$, formula (6) yields the estimator

$$|\xi(\tau_1)-\xi(\tau_2)|\le\frac{1}{T}\int_0^T|X_\Delta(t)||Y_\Delta(t+\tau_1)-Y_\Delta(t+\tau_2)|dt+\mathbf{E}|X_\Delta(0)||Y_\Delta(\tau_1)-Y_\Delta(\tau_2)|\le$$

$$\le\Big(\sup_{t\in[0,T]}|X_\Delta(t)|\Big)\cdot\Big(\sup_{t\in[0,T]}|Y_\Delta(t+\tau_1)-Y_\Delta(t+\tau_2)|\Big)+\mathbf{E}|X_\Delta(0)||Y_\Delta(\tau_1)-Y_\Delta(\tau_2)|\le$$

$$\le\left[\Big(\sup_{t\in[0,T]}|X_\Delta(t)|\Big)\cdot\sup_{t\in[0,T]}\left(\sup_{\substack{\tau_1\ne\tau_2\\ |\tau_1-\tau_2|<u_0\\ \tau_1,\tau_2\in[0,T_1]}}\frac{|Y_\Delta(t+\tau_1)-Y_\Delta(t+\tau_2)|}{g(|\tau_1-\tau_2|)}\right)+\right.$$

$$\left.+\mathbf{E}|X_\Delta(0)|\cdot\left(\sup_{\substack{\tau_1\ne\tau_2\\ |\tau_1-\tau_2|<u_0\\ \tau_1,\tau_2\in[0,T_1]}}\frac{|Y_\Delta(\tau_1)-Y_\Delta(\tau_2)|}{g(|\tau_1-\tau_2|)}\right)\right]\cdot g(|\tau_1-\tau_2|)=$$

$$=\left[\Big(\sup_{t\in[0,T]}|X_\Delta(t)|\Big)\cdot\left(\sup_{\substack{t\ne s\\ |t-s|<u_0\\ t,s\in[0,T+T_1]}}\frac{|Y_\Delta(t)-Y_\Delta(s)|}{g(|t-s|)}\right)+\right.$$

$$\left.+\mathbf{E}|X_\Delta(0)|\cdot\left(\sup_{\substack{t\ne s\\ |t-s|<u_0\\ t,s\in[0,T_1]}}\frac{|Y_\Delta(t)-Y_\Delta(s)|}{g(|t-s|)}\right)\right]\cdot g(|\tau_1-\tau_2|)=$$

$$=\left[\sup_{t\in[0,T]}|X_\Delta(t)|\cdot\eta+\mathbf{E}|X_\Delta(0)|\cdot\eta\right]\cdot g(|\tau_1-\tau_2|).$$

We note that $V=\sup_{t\in[0,T]}|X_\Delta(t)|\cdot\eta+\mathbf{E}|X_\Delta(0)|\cdot\eta$ and evaluate the second moment of this variable.

Since X_Δ and Y_Δ are Gaussian a.s. sample continuous processes, the relations

$$\mathbf{E}\left(\sup_{t\in[0,T]}|X_\Delta(t)|\right)^\alpha<\infty;\qquad \mathbf{E}\left(\sup_{t\in[0,T+T_1]}|Y_\Delta(t)|\right)^\alpha<\infty, \tag{9}$$

hold true for all $\alpha>0$. Thus, together with (8), we get

$$\mathbf{E}V^2=\mathbf{E}\left[\sup_{t\in[0,T]}|X_\Delta(t)|\cdot\eta+\mathbf{E}|X_\Delta(0)|\cdot\eta\right]^2\le$$

$$\le 2\left[\mathbf{E}\Big(\sup_{t\in[0,T]}|X_\Delta(t)|\cdot\eta\Big)^2+\mathbf{E}\Big(\mathbf{E}|X_\Delta(0)|\cdot\eta\Big)^2\right]\le$$

$$\le 2\left[\left(\mathbf{E}\Big(\sup_{t\in[0,T]}|X_\Delta(t)|\Big)^4\cdot\mathbf{E}\eta^4\right)^{\frac12}+\Big(\mathbf{E}|X_\Delta(0)|\Big)^2\cdot\mathbf{E}\eta^2\right]<\infty.$$

So, all conditions of Statement 2.2 are satisfied. This means that the distributions of the processes $\{\frac{1}{\sqrt{n}}\sum_{j=1}^n\xi^{(j)}(\tau),\ \tau\in[0,T_1]\}$ converge weakly as $n\to\infty$ to the Gaussian zero-mean process with the correlation function $C_{T,\Delta}$. Lemma 3.1 is proved completely. □

The following lemma gives the connection between the mean-square deviations of the limiting process $Z_{T,\Delta}$ and the mean-square deviations of the process Y_Δ.

Lemma 3.2. *For all* $\tau_1,\ \tau_2\in[0,T_1]$, *the inequality*

$$\boldsymbol{E}\Big|Z_{T,\Delta}(\tau_1)-Z_{T,\Delta}(\tau_2)\Big|^2\le 2K_{X_\Delta}(0)\boldsymbol{E}\Big|Y_\Delta(\tau_1)-Y_\Delta(\tau_2)\Big|^2 \tag{10}$$

holds true.

Proof of Lemma 3.2. First of all, we consider the expression that appears in the representation of the correlation function (5) of the limiting process $Z_{T,\Delta}$:

$$\begin{aligned}(11)\quad & G_{T,\Delta}(\tau_1,\tau_2)=\\ &=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Big[e^{i(\tau_1-\tau_2)\lambda_2}|H^*(\lambda_2)|^2+e^{i(\tau_1\lambda_1+\tau_2\lambda_2)}H^*(\lambda_1)H^*(\lambda_2)\Big]f_\Delta(\lambda_1)f_\Delta(\lambda_2)d\lambda_1d\lambda_2=\\ &=\int_{-\infty}^{\infty}f_\Delta(\lambda_1)d\lambda_1\cdot\int_{-\infty}^{\infty}e^{i(\tau_1-\tau_2)\lambda_2}|H^*(\lambda_2)|^2f_\Delta(\lambda_2)d\lambda_2+\\ &+\int_{-\infty}^{\infty}e^{i\tau_1\lambda_1}H^*(\lambda_1)f_\Delta(\lambda_1)d\lambda_1\cdot\int_{-\infty}^{\infty}e^{i\tau_2\lambda_2}H^*(\lambda_2)f_\Delta(\lambda_2)d\lambda_2=\\ &=K_{X_\Delta}(0)\cdot K_{Y_\Delta}(\tau_1-\tau_2)+K_{Y_\Delta,X_\Delta}(\tau_1)\cdot K_{Y_\Delta,X_\Delta}(\tau_2),\end{aligned}$$

where $K_{X_\Delta},\ K_{Y_\Delta}$ and K_{Y_Δ,X_Δ} are the correlation functions of the processes $X_\Delta,\ Y_\Delta$ and the joint correlation function of these processes, respectively.

Let us write the mean-square deviation of $Z_{T,\Delta}$ for all $\tau_1,\ \tau_2\in[0,T_1]$ as

$$\mathbf{E}\Big|Z_{T,\Delta}(\tau_1)-Z_{T,\Delta}(\tau_2)\Big|^2=C_{T,\Delta}(\tau_1,\tau_1)-2C_{T,\Delta}(\tau_1,\tau_2)+C_{T,\Delta}(\tau_2,\tau_2)=$$

$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left\{\Big[e^{i(\tau_1-\tau_1)\lambda_2}|H^*(\lambda_2)|^2+e^{i(\tau_1\lambda_1+\tau_1\lambda_2)}H^*(\lambda_1)H^*(\lambda_2)\Big]f_\Delta(\lambda_1)f_\Delta(\lambda_2)-\right.$$

$$- 2\Big[e^{i(\tau_1-\tau_2)\lambda_2}|H^*(\lambda_2)|^2 + e^{i(\tau_1\lambda_1+\tau_2\lambda_2)}H^*(\lambda_1)H^*(\lambda_2)\Big] f_\Delta(\lambda_1)f_\Delta(\lambda_2)+$$

$$+ \Big[e^{i(\tau_2-\tau_2)\lambda_2}|H^*(\lambda_2)|^2 + e^{i(\tau_2\lambda_1+\tau_2\lambda_2)}H^*(\lambda_1)H^*(\lambda_2)\Big] f_\Delta(\lambda_1)f_\Delta(\lambda_2)\Big\}\times$$

$$\times \frac{1}{T^2}\bigg|\int_0^T e^{it(\lambda_2-\lambda_1)}\,dt\bigg|^2\, d\lambda_1 d\lambda_2.$$

In view of (11) and since $\frac{1}{T^2}\bigg|\int_0^T e^{it(\lambda_2-\lambda_1)}\,dt\bigg|^2 \leq 1$, we obtain

$$\begin{aligned}
\mathbf{E}\Big|Z_{T,\Delta}(\tau_1) - Z_{T,\Delta}(\tau_2)\Big|^2 &\leq G_{T,\Delta}(\tau_1,\tau_1) - 2G_{T,\Delta}(\tau_1,\tau_2) + G_{T,\Delta}(\tau_2,\tau_2) = \\
&= K_{X_\Delta}(0)\cdot K_{Y_\Delta}(\tau_1-\tau_1) + K_{Y_\Delta,X_\Delta}(\tau_1)\cdot K_{Y_\Delta,X_\Delta}(\tau_1) - \\
&-2\Big[K_{X_\Delta}(0)\cdot K_{Y_\Delta}(\tau_1-\tau_2) + K_{Y_\Delta,X_\Delta}(\tau_1)\cdot K_{Y_\Delta,X_\Delta}(\tau_2)\Big] + \\
&+K_{X_\Delta}(0)\cdot K_{Y_\Delta}(\tau_2-\tau_2) + K_{Y_\Delta,X_\Delta}(\tau_2)\cdot K_{Y_\Delta,X_\Delta}(\tau_2) = \\
&= 2K_{X_\Delta}(0)\Big[K_{Y_\Delta}(0) - K_{Y_\Delta}(\tau_1-\tau_2)\Big] + \\
&+\bigg[\Big(K_{Y_\Delta,X_\Delta}(\tau_1)\Big)^2 - 2K_{Y_\Delta,X_\Delta}(\tau_1)\cdot K_{Y_\Delta,X_\Delta}(\tau_2) + \Big(K_{Y_\Delta,X_\Delta}(\tau_2)\Big)^2\bigg] = \\
&= 2K_{X_\Delta}(0)\Big[K_{Y_\Delta}(0) - K_{Y_\Delta}(\tau_2-\tau_2)\Big] + \bigg[K_{Y_\Delta,X_\Delta}(\tau_1) - K_{Y_\Delta,X_\Delta}(\tau_2)\bigg]^2.
\end{aligned}$$

Thus,

$$\mathbf{E}\Big|Z_{T,\Delta}(\tau_1) - Z_{T,\Delta}(\tau_2)\Big|^2 \leq 2K_{X_\Delta}(0)\Big[K_{Y_\Delta}(0) - K_{Y_\Delta}(\tau_2-\tau_2)\Big] + \Big|K_{Y_\Delta,X_\Delta}(\tau_1) - K_{Y_\Delta,X_\Delta}(\tau_2)\Big|^2. \tag{12}$$

The Cauchy–Schwartz inequality yields

$$\begin{aligned}
\Big|K_{Y_\Delta,X_\Delta}(\tau_1) - K_{Y_\Delta,X_\Delta}(\tau_2)\Big| &\leq \int_{-\infty}^{\infty} |1-e^{i(\tau_1-\tau_2)\lambda}||H^*(\lambda)|f_\Delta(\lambda)d\,\lambda = \\
&= 2\int_{-\infty}^{\infty}\Big|\sin\frac{(\tau_1-\tau_2)\lambda}{2}\Big||H^*(\lambda)|\sqrt{f_\Delta(\lambda)}\cdot\sqrt{f_\Delta(\lambda)}d\,\lambda \leq \\
&\leq 2\bigg(\int_{-\infty}^{\infty}\sin^2\Big(\frac{(\tau_1-\tau_2)\lambda}{2}\Big)|H^*(\lambda)|^2 f_\Delta(\lambda)d\lambda\bigg)^{\frac12}\cdot\bigg(\int_{-\infty}^{\infty} f_\Delta(\lambda)d\,\lambda\bigg)^{\frac12}.
\end{aligned}$$

Since

$$\begin{aligned}
\mathbf{E}\Big|Y_\Delta(\tau_1) - Y_\Delta(\tau_2)\Big|^2 &= \\
&= 2\Big[K_{Y_\Delta}(0) - K_{Y_\Delta}(\tau_1-\tau_2)\Big] = 4\int_{-\infty}^{\infty}\sin^2\Big(\frac{(\tau_1-\tau_2)\lambda}{2}\Big)|H^*(\lambda)|^2 f_\Delta(\lambda)d\lambda
\end{aligned}$$

and

$$K_{X_\Delta}(0) = \int_{-\infty}^{\infty} f_\Delta(\lambda)d\,\lambda,$$

formula (12) yields

$$\mathbf{E}\Big|Z_{T,\Delta}(\tau_1) - Z_{T,\Delta}(\tau_2)\Big|^2 \leq 2K_{X_\Delta}(0)\Big[K_{Y_\Delta}(0) - K_{Y_\Delta}(\tau_2-\tau_2)\Big] +$$

$$+\left[2\left(\int_{-\infty}^{\infty}\sin^2\left(\frac{(\tau_1-\tau_2)\lambda}{2}\right)|H^*(\lambda)|^2 f_\Delta(\lambda)d\lambda\right)^{\frac{1}{2}}\cdot\left(\int_{-\infty}^{\infty}f_\Delta(\lambda)d\,\lambda\right)^{\frac{1}{2}}\right]^2=$$

$$=4K_{X_\Delta}(0)\big[K_{Y_\Delta}(0)-K_{Y_\Delta}(\tau_2-\tau_2)\big]=2K_{X_\Delta}(0)\cdot\mathbf{E}\big|Y_\Delta(\tau_1)-Y_\Delta(\tau_2)\big|^2.$$

Thus, Lemma 3.2 is proved completely. □

In the statement given below, the asymptotic properties of the processes $\widehat{H}^{(n)}_{T,\Delta}$ and $\widehat{Z}^{(n)}_{T,\Delta}$ as $n\to\infty$ are established. (We suppose that the parameters T and Δ have fixed values.)

Theorem 3.1. *(i)* $P\Big\{\lim_{n\to\infty}\sup_{\tau\in[0,T_1]}\Big|\widehat{H}^{(n)}_{T,\Delta}(\tau)-\boldsymbol{E}\widehat{H}^{(n)}_{T,\Delta}(\tau)\Big|=0\Big\}=1$;

(ii) $\widehat{Z}^{(n)}_{T,\Delta}\in C[0,T_1]$ *a.s. for all* $n\geq 1$;

(iii) $Z_{T,\Delta}\in C[0,T_1]$ *a.s.;*

(iv) If the process Y_Δ *is such that, for some* $\varepsilon>0$, *the relation*

$$P\left\{\sup_{\substack{t\neq s\\ t,s\in[0,T+T_1]}}\frac{|Y_\Delta(t)-Y_\Delta(s)|}{|\ln|t-s||^{\frac{1}{2}+\varepsilon}}<\infty\right\}=1 \qquad (13)$$

holds true, then the distributions of stochastic processes $\{\widehat{Z}^{(n)}_{T,\Delta}(\tau),\ \tau\in[0,T_1]\}$ *converge weakly as* $n\to\infty$ *in the space* $C[0,T_1]$ *to the distributions of the Gaussian process* $\{Z_{T,\Delta}(\tau),\ \tau\in[0,T_1]\}$ *with a zero mean and the correlation function* $C_{T,\Delta}$.

Proof of Theorem 3.1. **Proof of (i)**. Let us verify the conditions of Statement 2.1. Note that, for all $j=1,...,n$,

$$\xi^{(j)}(\tau)=\frac{1}{T}\int_0^T Y^{(j)}_\Delta(t+\tau)X^{(j)}_\Delta(t)dt,\quad \tau\in[0,T_1].$$

According to the problem considered, the sequence $\Big\{(\xi^{(j)}(\tau),\ \tau\in[0,T_1]),\ j\geq 1\Big\}$ is a sequence of independent a.s. sample continuous stochastic processes that are the copies of the process

$$\xi(\tau)=\frac{1}{T}\int_0^T Y_\Delta(t+\tau)X_\Delta(t)dt,\quad \tau\in[0,T_1].$$

In particular, it is known that

$$\mathbf{E}\xi(\tau)=\mathbf{E}\widehat{H}^{(n)}_{T,\Delta}(\tau),\quad \tau\in[0,T_1].$$

Since X_Δ and Y_Δ are Gaussian a.s. sample continuous processes, the relations

$$\mathbf{E}\left(\sup_{t\in[0,T]}|X_\Delta(t)|\right)^\alpha<\infty;\qquad \mathbf{E}\left(\sup_{t\in[0,T+T_1]}|Y_\Delta(t)|\right)^\alpha<\infty,$$

holds true for any $\alpha>0$; this yields

$$\mathbf{E}\sup_{\tau\in[0,T_1]}|\xi(\tau)|=\mathbf{E}\sup_{\tau\in[0,T_1]}\left|\frac{1}{T}\int_0^T Y_\Delta(t+\tau)X_\Delta(t)dt\right|\leq$$

$$\leq\mathbf{E}\sup_{\tau\in[0,T_1]}\left(\sup_{t\in[0,T]}|Y_\Delta(t+\tau)X_\Delta(t)|\right)\leq$$

$$\leq\mathbf{E}\left(\sup_{t\in[0,T+T_1]}|Y_\Delta(t)|\times\sup_{t\in[0,T]}|X_\Delta(t)|\right)\leq$$

$$\leq \left[\mathbf{E}\Big(\sup_{t\in[0,T+T_1]}|Y_\Delta(t)|\Big)^2\right]^{\frac{1}{2}} \times \left[\mathbf{E}\Big(\sup_{t\in[0,T]}|X_\Delta(t)|\Big)^2\right]^{\frac{1}{2}} < \infty.$$

Thus, for the sequence $\Big\{(\xi^{(j)}(\tau),\ \tau \in [0,T_1]),\ j \geq 1\Big\}$, all conditions of Statement 2.1 are satisfied. Therefore,

$$\mathsf{P}\left\{\lim_{n\to\infty}\sup_{\tau\in[0,T_1]}\Big|\frac{1}{n}\sum_{j=1}^{n}\xi^{(j)}(\tau) - \mathbf{E}\xi(\tau)\Big| = 0\right\} = 1.$$

Since $\frac{1}{n}\sum_{j=1}^{n}\xi^j(\tau) = \widehat{H}^{(n)}_{T,\Delta}(\tau)$ and $\mathbf{E}\xi(\tau) = \mathbf{E}\widehat{H}^{(n)}_{T,\Delta}(\tau)$, $\tau \in [0,T_1]$, statement (i) of Theorem 3.1 is proved completely. We remark that $\mathbf{E}\widehat{H}^{(n)}_{T,\Delta}(\cdot) \in C[0,T_1]$.

Proof of (ii). The proof follows immediately from the representation

$$\widehat{Z}^{(n)}_{T,\Delta}(\tau) = \sqrt{n}\Big(\widehat{H}^{(n)}_{T,\Delta}(\tau) - \mathbf{E}\widehat{H}^{(n)}_{T,\Delta}(\tau)\Big), \tau \in [0,T_1],$$

where the estimator $\widehat{H}^{(n)}_{T,\Delta}$ is an a.s. sample continuous process, and $\mathbf{E}\widehat{H}^{(n)}_{T,\Delta}(\cdot) \in C[0,T_1]$.

Proof of (iii). Let us rewrite relation (11) from Lemma 3.2 in the form

$$\sigma_{Z_{T,\Delta}}(\tau_1,\tau_2) \leq \sqrt{2K_{X_\Delta}(0)}\sigma_{Y_\Delta}(\tau_1,\tau_2),\ \tau_1,\ \tau_2 \in [0,T_1],$$

where $\sigma_{Z_{T,\Delta}}$ is the pseudometric generated by the mean-square deviations of the process $Z_{T,\Delta}$. Since Y_Δ and $Z_{T,\Delta}$ are separable zero-mean Gaussian processes and $Y_\Delta \in C[0,T_1]$ a.s., the Marcus–Shepp comparison inequality for Gaussian processes [18] yields $Z_{T,\Delta} \in C[0,T_1]$ a.s. So, we proved statement (iii) of Theorem 3.1.

Proof of (iv). The proof follows immediately from Lemma 3.1, if we denote

$$g(u) = |\ln u|^{-(\frac{1}{2}+\varepsilon)}.$$

Thus, Theorem 3.1 is proved completely. □

Remark 3.1. It is well-known (see, e.g., [12]) that condition (13) is satisfied if, for sufficiently small h, the inequality

$$\mathbf{E}|Y_\Delta(t) - Y_\Delta(s)|^2 = O\Big(|\ln|t-s||^{-(2+\varepsilon)}\Big) \tag{14}$$

holds true uniformly on $|t-s| < h$.

Remark 3.2. It can be shown (see, e.g., [16]) that condition (14) holds true if, for some $\delta > 0$,

$$\int_0^\infty \ln^{2+\delta}(1+\lambda)|H^*(\lambda)|^2 f_\Delta(\lambda)\,d\lambda < \infty. \tag{15}$$

3.2. **Construction of confidence bands for the limiting process.** In addition to Theorem 3.1 of the asymptotic normality of $\widehat{Z}^{(n)}_{T,\Delta}$, we have that, for all $x > 0$,

$$\lim_{n\to\infty}\mathsf{P}\Big\{\sup_{\tau\in[0,T_1]}\Big|\sqrt{n}\big(\widehat{H}^{(n)}_{T,\Delta}(\tau) - \mathbf{E}\widehat{H}^{(n)}_{T,\Delta}(\tau)\big)\Big| > x\Big\} = \mathsf{P}\Big\{\sup_{\tau\in[0,T_1]}\Big|Z_{T,\Delta}(\tau)\Big| > x\Big\}.$$

This relation gives the immediate cause for the construction of the confidence bands for the limiting process $Z_{T,\Delta}$, which are sufficiently accurate if n is a large number. To seek the estimators of the corresponding probability, we use the technique of work [3].

The following theorem holds true.

Theorem 3.2. *For the Gaussian zero-mean process* $\{Z_{T,\Delta}(\tau),\ \tau \in [0,T_1]\}$ *with the correlation function* $C_{T,\Delta}$, *the inequality*

$$P\left\{\sup_{\tau\in[0,T_1]}\Big|Z_{T,\Delta}(\tau)\Big| > x\right\} \leq 2P\left\{\sup_{\tau\in[0,T_1]}\Big(\sqrt{2K_{X_\Delta}(0)}Y_\Delta(\tau) + \xi g(\tau)\Big) > x\right\}, \tag{16}$$

holds true for all $x > 0$; here, for $\tau \in [0, T_1]$,

$$g^2(\tau) = \sup_{\tau\in[0,T_1]} \big(\max\{0, 2K_{X_\Delta}(0)K_{Y_\Delta}(0) - C_{T,\Delta}(\tau,\tau)\}\big) - \big(2K_{X_\Delta}(0)K_{Y_\Delta}(0) - C_{T,\Delta}(\tau,\tau)\big),$$

and ξ is an $N(0,1)$-distributed random variable, which is independent of the process Y_Δ.

Proof of Theorem 3.2. The proof follows immediately from the comparison inequality for the distributions of the maxima of Gaussian processes [3], which is applied to the separable zero-mean Gaussian processes $Z_{T,\Delta}$ and Y_Δ satisfying condition (10). □

The statements below are the analogs of some facts considered in [3] and [11]. Therefore, we give them without proofs.

Remark 3.3. For the Gaussian zero-mean process $\{Z_{T,\Delta}(\tau),\ \tau \in [0, T_1]\}$ with the correlation function $C_{T,\Delta}$, the inequality

$$\mathsf{P}\left\{\sup_{\tau\in[0,T_1]} \Big|Z_{T,\Delta}(\tau)\Big| > x\right\} \le \le 2\mathsf{P}\left\{\sup_{\tau\in[0,T_1]} Y_\Delta(\tau) > \frac{\gamma}{\sqrt{2K_{X_\Delta}(0)}}x\right\} + 2\mathsf{P}\left\{\xi \sup_{\tau\in[0,T_1]} g(\tau) > (1-\gamma)x\right\} \tag{17}$$

holds true for all $x > 0$; here,

$$g(\tau) = \sqrt{\sup_{\tau\in[0,T_1]}\left(\max\{0,\ 2K_{X_\Delta}(0)K_{Y_\Delta}(0) - C_{T,\Delta}(\tau,\tau)\}\right) - \Big(2K_{X_\Delta}(0)K_{Y_\Delta}(0) - C_{T,\Delta}(\tau,\tau)\Big)},\quad \tau \in [0, T_1],$$

and γ is any number from $[0, 1]$.

Remark 3.4. For the Gaussian zero-mean process $\{Z_{T,\Delta}(\tau),\ \tau \in [0, T_1]\}$ with the correlation function $C_{T,\Delta}$, the inequality

$$\mathsf{P}\left\{\sup_{\tau\in[0,T_1]} \Big|Z_{T,\Delta}(\tau)\Big| > 2x\right\} \le$$

$$\le 2\mathsf{P}\left\{\sup_{\tau\in[0,T_1]} |Y_\Delta(\tau)| > \frac{x}{\sqrt{2K_{X_\Delta}}}\right\} + 4\exp\left\{-\frac{x^2}{\Theta_{T_1}}\right\} \tag{18}$$

holds true for all $x > 0$; here, $\Theta_{T_1} = 4\sup_{\tau\in[0,T_1]} \big|2K_{X_\Delta}(0)K_{Y_\Delta}(0) - C_{T,\Delta}(\tau,\tau)\big|$.

Remark 3.5. We can use all results considered in this paper for the estimation of the error term

$$\sqrt{n}|\widehat{H}^{(n)}_{T,\Delta}(\tau) - H(\tau)|,\quad \tau \in [0, T_1].$$

This fact follows from the inequality

$$|\widehat{H}^{(n)}_{T,\Delta}(\tau) - H(\tau)| \le |\widehat{H}^{(n)}_{T,\Delta}(\tau) - \mathbf{E}\widehat{H}^{(n)}_{T,\Delta}(\tau)| + |\mathbf{E}\widehat{H}^{(n)}_{T,\Delta}(\tau) - H(\tau)|,\quad \tau \in [0, T_1],$$

if the term $|\mathbf{E}\widehat{H}^{(n)}_{T,\Delta}(\tau) - H(\tau)|$ is made sufficiently small, by choosing the corresponding parameters T and Δ and assuming some additional conditions on the local smoothness of the response function H (see, e.g., [2]).

Conclusion

In this paper, we have studied the estimators of the response real-valued function $H \in L_2(\mathbf{R})$ of the linear system that is disturbed by a family of stationary zero-mean Gaussian a.s. sample continuous processes, by applying the scheme of some independent samples, when the pair of inputs and outputs are observed. We have investigated the conditions for the asymptotic normality of corresponding centered cross-correlogram estimators in the space of continuous functions and constructed the confidence bands for the limiting process. In particular, under some conditions on the parameters T, Δ and the local smoothness of the response function H, all results of this paper hold true for the corresponding error term of the estimation.

References

1. J. S. Bendat and A. G. Piersol, *Engineering Applications of Correlation and Spectral Analysis*, New York, Wiley, 1993.
2. I. P. Blazhievska, *On asymptotic behavior of the error term in cross-correlogram estimation of response functions in linear systems*, Theory of Stochastic Processes, Vol. **16 (32)**, no. 2 (2010), 5–11.
3. V. V. Buldygin, *On some comparison inequality for the distribution of the maximum of the Gaussian process*, Theor. Probab. and Math. Statist., Vol. **28** (1983), 9-14 (in Russian).
4. V. V. Buldygin, *Limiting theorems in functional spaces and one problem in statistics of stochastic processes*, Probabilistic Methods of Infinite-Dimensional Analysis, Kiev, Inst. of Math. AN USSR (1980), 24-36 (in Russian).
5. V. V. Buldygin and I. P. Blazhievska, *On asymptotic behavior of cross-correlogram estimators of response functions in linear Volterra systems*, Theory of Stochastic Processes, Vol. **15 (31)**, no. 2 (2009), 84–98.
6. V. V. Buldygin and E. V. Ilarionov *On some problem in statistics of stochastic fields*, Probabilistic Infinite-Dimensional Analysis, Kiev, Inst. of Math. AN USSR (1981), 6 - 14 (in Russian).
7. V. V. Buldygin and Yu. V. Kozachenko, *Metric Characterization of Random Variables and Random Processes*, Providence, RI, Amer. Math. Soc., 2000.
8. V. V. Buldygin and V. G. Kurotschka, *On cross-correlogram estimators of the response function in continuous linear systems from discrete observations*, Random Oper. and Stoch. Eq., **7** (1999), no. 1, 71–90.
9. V. V. Buldygin and Fu Li, *On asymptotical normality of an estimation of unit impulse responses of linear systems* I, II, Theor. Probab. and Math. Statist., Vol. **54** (1997), 17–24; Vol. **55** (1997), 29–36.
10. V. Buldygin, F. Utzet, and V. Zaiats, *Asymptotic normality of cross-correlongram estimates of the response function*, Statistical Interference for Stochastic Processes, Vol. **7** (2004), 1–34.
11. V. V. Buldygin and A. B. Charazishvili, *The Brunn–Minkowski Inequality and Its Applications*, Kiev, Naukova Dumka, 1985 (in Russian).
12. R. M. Dudley, *Sample functions of the Gaussian process*, Ann. of Probab., Vol. **1** (1973), 66–103.
13. K. Fernigue, in *Regularite des trajectoires des fonctions aleatoires gaussiennes*, Lect. Notes in Math., Vol. 450, Berlin, Springer, 1975, pp. 1–96.
14. U. Grenander, *Probabilities on Algebraic Structures*, New York, Wiley, 1963.
15. N. C. Jain and M. B. Marcus, *Central limit theorem for CS-valued ramdom variables*, J. Funct. Analysis, Vol. **19**, no. 3 (1975), 216–231.
16. H. Cramér and M. Leadbetter *Stationary and Related Stochastic Processes: Sample Function Properties and Their Applications*, New York, Wiley, 1967.
17. N. N. Leonenko and A. V. Ivanov *Statistical Analysis of Stochastic Fields*, Kiev, Vyshcha Shkola, 1986 (in Russian).
18. M. B. Marcus and L. A. Shepp *Countinuity of Gaussian processes*, Trans. Amer. Math. Soc., **151** (1970), 377–391.

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