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INHOMOGENEOUS DIFFUSION PROCESSES ON A HALF-LINE WITH JUMPS ON ITS BOUNDARY

By means of the method of classical potential theory, we construct a multiplicative operator family that describes an inhomogeneous diffusion process on a half-line with the Feller–Wentzel boundary condition which corresponds to the absorption and jumps of the process.

1. Introduction

In this paper, we found an integral representation of the multiplicative operator family which describes an inhomogeneous diffusion process on a half-line with the Feller–Wentzel boundary condition [1, 2] represented in the form of a combination of the two terms: a local term, which corresponds to the absorption of the process after its reaching the domain boundary, and a nonlocal one, which indicates that, at a zero point, the discontinuities of a process path are possible. A construction of the required operator family is performed by analytical methods with the use of classical potential theory ([3], [4]), which is applied for the solution to the corresponding boundary-value problem for a linear parabolic equation of the second order with variable coefficients.

We note that we derived a nontrivial generalization of the corresponding result obtained earlier in [5], where a similar problem was analyzed within similar methods for in case of a homogeneous diffusion process without local terms in the Feller–Wentzel boundary condition. In addition, a problem of existence of the Feller semigroup for the multidimensional diffusion process with a nonlocal Wentzel boundary condition was analyzed in work [6] and was studied by means of the methods of functional analysis. We should also mention works [7, 8], where the diffusion processes in a half-space with Wentzel boundary conditions were obtained as the weak solutions of some stochastic differential equations.

2. Statement of the problem and some auxiliary facts

Let $D = \{x \in \mathbb{R} : \quad x > 0\}$ be a domain on the line $\mathbb{R}$ with a boundary $\partial D = \{0\}$, and let a closure $\overline{D} = D \cup \{0\}; T > 0$ be fixed. If Γ is $\overline{D}$ or $\mathbb{R}$, then $C_b(\Gamma)$ is a Banach space of all functions $f(x)$ real-valued, bounded, and continuous on Γ with the norm $\|f\| = \sup_{x\in\Gamma} |f(x)|$, and $C^{(2)}_{\text{unif}}(\Gamma)$ is the set of all functions f bounded and uniformly continuous on Γ together with their first- and second-order derivatives. Assume that an inhomogeneous diffusion process is given in D, and it is generated by a second-order differential operator $A_s, s \in [0, T]$ that acts on $C^{(2)}_{\text{unif}}(\overline{D})$:

$$A_s f(x) = \frac{1}{2} b(s, x) \frac{d^2 f}{dx^2}(x) + a(s, x) \frac{df}{dx}(x), \tag{1}$$

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where $a(s,x)$ and $b(s,x)$ are real continuous bounded functions in the domain $[0,T]\times\overline{D}$, and $b(s,x)\geq 0$ for all $(s,x)\in[0,T]\times\overline{D}$. We also assume that the boundary operator $L_s, s\in[0,T]$, is given, and it is defined by the formula

$$L_s f(0)=\gamma(s)f(0)+\int_D [f(0)-f(y)]\mu(s,dy), \tag{2}$$

where the function $\gamma(s)$ and the measure $\mu(s,dy)$ satisfy the following conditions:

a) $\gamma(s)$ is nonnegative and continuous on a closed interval $[0,T]$;
b) $\mu(s,\cdot)$ is a nonnegative measure on D such that it is continuous on $[0,T]$ as a function of the variable s;
c) $\gamma(s)+\mu(s,D)>0$ for all $s\in[0,T]$.

Note that the operator in (2) is a particular case of the Feller–Wentzel operator ([1, 2]), which describes the process behavior after it reaches the boundary of the domain. The coefficient γ and the measure μ correspond to such properties of the process as its absorption at zero and the jump departure from zero, respectively. Let us recall that the general Feller–Wentzel operator consists of two more terms that correspond to the instantaneous reflection of the process and its viscosity at the zero point.

The problem is to build a multiplicative operator family $T_{st}, 0\leq s<t\leq T$, that describes the inhomogeneous Feller process on $\overline{D}$, whose generator $\widetilde{A}_s$ is defined on the functions $f\in C^{(2)}_{\text{unif}}(\overline{D})$, such that

$$L_s f(0)=0, \tag{3}$$

and $\widetilde{A}_s f=A_s f$ for them.

According to the analytical approach to the solution of this problem, the required operator family $T_{st}, 0\leq s<t\leq T$ is determined by solving the following boundary-value problem:

$$\frac{\partial u(s,x,t)}{\partial s}+\frac{1}{2}b(s,x)\frac{\partial^2 u(s,x,t)}{\partial x^2}+a(s,x)\frac{\partial u(s,x,t)}{\partial x}=0,\quad 0\leq s<t\leq T, x\in D, \tag{4}$$

$$\lim_{s\uparrow t}u(s,x,t)=\varphi(x),\quad x\in D, \tag{5}$$

$$\gamma(s)u(s,0,t)+\int_D[u(s,0,t)-u(s,y,t)]\mu(s,dy)=0,\quad 0\leq s<t\leq T, \tag{6}$$

where $\varphi\in C_b(\overline{D})$ is the given function.

In the present paper, problem (4)-(6) is studied under the condition that the next additional assumptions hold:

1) there exist constants b and B such that $0<b\leq b(s,x)\leq B$ for all $(s,x)\in[0,T]\times\mathbb{R}$;
2) the function $a(s,x)$ is bounded on the domain $[0,T]\times\mathbb{R}$ and, in addition, for all $s,s'\in[0,T], x,x'\in\mathbb{R}$ the following inequalities hold:

$$|b(s,x)-b(s',x')|\leq c\left(|s-s'|^{\frac{\alpha}{2}}+|x-x'|^{\alpha}\right),$$

$$|a(s,x)-a(s',x')|\leq c\left(|s-s'|^{\frac{\alpha}{2}}+|x-x'|^{\alpha}\right),$$

where c and α are positive constants, $0<\alpha<1$;
3) the function φ belongs to a class $C_b(\mathbb{R})$;
4) the function $\gamma(s)$ is Hölder continuous with exponent $\frac{1+\alpha}{2}$ on a closed interval $[0,T]$;
5) $\mu(s,D)=1$ for all $s\in[0,T]$;
6) for an arbitrary function $f\in C_b(\overline{D})$, the function $G_f(s)=\int_D f(y)\mu(s,dy)$ is Hölder continuous with exponent $\frac{1+\alpha}{2}$ on $[0,T]$.

Assumptions 1) and 2) guarantee (see [3, 4, 9]) the existence of the fundamental solution to Eq. (4) in the domain $[0,T]\times\mathbb{R}$ which is denoted by $G(s,x,t,y)(0\le s< t\le T, x,y\in\mathbb{R})$. Let us recall that the function $G(s,x,t,y)$ is nonnegative, continuous in the aggregate of the variables, continuously differentiable with respect to s, and twice continuously differentiable with respect to x, and the following estimations hold ($0\le s<t\le T, x,y\in\mathbb{R}$):

$$|D_s^r D_x^p G(s,x,t,y)|\le c(t-s)^{-\frac{1+2r+p}{2}}\exp\left\{-h\frac{(y-x)^2}{t-s}\right\}, \tag{7}$$

where r and p are nonnegative integers such that $2r+p\le 2$; D_s^r is the partial derivative with respect to s of order r; D_x^p is the partial derivative with respect to x of order p; c, and h are positive constants. Furthermore, $G(s,x,t,y)$ is represented as

$$G(s,x,t,y)=Z_0(s,y-x,t,y)+Z_1(s,x,t,y), \tag{8}$$

where

$$Z_0(s,x,t,y)=[2\pi b(t,y)(t-s)]^{-\frac{1}{2}}\exp\left\{-\frac{(y-x)^2}{2b(t,y)(t-s)}\right\}. \tag{9}$$

Moreover, the function $Z_1(s,x,t,y)$ satisfies the inequalities

$$|D_s^r D_x^p Z_1(s,x,t,y)|\le c(t-s)^{-\frac{1+2r+p-\alpha}{2}}\exp\left\{-h\frac{(y-x)^2}{t-s}\right\}, \tag{10}$$

where $0\le s<t\le T$, $x,y\in\mathbb{R}$, $2r+p\le 2$, c and h are positive constants, and α is the constant from 2).

It follows from condition 6) that, for an arbitrary function $f\in C_b(\overline{D})$, there exists a Hölder constant such that, for all $s,s'\in[0,T]$, the inequality

$$|G_f(s)-G_f(s')|\le c_f|s-s'|^{\frac{1+\alpha}{2}}$$

holds.

Let us consider c_f as a functional acting on the linear space $C_b(\overline{D})$. It is easy to verify that, for this functional, the following conditions hold:

- $c_{f_1+f_2}\le c_{f_1}+c_{f_2}$, for all $f_1,f_2\in C_b(\overline{D})$.
- $c_{\lambda f}=|\lambda|\cdot c_f$, for an arbitrary $\lambda\in\mathbb{R}$;

So the functional c_f is a seminorm (see [11]), and the next lemma is valid.

Lemma 2.1. *Assume that the measure μ from (2) satisfies condition 6). Then, for an arbitrary constant $M>0$, there exists a constant $c>0$ such that, for all functions $f\in C_b(\overline{D})$ bounded by M and for all $s,s'\in[0,T]$, the function $G_f(s)$ satisfies the relation*

$$|G_f(s)-G_f(s')|\le c|s-s'|^{\frac{1+\alpha}{2}}.$$

3. Solving the boundary-value problem (4)-(6)

In this section, we establish the classical solvability of the boundary-value problem (4)-(6). We say that a solution to this problem is a classical one if, for all $t\in(0,T]$, it belongs to the class

$$\mathcal{C}^{1,2}([0,t)\times D)\cap\mathcal{C}([0,t)\times\overline{D}). \tag{11}$$

Theorem 3.1. *Assume that the coefficients of the operator A_s from (1), the function φ from (5), the function γ, and the measure μ from (2) satisfy conditions a)-c) and 1)-6). Then there exists a classical solution to problem (4)-(6) which can be represented as follows ($0\le s<t\le T, x\in\overline{D}$):*

$$u(s,x,t)=\int_{\mathbb{R}}G(s,x,t,y)\varphi(y)dy+\int_s^t G(s,x,\tau,0)V(\tau,t,\varphi)d\tau, \tag{12}$$

where $V(s,t,\varphi)$ is a solution to some Volterra integral equation of the second kind. In addition, this solution satisfies the inequality

$$|u(s,x,t)| \le c\|\varphi\|, \tag{13}$$

where $0 \le s < t \le T$, $x \in \overline{D}$, and c is a positive constant.

Proof. We find a solution to problem (4)-(6) in the form (12). We denote the Poisson potential on the right-hand side of equality (12) by $u_0(s,x,t)$ and the simple-layer potential by $u_1(s,x,t)$. Consider *a priori* that an unknown density V from the potential u_1 is continuous in $s \in [0,t)$.

By substituting the expression for $u(s,x,t)$ from (12) into (6), we obtain the first-kind Volterra integral equation for V:

$$\Phi_0(s,t,\varphi) = \int_s^t K_0(s,\tau)V(\tau,t,\varphi)d\tau, \quad 0 \le s < t \le T. \tag{14}$$

Here,

$$K_0(s,\tau) = \gamma(s)G(s,0,\tau,0) + \int_D [G(s,0,\tau,0) - G(s,y,\tau,0)]\mu(s,dy),$$
$$\Phi_0(s,t,\varphi) = -\gamma(s)u_0(s,0,t) - \int_D [u_0(s,0,t) - u_0(s,y,t)]\mu(s,dy).$$

By means of the Holmgren's method, we reduce this equation to an equivalent Volterra integral equation of the second kind. To do this, we define the operator

$$\mathcal{E}(s,t)\psi_0 = \sqrt{\frac{2}{\pi}}\frac{d}{ds}\int_s^t (\rho - s)^{-\frac{1}{2}}\psi_0(\rho,t,\varphi)d\rho, \quad 0 \le s < t \le T,$$

and apply it to both sides of Eq. (14). After simple transformations, we obtain

$$\mathcal{E}(s,t)\Phi_0 = -\frac{V(s,t,\varphi)}{\sqrt{b(s,0)}} + \sqrt{\frac{2}{\pi}}\frac{d}{ds}\int_s^t V(\tau,t,\varphi)d\tau \int_s^\tau (\rho - s)^{-\frac{1}{2}}\Big[\gamma(\rho)Z_0(\rho,0,\tau,0)+$$
$$+ (\gamma(\rho)+1)Z_1(\rho,0,\tau,0) - \int_D G(s,y,\tau,0)\mu(\rho,dy)\Big]d\rho. \tag{15}$$

To simplify the expression in the second term on the right-hand side of (15), we introduce the notations

$$I_1(s,\tau) = \int_s^\tau (\rho - s)^{-\frac{1}{2}}\Big[\gamma(\rho)Z_0(\rho,0,\tau,0) + (\gamma(\rho)+1)Z_1(\rho,0,\tau,0)\Big]d\rho,$$
$$I_2(s,\tau) = \int_s^\tau (\rho - s)^{-\frac{1}{2}}d\rho \int_D G(s,y,\tau,0)\mu(\rho,dy),$$

and investigate the behavior of these integrals as $\tau \downarrow s$.

Consider firstly the function I_1 and rewrite it in the following way:

$$I_1(s,\tau) = \sqrt{\frac{\pi}{2b(\tau,0)}} \cdot \gamma(s) + J_1(s,\tau), \tag{16}$$

where

$$J_1(s,\tau) = \frac{1}{\sqrt{2\pi b(\tau,0)}}\int_s^\tau (\rho - s)^{-\frac{1}{2}}(\tau - \rho)^{-\frac{1}{2}}(\gamma(\rho) - \gamma(s))d\rho+$$
$$+ \int_s^\tau (\rho - s)^{-\frac{1}{2}}(\gamma(\rho)+1)Z_1(\rho,0,\tau,0)d\rho. \tag{17}$$

From condition 4) and inequality (10) in the case of $r = p = 0$, we obtain

$$\lim_{\tau \downarrow s} J_1(s,\tau) = 0. \tag{18}$$

We now consider the function I_2 and prove that

$$\lim_{\tau\downarrow s} I_2(s,\tau)=0. \tag{19}$$

To this end, we represent I_2 as follows:

$$\begin{aligned} I_2(s,\tau) &= \frac{1}{\sqrt{2\pi b(\tau,0)}}\int_s^\tau (\rho-s)^{-\frac{1}{2}}(\tau-\rho)^{-\frac{1}{2}}d\rho\int_D e^{-\frac{y^2}{2b(\tau,0)(\tau-\rho)}}\left(\mu(\rho,dy)-\mu(s,dy)\right)+\\ &+\frac{1}{\sqrt{2\pi b(\tau,0)}}\int_s^\tau (\rho-s)^{-\frac{1}{2}}(\tau-\rho)^{-\frac{1}{2}}d\rho\int_D e^{-\frac{y^2}{2b(\tau,0)(\tau-\rho)}}\mu(s,dy)+\\ &+\int_s^\tau(\rho-s)^{-\frac{1}{2}}d\rho\int_D Z_1(\rho,y,\tau,0)\mu(\rho,dy). \end{aligned}\tag{20}$$

We note that the functions $f_{\tau,\rho}(y)=e^{-\frac{y^2}{2b(\tau,0)(\tau-\rho)}}$ belong to the class $C_b(\overline{D})$ for all $0\le s<\rho<\tau<t\le T$ and are bounded by 1. According to Lemma 2.1, the next inequality holds:

$$\left|\int_D e^{-\frac{y^2}{2b(\tau,0)(\tau-\rho)}}\left(\mu(\rho,dy)-\mu(s,dy)\right)\right|\le c|\rho-s|^{\frac{1+\alpha}{2}}, \tag{21}$$

where $0\le s<\rho<\tau<t\le T$, c are some positive constant. We will further use c to denote any positive constant, whose specific value is not of interest.

Estimations (21) and (10) imply that the first and third terms on the right-hand side of (20) converge to zero as $\tau\downarrow s$. It remains to investigate the second item on the right-hand side of (20). We denote it by $J_2(s,\tau)$. It can be expressed in the form

$$\begin{aligned} &J_2(s,\tau)=\\ &=\frac{1}{\sqrt{2\pi b(\tau,0)}}\int_D e^{-\frac{y^2}{2b(\tau,0)(\tau-s)}}\mu(s,dy)\int_s^\tau(\rho-s)^{-\frac{1}{2}}(\tau-\rho)^{-\frac{1}{2}}e^{-\frac{y^2}{2b(\tau,0)(\tau-s)}\cdot\frac{\rho-s}{\tau-\rho}}d\rho=\\ &=\frac{1}{\sqrt{2\pi b(\tau,0)}}\int_D e^{-\frac{y^2}{2b(\tau,0)(\tau-s)}}\mu(s,dy)\int_0^\infty z^{-\frac{1}{2}}(z+1)^{-1}e^{-\frac{y^2}{2b(\tau,0)(\tau-s)}\cdot z}. \end{aligned}\tag{22}$$

In view of (22), we obtain

$$J_2(s,\tau)\le\frac{4}{\sqrt{2\pi b(\tau,0)}}\int_D e^{-\frac{y^2}{2b(\tau,0)(\tau-s)}}\mu(s,dy)\le\frac{4}{\sqrt{2\pi b}}\left(\mu(s,(0,\delta))+e^{-\frac{\delta^2}{2B(\tau-s)}}\right), \tag{23}$$

where $\delta>0$ is an arbitrary positive number, and b and B are the constants from 1). Further, it follows from the properties of the measure μ that, for an arbitrary constant $\varepsilon>0$, there exists $\delta=\delta_0>0$ such that, for all $s\in[0,T]$, the inequality $\mu(s,(0,\delta_0))<\varepsilon$ holds.

In view of the last inequality and estimate (23), we establish that $\lim_{s\downarrow\tau}J_2(s,\tau)=0$. The proof of (19) is completed.

With regard for relations (16)-(19), equality (15) can be reduced to

$$\begin{aligned}\mathcal{E}(s,t)\Phi_0 = &-\frac{V(s,t,\varphi)}{\sqrt{b(s,0)}} + \frac{d}{ds}\int_s^t \frac{V(\tau,t,\varphi)}{\sqrt{b(\tau,0)}}\gamma(s_0)d\tau\Big|_{s_0=s} + \\ &+\frac{1}{\pi}\int_s^t \frac{V(\tau,t,\varphi)}{\sqrt{b(\tau,0)}}d\tau\frac{d}{ds}\int_s^\tau (\rho-s)^{-\frac{1}{2}}(\tau-\rho)^{-\frac{1}{2}}\left(\gamma(\rho)-\gamma(s_0)\right)d\rho\Big|_{s_0=s} + \\ &+\sqrt{\frac{2}{\pi}}\int_s^t V(\tau,t,\varphi)d\tau\frac{d}{ds}\int_s^\tau (\rho-s)^{-\frac{1}{2}}(\gamma(\rho)+1)Z_1(\rho,0,\tau,0)d\rho - \\ &-\sqrt{\frac{2}{\pi}}\int_s^t V(\tau,t,\varphi)d\tau\frac{d}{ds}\int_s^\tau (\rho-s)^{-\frac{1}{2}}d\rho\int_D G(\rho,y,\tau,0)\mu(\rho,dy).\end{aligned}$$

Hence, for an unknown function V, we obtain the Volterra integral equation of the second kind

$$V(s,t,\varphi) = \int_s^t K(s,\tau)V(\tau,t,\varphi)d\tau + \psi(s,t,\varphi), \quad 0 \le s < t \le T, \tag{24}$$

where

$$\begin{aligned}K(s,\tau) = &\frac{1}{2\pi(\gamma(s)+1)}\sqrt{\frac{b(s,0)}{b(\tau,0)}}\int_s^\tau (\rho-s)^{-\frac{3}{2}}(\tau-\rho)^{-\frac{1}{2}}(\gamma(\rho)-\gamma(s))d\rho + \\ &+\frac{1}{\gamma(s)+1}\sqrt{\frac{2b(s,0)}{\pi}}\frac{d}{ds}\int_s^\tau (\rho-s)^{-\frac{1}{2}}\left[(\gamma(\rho)+1)Z_1(\rho,0,\tau,0) - \int_D G(\rho,y,\tau,0)\mu(\rho,dy)\right]d\rho, \\ \psi(s,t,\varphi) = &-\frac{1}{\gamma(s)+1}\sqrt{b(s,0)}\cdot\mathcal{E}(s,t)\Phi_0.\end{aligned}$$

We now show that there exists a solution to Eq. (24) which can be found by means of the convergence method:

$$V(s,t,\varphi) = \sum_{k=0}^{\infty} V^{(k)}(s,t,\varphi), \quad 0 \le s < t \le T, \tag{25}$$

where

$$V^{(0)}(s,t,\varphi) = \psi(s,t,\varphi), \quad V^{(k)}(s,t,\varphi) = \int_s^t K(s,\tau)V^{(k-1)}(\tau,t,\varphi)d\tau, \quad k = 1,2,\ldots.$$

For this purpose, we firstly investigate the kernel $K(s,\tau)$ of Eq. (24). We denote the first term in the expression for $K(s,\tau)$ by $P_1(s,\tau)$ and the second one by $P_2(s,\tau)$.

Taking condition 4) into consideration, we have

$$|P_1(s,\tau)| \le c(\tau-s)^{-\frac{1}{2}+\frac{\alpha}{2}}. \tag{26}$$

Before the investigation of the function $P_2(s,\tau)$, we write it as follows:

$$P_2(s,\tau) = \frac{1}{\gamma(s)+1}\sqrt{\frac{2b(s,0)}{\pi}}\left(P_{21}(s,\tau) - P_{22}(s,\tau)\right), \tag{27}$$

where

$$P_{21}(s,\tau) = -\frac{d}{ds}\int_s^\tau (\rho-s)^{-\frac{1}{2}}d\rho\int_D Z_1(\rho,y,\tau,0)(\mu(\rho,dy)-\mu(s_0,dy))\Big|_{s_0=s} +$$

$$+\frac{d}{ds}\int_s^\tau (\rho-s)^{-\frac{1}{2}}\Big[(\gamma(\rho)+1)Z_1(\rho,0,\tau,0)-\int_D Z_1(\rho,y,\tau,0)\mu(s_0,dy)\Big]d\rho\Big|_{s_0=s},$$

$$P_{22}(s,\tau) = \frac{d}{ds}\int_s^\tau (\rho-s)^{-\frac{1}{2}}d\rho\int_D Z_0(\rho,y,\tau,0)(\mu(\rho,dy)-\mu(s_0,dy))\Big|_{s_0=s} +$$

$$+\frac{d}{ds}\int_s^\tau (\rho-s)^{-\frac{1}{2}}d\rho\int_D Z_0(\rho,y,\tau,0)\mu(s_0,dy)\Big|_{s_0=s}.$$

For the function $P_{21}(s,\tau)$, after simple transformations, we obtain the formula

$$P_{21}(s,\tau) = -\frac{1}{2}\int_s^\tau (\rho-s)^{-\frac{3}{2}}d\rho\Big(\int_D Z_1(\rho,y,\tau,0)(\mu(\rho,dy)-\mu(s,dy))+$$

$$+(\gamma(\rho)-\gamma(s))Z_1(\rho,0,\tau,0)-\int_D (Z_1(\rho,y,\tau,0)-Z_1(s,y,\tau,0))\mu(s,dy)+$$

$$+(\gamma(s)+1)(Z_1(\rho,0,\tau,0)-Z_1(s,0,\tau,0))\Big). \tag{28}$$

Now, while estimating each term on the right-hand side of (28) by means of (10), and using therewith the assertion of Lemma 2.1, condition 4), as well as the Lagrange formula for the differences $Z_1(\rho,y,\tau,0)-Z_1(s,y,\tau,0)$ and $Z_1(\rho,0,\tau,0)-Z_1(s,0,\tau,0)$, we obtain

$$|P_{21}(s,\tau)| \le c(\tau-s)^{-1+\frac{\alpha}{2}}. \tag{29}$$

Consider the function $P_{22}(s,\tau)$. It can be represented as follows:

$$P_{22}(s,\tau) = \frac{1}{4\sqrt{\pi b(\tau,0)}}\int_s^\tau (\rho-s)^{-\frac{3}{2}}(\tau-\rho)^{-\frac{1}{2}}\int_D e^{-\frac{y^2}{2b(\tau,0)(\tau-\rho)}}(\mu(\rho,dy)-\mu(s,dy))d\rho+$$

$$+\sqrt{\frac{\pi b(\tau,0)}{2}}\int_D \frac{\partial Z_0}{\partial y}(s,y,\tau,0)\mu(s,dy) = L_1(s,\tau)+L_2(s,\tau). \tag{30}$$

As a consequence of inequality (21) for $L_1(s,\tau)$, the estimation

$$|L_1(s,\tau)| \le c(\tau-s)^{-\frac{1}{2}+\frac{\alpha}{2}} \tag{31}$$

holds.

To estimate the function $L_2(s,\tau)$, we preliminarily represent it as follows:

$$L_2(s,\tau) = \sqrt{\frac{\pi b(\tau,0)}{2}}\int_0^\delta \frac{\partial Z_0}{\partial y}(s,y,\tau,0)\mu(s,dy)+\sqrt{\frac{\pi b(\tau,0)}{2}}\int_\delta^\infty \frac{\partial Z_0}{\partial y}(s,y,\tau,0)\mu(s,dy),$$

where δ is an arbitrary positive number.

According to our assumptions on the measure μ, it is easy to obtain that the second integral in the formula for L_2 satisfies the inequality

$$\int_\delta^\infty \frac{\partial Z_0}{\partial y}(s,y,\tau,0)\mu(s,dy) \le c(\delta)(\tau-s)^{-\frac{1}{2}}, \tag{32}$$

where the constant c depends on δ. In addition, $c(\delta)\to\infty$ as $\delta\to 0$.

As far as the estimation of the first integral in the expression for L_2 is concerned, it will be executed in a combination with functions used to determine the terms of series (25).

Further, for the kernel $K(s,\tau)$, we will use the representation

$$K(s,\tau) = K_1(s,\tau)+K_2(s,\tau), \quad 0\le s<\tau<t\le T, \tag{33}$$

where

$$K_1(s,\tau) = -\frac{1}{\gamma(s)+1}\sqrt{b(s,0)b(\tau,0)}\int_0^\delta \frac{\partial Z_0}{\partial y}(s,y,\tau,0)\mu(s,dy).$$

As follows from (26), (29), (31), and (32), $K_2(s,\tau)$ satisfies the inequality

$$|K_2(s,\tau)| \le p(\delta)(\tau-s)^{-1+\frac{\alpha}{2}},$$

where $p(\delta)$ is some positive constant depending on δ.

By means of the scheme used for the estimation of $K_2(s,\tau)$, we can also estimate the function $\psi(s,t,\varphi)$. We prove that, for all $0 \le s < t \le T$, it satisfies the inequality

$$|\psi(s,t\varphi)| \le r(t-s)^{-\frac{1}{2}}, \tag{34}$$

where r is some positive constant.

Then in the same way as in [5], by means of the mathematical induction method, we show that, for the terms of series (25) the next inequalities are valid ($0 \le s < t \le T$):

$$\left|V^{(k)}(s,t,\varphi)\right| \le r\|\varphi\|(t-s)^{-\frac{1}{2}}\sum_{n=0}^{k} C_k^n \cdot a^{(k-n)} m(\delta)^n, \quad k=0,1,2, \tag{35}$$

where

$$a^{(n)} = \frac{\left(p(\delta)T^{\frac{\alpha}{2}}\Gamma\left(\frac{\alpha}{2}\right)\right)^n \cdot \Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1+n\alpha}{2}\right)}, \quad m = 0,1,2,\dots,k,$$

$$m(\delta) = \frac{B}{b}\max_{s\in[0,T]}\mu(s,(0,\delta)).$$

Let us fix $\delta = \delta_0$ such that $m(\delta_0) < 1$. Then, in view of (35), we have

$$\sum_{k=0}^{\infty}\left|V^{(k)}(s,t,\varphi)\right| \le r\|\varphi\|(t-s)^{-\frac{1}{2}}\sum_{k=0}^{\infty}\sum_{n=0}^{k} C_k^n a^{(k-n)} m(\delta_0)^n =$$

$$= r\|\varphi\|(t-s)^{-\frac{1}{2}}\sum_{k=0}^{\infty} a^{(k)}\sum_{n=0}^{\infty} C_{k+n}^n m(\delta_0)^n =$$

$$= r\|\varphi\|(t-s)^{-\frac{1}{2}}\sum_{k=0}^{\infty}\frac{a^{(k)}}{(1-m(\delta_0))^{k+1}} =$$

$$= r\|\varphi\|(t-s)^{-\frac{1}{2}}\sum_{k=0}^{\infty}\frac{\left(\frac{p(\delta_0)}{1-m(\delta_0)}T^{\frac{\alpha}{2}}\Gamma\left(\frac{\alpha}{2}\right)\right)^k}{\Gamma\left(\frac{1+k\alpha}{2}\right)}\cdot\frac{\Gamma(\frac{1}{2})}{1-m(\delta_0)}. \tag{36}$$

Estimation (36) ensures the absolute and uniform convergence of series (25) in $0 \le s < t \le T$. Thus, the function V exists. Moreover, it is continuous in $s \in [0,t]$, and the inequality

$$|V(s,t,\varphi)| \le c\|\varphi\|(t-s)^{-\frac{1}{2}}, \quad 0 \le s < t \le T, \tag{37}$$

holds. Note that our assumption on V is valid. Inequalities (7) and (37) yield the existence of a solution to problem (4)-(6) which is represented by formula (12) and satisfies estimation (13).

The proof of Theorem 3.1 is now completed. □

Remark 3.1. If we additionally assume in Theorem 3.1 that the function φ satisfies the fitting condition

$$L_s\varphi(0) = 0, \quad s \in [0,T], \tag{38}$$

then the constructed solution to problem (4)-(6) belongs to the class

$$\mathcal{C}^{1,2}([0,t)\times D)\cap\mathcal{C}([0,t]\times\overline{D}).$$

Theorem 3.2. *If the coefficients of the operator A_s from (1), the function γ, and the measure μ from (2) satisfy the conditions of Theorem 3.1, then there cannot exist more than one classical solution to problem (4)-(6).*

Proof. Let $u^{(1)}(s,x,t)$ and $u^{(2)}(s,x,t)$ be the solutions to problem (4)-(6) from class (11). Then the function $\overline{u}(s,x,t)=u^{(1)}(s,x,t)-u^{(2)}(s,x,t)$ is the solution to the following first boundary-value parabolic problem:

(39) $$\frac{\partial u(s,x,t)}{\partial s}+\frac{1}{2}b(s,x)\frac{\partial^2u(s,x,t)}{\partial x^2}+a(s,x)\frac{\partial u(s,x,t)}{\partial x}=0,\quad 0\le s<t\le T, x\in D,$$

(40) $$\lim_{s\uparrow t}u(s,x,t)=0,\quad x\in D,$$

(41) $$u(s,0,t)=\upsilon(s,t),\quad 0\le s<t\le T,$$

where

$$\upsilon(s,t)=\frac{1}{\gamma(s)+1}\int_D\overline{u}(s,y,t)\mu(s,dy).$$

We note that the function $\overline{u}$ belongs to class (11). Taking conditions a) and b) into account, we can assert that $\upsilon(s,t)$ is continuous in $s\in[0,t)$. In addition, it satisfies the fitting condition

(42) $$\lim_{s\uparrow t}\upsilon(s,t)=0.$$

Thus, $\overline{u}(s,x,t)$ is the unique solution to problem (39)-(41) and can be expressed by the formula (see [3], [4])

(43) $$\overline{u}(s,x,t)=\int_s^tG(s,x,\tau,0)V(\tau,t)d\tau,$$

where V is an unknown function which is unambiguously determined from (41). By substituting the right-hand side of equality (43) in the boundary condition (41), we obtain the Volterra integral equation of the second kind for V (24), where $\psi\equiv0$. Taking into account that the function $V(s,t)\equiv0$ is the unique solution to this equation, it becomes clear from (43) that

$$\overline{u}(s,x,t)\equiv0.$$

The proof of Theorem 3.2 is now completed. □

4. Construction of the process

We define a two-parameter operator family $T_{st}, 0\le s<t\le T$ acting on the function $\varphi\in C_b(\mathbb{R})$ by the formula

(44) $$T_{st}\varphi(x)=\int_{\mathbb{R}}G(s,x,t,y)\varphi(y)dy+\int_s^tG(s,x,\tau,0)V(\tau,t,\varphi)d\tau,$$

where the function V is the solution to the Volterra integral equation of the second kind (24). Let us study the properties of the operator family T_{st}, $0\le s<t\le T$, in assumption that the conditions of Theorem 3.1 are satisfied.

Note that the operators T_{st} are linear and bounded for all $0\le s<t\le T$. This fact follows from the representation of the function V and estimation (13).

We mention one more property of the operator family T_{st}. If the sequence $\varphi_n\in C_b(\mathbb{R})$ is such that $\lim_{n\to\infty}\varphi_n(x)=\varphi(x)$ for all $x\in\mathbb{R}$ and, in addition, $\sup_n\|\varphi_n\|<\infty$, then $\lim_{n\to\infty}T_{st}\varphi_n(x)=T_{st}\varphi(x)$ for all $0\le s<t\le T, x\in\overline{D}$. This assertion is an obvious consequence of the Lebesgue theorem on the limiting transition under the integral sign

and the theorem on the rearrangement of limits for a functional series. Taking into consideration this property, all the following reasoning can be done, without loss of generality, under condition that the function φ is finite.

Let us prove that the operators $T_{st}, 0 \le s < t \le T$, remain a cone of nonnegative functions invariant.

Lemma 4.1. *Assume that the coefficients of the operator A_s from (1), the function γ, and the measure μ from (2) satisfy the conditions of Theorem 3.1. Then, if the function $\varphi \in C_b(\mathbb{R})$ is nonnegative for all $x \in \overline{D}$, then the function $T_{st}\varphi(x)$ is also nonnegative for all $0 \le s < t \le T, x \in \overline{D}$.*

Proof. We fix an arbitrary $t \in (0, T]$ and the function $\varphi \in C_b(\mathbb{R})$ which is finite and such that $\varphi(x) \ge 0$ on the domain $\overline{D}$.

In the case of $\varphi(x) = 0$, it follows for all $x \in \overline{D}$ from Theorem 3.2 that $T_{st}\varphi(x) = 0$ for all $x \in \overline{D}, s \in [0, t]$. Thus, in this case, the assertion of the lemma is obvious.

Further, we can consider the function φ not everywhere being equal to zero on $\overline{D}$. Let m be a minimum of the function $T_{st}\varphi(x)$ on the domain $(s, x) \in [0, t] \times \overline{D}$. Let us assume that $m < 0$. Then, according to the principle of maximum ([4]), it follows that the value m can be possessed only on $(s, x) \in (0, t) \times \{0\}$. Fix $s_0 \in (0, t)$ such that $T_{s_0t}\varphi(0) = m$. Then the following inequalities hold:

$$\gamma(s_0)T_{s_0t}\varphi(0) \le 0, \quad \int_{\overline{D}}[T_{s_0t}\varphi(0) - T_{s_0t}\varphi(y)]\mu(s, dy) < 0. \tag{45}$$

Thus, in case of $s = s_0$, the fulfillment of the boundary condition (6) is impossible. A contradiction we arrived at indicates that $m \ge 0$.

The proof of Lemma 4.1 is now completed. □

Let us show that the operators $T_{st}, 0 \le s < t \le T$, are contractive, i.e., they do not increase the norm of an element.

Lemma 4.2. *Assume that the coefficients of the operator A_s from (1), the function γ, and the measure μ from (2) satisfy the conditions of Theorem 3.1. Then, for an arbitrary function $\varphi \in C_b(\mathbb{R})$, the following inequality holds:*

$$|T_{st}\varphi(x)| \le \|\varphi\|, \tag{46}$$

where $0 \le s < t \le T, x \in \overline{D}$.

Proof. If $\varphi(x) = 0$ for all $x \in \overline{D}$, then inequality (46) obviously holds. Thus, we can consider that the function φ is not everywhere equal to zero on $\overline{D}$. Assume that $M > \|\varphi\|$. Then, by means of similar considerations used in the proof of Lemma 4.1, we arrive at a contradiction. Consequently, $T_{st}\varphi(x) \le \|\varphi\|$ for all $0 \le s < t \le T, x \in \overline{D}$. Replacing φ by $-\varphi$ in the last inequality, we obtain that $T_{st}\varphi(x) \ge -\|\varphi\|$ for all $0 \le s < t \le T, x \in \overline{D}$.

The proof of Lemma 4.2 is now completed. □

Further, we observe that the operator family T_{st} is multiplicative, i.e., for all $0 \le s < u < t \le T, x \in \overline{D}$, the relation

$$T_{st}\varphi(x) = T_{su}T_{ut}\varphi(x). \tag{47}$$

holds. Equality (47) follows from Theorem 3.2 and the fact that the function $\widetilde{u}(s, x, t) = T_{su}T_{ut}\varphi(x), 0 \le s < u < t \le T, x \in \overline{D}$, is a solution to problem (4)-(6) from class (11).

The next theorem is the consequence of Lemmas 4.1 and 4.2 and relation (47) (see [10]).

Theorem 4.1. *Assume that the conditions of Theorem 3.1 are satisfied. Then the two-parameter operator family $T_{st}, 0 \le s < t \le T$, defined by formula (44) describes the* *inhomogeneous Feller process on $\overline{D}$ such that it coincides on D with the diffusion process generated by the operator A_s from (1), and its behavior on ∂D is determined by the boundary condition (3). If $P(s, x, t, dy)$ is the transition probability of this process, then, for all $\varphi \in C_b(\overline{D})$, the following equality holds:*

$$T_{st}\varphi(x) = \int_{\overline{D}} P(s, x, t, dy)\varphi(y).$$

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