Theory of Stochastic Processes
Vol. 17 (33), no. 1, 2011, pp. 100–108

G. V. RIABOV

FINITE ABSOLUTE CONTINUITY ON AN ABSTRACT WIENER SPACE

The finite absolute continuity of probability measures on an abstract Wiener space (X, H, μ) with respect to a Gaussian measure μ is studied. The limit theorem for the tails of such measures is proved.

1. Introduction

The main object of consideration in the present paper is the notion of finite absolute continuity of probability measures on Banach spaces. We study the measures that are finitely absolutely continuous with respect to Gaussian measures and the null sets of such measures.

The notion of finite absolute continuity was introduced by A.A. Dorogovtsev in [1] as a generalization of the absolute continuity in terms of moments of measures. Precisely, given two probabilities μ and ν on a Banach space X, ν is called finitely absolutely continuous with respect to μ if for each $n \geq 0$, there exists $C_n > 0$ such that the inequality

$$\left| \int_X Q d\nu \right| \leq C_n \left(\int_X Q^2 d\mu \right)^{1/2}$$

holds for an arbitrary polynomial Q of a degree at most n.

In [1], it is shown that the finite absolute continuity of ν with respect to μ is a necessary and sufficient condition for the existence of a certain expansion of ν with respect to μ. Let $Z_n(\mu)$ be the space of μ-orthogonal polynomials of a degree n. The sequence $(A_n)_{n\geq 0}$, $A_n \in Z_n(\mu)$ is called an orthogonal polynomial expansion of ν with respect to μ if for an arbitrary polynomial Q

$$\int_X Q d\nu = \sum_{n=0}^{\infty} \int_X Q A_n d\mu.$$

Then the following statement holds.

Theorem. ([1, Th.2] *ν is finitely absolutely continuous with respect to μ if and only if there exists an orthogonal polynomial expansion of ν with respect to μ.*

In the case where μ is a Gaussian measure, the orthogonal polynomial expansion is an analog of the Itô–Wiener expansion for measures. In [2] in such Gaussian setting, a class of measures is determined via positive generalized Wiener functions, and the problem of characterizing null sets for such measures is solved with the help of capacities on an abstract Wiener space.

In this article, we show that a class of measures which are finitely absolutely continuous with respect to a Gaussian measure μ is, in general, wider than that defined by positive generalized Wiener functions, and the finitely absolutely continuous measure can be

2000 *Mathematics Subject Classification.* Primary 60B11; Secondary 28C20, 46G12, 60G30.
Key words and phrases. Finite absolute continuity, Itô–Wiener expansion, Gaussian measure, capacity, slim set, weak convergence.
This work was partially supported by the State fund for fundamental researches of Ukraine and the Russian foundation for basic research under project F40.1/023.

concentrated on a slim set. We investigate relations between the Itô–Wiener expansion, weak convergence, and null sets for such measures.

2. Definitions and examples

Let X be a real separable Frechet space, and let $\mathcal{B}(X)$ be its Borelian σ-field. By $\mathcal{P}_n(X)$ we denote the totality of all polynomials on X of a degree of at most n, i.e. $\mathcal{P}_n(X)$ is the set of all functions $x \to P(l_1(x), \ldots, l_N(x))$, $l_1, \ldots, l_N \in X^*$, $P : \mathbb{R}^N \to \mathbb{R}$ is a polynomial of a degree of at most n. Denote $\mathcal{P}(X) = \cup_{n=0}^{\infty} \mathcal{P}_n(X)$ the totality of all polynomials on X.

Consider the probabilities μ and ν on $(X, \mathcal{B}(X))$ which have all weak moments of an arbitrary order:

$$\forall\, l \in X^* \;\forall\, p \geq 1 \;\; l \in L_p(X, \mu) \cap L_p(X, \nu).$$

Definition 1. [1] ν is called finitely absolutely continuous with respect to μ (this fact is denoted by $\nu \ll_0 \mu$), if
$\forall\, n \geq 0 \;\exists\, C_n > 0 : \;\forall\, Q \in \mathcal{P}_n(X)$

$$\left| \int_X Q d\nu \right| \leq C_n \left(\int_X Q^2 d\mu \right)^{1/2}.$$

The notion of finite absolute continuity was investigated in [1, 3, 4]. In [3] the relations between the finite absolute continuity and the equivalence of measures are shown, and the criteria of finite absolute continuity for Gaussian measures are proved. For a certain class of measures on the space of continuous functions, the finite absolute continuity with respect to a Wiener measure is proved in [4].

From now on, we suppose that (X, H, μ) is an abstract Wiener space, i.e. μ is a centered Gaussian measure on $(X, \mathcal{B}(X))$, $\operatorname{supp} \mu = X$, and H is a Hilbert space which is continuously imbedded in X and satisfies the property

$$\int_X \exp\{il(x)\}\mu(dx) = \exp\{-\frac{1}{2}|R_\mu(l)|_H^2\}, \quad l \in X^*.$$

Here, $R_\mu : \; X^* \to H, \; R_\mu(l) = \int_X l(x) x \mu(dx)$ [5, § 3.2].

A large class of measures finitely absolutely continuous with respect to μ is given by measures which correspond to positive generalized Wiener functions in the sense of [2]. Define the orthogonal complement of $\mathcal{P}_{n-1}(X)$ in $\overline{\mathcal{P}_n(X)}_{L_2(\mu)}$ as $Z_n(\mu)$, $Z_0(\mu) = \mathbb{R}$. Let $J_n : \; L_2(\mu) \to Z_n(\mu)$ be the orthogonal projector. For any $p > 1$, $r \in \mathbb{R}$, a Sobolev space $\mathbb{D}_p^r$ is defined as the completion of $\mathcal{P}(X)$ under the norm

$$\|Q\|_{p,r} = \| \sum_{n=0}^{\infty} (1+n)^{r/2} J_n Q \|_{L_p(\mu)}.$$

$\mathbb{D}^\infty = \cap_{\substack{p>1 \\ r>0}} \mathbb{D}_p^r$, $\mathbb{D}^{-\infty} = \cap_{\substack{q>1 \\ r>0}} \mathbb{D}_q^{-r}$. $\mathbb{D}^\infty$ is countably normed space and $\mathbb{D}^{-\infty}$ is its conjugate one. In [2], it is proved that all positive elements of $\mathbb{D}^{-\infty}$ are represented by measures on $(X, \mathcal{B}(X))$.

Theorem. [2, Th.4.1] *For each positive $\Phi \in \mathbb{D}^{-\infty}$, there exists the unique finite measure ν_Φ on $(X, \mathcal{B}(X))$ which satisfies the following property:*

for every cylindrical smooth function f of the form $f(x) = g(l_1(x), \ldots, l_N(x))$, where $l_1, \ldots, l_N \in X^*$ and $g \in C_b^\infty(\mathbb{R}^N)$

$$\int_X f d\nu_\Phi = \Phi(f).$$

Proposition 1. $\nu_\Phi \ll_0 \mu$.

Proof. If $\Phi \in \mathbb{D}_q^{-r}$, then

$$\left|\int_X Q d\nu_\Phi\right| \le \|\Phi\|_{q,-r}\|Q\|_{p,r} = \|\Phi\|_{q,-r}\|\sum_{n=0}^{\infty}(1+n)^{r/2}J_nQ\|_{L_p(\mu)}$$

for any $Q \in \mathcal{P}(X)$ [2, Th.6.1]. The finite absolute continuity follows from the equivalence of all $L_p(\mu)$-norms on $\mathcal{P}_n(X)$ [5, Corollary 5.6.5]. □

In general, $\nu_\Phi \perp \mu$. Still, the notion of capacity introduced by P. Malliavin makes it possible to characterize null sets for ν_Φ. For $p > 1, r \ge 0$, and an open $U \subset X$, we define

$$C_p^r(U) = \inf\{\|f\|_{p,r}^p \,|\, f \in \mathbb{D}_p^r,\ f \ge 1\!\!1_U\}.$$

For arbitrary $A \subset X$, we define

$$C_p^r(A) = \inf\{C_p^r(U) \,|\, A \subset U,\ U \text{ is open}\}.$$

A set $A \subset X$ is called slim, if $C_p^r(A) = 0$ for all $p > 1, r \ge 0$.

Theorem ([2, Th.4.2]). *For all $\Phi \in \mathbb{D}_q^{-r}$ and $A \in \mathcal{B}(X)$, the following inequality holds:*

$$\nu_\Phi(A) \le \|\Phi\|_{q,-r}(C_p^r(A))^{1/p}.$$

Furthermore, for every $A \in \mathcal{B}(X), C_p^r(A) > 0$, there exists $\Phi \in \mathbb{D}_{q,-r}$ such that $\nu_\Phi(A) > 0$.

As a corollary, a measure defined by a positive generalized Wiener function cannot carry a mass on the slim set. In this sense, such a measure inherits certain properties of the initial Gaussian measure on an abstract Wiener space. Consider such examples of slim sets [6]:

1) If X is finite-dimensional, there exist $p > 1, r > 0$ such that $C_p^r(\{x_0\}) > 0$;
2) If X is infinite-dimensional, then every finite set is slim.
3) $X = C_0([0,1])$, μ is a Wiener measure. The following sets are slim:
3.1) $\{x$ is differentiable at some point $t \in [0,1]\}$;
3.2) $\left\{\overline{\lim_{t\to 0}} \frac{x(t)}{\sqrt{2t \ln\ln t^{-1}}} \ne 1\right\}$;
3.3) $\left\{\lim_{\delta\to 0}\sup_{0<t_2-t_1<\delta} \frac{|x(t_2)-x(t_1)|}{\sqrt{2t\ln t^{-1}}} \ne 1\right\}$.

In the following section, we will show that, in general, not all measures finitely absolutely continuous with respect to μ are defined by positive generalized Wiener functions, and the measure $\nu, \nu \ll_0 \mu$ can be concentrated on a slim set.

3. Finite absolute continuity on the space of sequences

Consider the space $X = \mathbb{R}^\infty$ with a metric of coordinate-wise convergence and a measure $\mu = \gamma_1^{\otimes\infty}$, γ_1 is the standard Gaussian measure on $\mathbb{R}$. We need a criterion for the product-measure on X to be finitely absolutely continuous with respect to μ.

Theorem 1. *Let ν be a product-measure on X that has all weak moments of an arbitrary order. Then $\nu \ll_0 \mu$ if and only if, for any integer $d \ge 0$,*

$$\sum_{n=1}^{\infty}\left(\int_X x_n^d \nu(dx) - \int_X x_n^d \mu(dx)\right)^2 < \infty.$$

Proof. Necessity. By H_n, we denote the n-th normed Hermite polynomial [5, § 1.3], i.e.,

$$H_n(x) = \frac{(-1)^n}{\sqrt{n!}} e^{\frac{x^2}{2}} \frac{d^n}{dx^n} e^{-\frac{x^2}{2}}.$$

For fixed $N \geq 1, a_1, \dots, a_N \in \mathbb{R}$, consider $Q(x) = \sum_{n=1}^{N} a_n H_d(x_n)$. Then $Q \in \mathcal{P}_d(X)$ and $\nu \ll_0 \mu$ yield

$$\left|\sum_{n=1}^{N} a_n \int_X H_d(x_n)\nu(dx)\right| \leq C_d \left(\int_X (\sum_{n=1}^{N} a_n H_d(x_n))^2 \mu(dx)\right)^{1/2} = C_d \left(\sum_{n=1}^{N} a_n^2\right)^{1/2}.$$

Hence,

$$\sum_{n=1}^{\infty} \left(\int_X H_d(x_n)\nu(dx)\right)^2 \leq C_d^2 < \infty.$$

Choose $b_0, \dots, b_d \in \mathbb{R}$ such that $x^d = \sum_{k=0}^{d} b_k H_k(x), x \in \mathbb{R}$.

Note that $b_0 = \int_{\mathbb{R}} x^d \gamma_1(dx)$.

$$\sum_{n=1}^{\infty} \left(\int_X x_n^d \nu(dx) - \int_X x_n^d \mu(dx)\right)^2 = \sum_{n=1}^{\infty} \left(\sum_{k=1}^{d} b_k \int_X H_k(x_n)\nu(dx)\right)^2 < \infty.$$

Sufficiency. Choose $b_0, \dots, b_d \in \mathbb{R}$ such that $H_d(x) = \sum_{k=0}^{d} b_k x^k, x \in \mathbb{R}$. Note that $b_0 = -\sum_{k=1}^{d} b_k \int_{\mathbb{R}} x^k \gamma_1(dx)$.

$$\sum_{n=1}^{\infty} \left(\int_X H_d(x_n)\nu(dx)\right)^2 = \sum_{n=1}^{\infty} \left(\sum_{k=1}^{d} b_k \left(\int_X x_n^k \nu(dx) - \int_X x_n^k \mu(dx)\right)\right)^2 < \infty. \quad (1)$$

For each $d \geq 0$, we define

$$A_d(x) = \sum_{\sum_{n=1}^{\infty} i_n = d} \left(\int_X \prod_{n=1}^{\infty} H_{i_n}(y_n)\nu(dy)\right) \prod_{n=1}^{\infty} H_{i_n}(x_n).$$

A_d is well-defined as an element of $Z_d(\mu)$ because of the structure of ν and (1). If $\sum_{n=1}^{\infty} i_n = d$, then

$$\int_X \prod_{n=1}^{\infty} H_{i_n}(x_n)\nu(dx) = \int_X \prod_{n=1}^{\infty} H_{i_n}(x_n) A_d(x)\mu(dx).$$

Hence, for every $Q \in \mathcal{P}_d(X)$,

$$\int_X Q(x)\nu(dx) = \sum_{k=0}^{d} \int_X J_k Q(x)\nu(dx) =$$

$$= \sum_{k=0}^{d} \int_X J_k Q(x) A_k(x)\mu(dx) = \sum_{k=0}^{\infty} \int_X Q(x) A_k(x)\mu(dx).$$

$(A_d)_{d=0}^{\infty}$ is an orthogonal polynomial expansion of ν with respect to μ. Consequently, $\nu \ll_0 \mu$. ☐

Corollary. *Consider the product-measure* $\nu = \bigotimes_{n=1}^{\infty} (q_n \delta_{c_n} + (1-q_n)\gamma_1)$. *If, for any integer* $d \geq 0$,

$$\sum_{n=1}^{\infty} q_n^2 c_n^{2d} < \infty,$$

then $\nu \ll_0 \mu$.

In the Corollary, we put $q_n = \frac{1}{n}, c_n = \ln n$. Then the measure $\nu = \bigotimes_{n=1}^{\infty} (\frac{1}{n}\delta_{\ln n} + (1 - \frac{1}{n})\gamma_1)$ is finitely absolutely continuous with respect to μ and is such that

$$\overline{\lim_{n\to\infty}} \frac{x_n}{\sqrt{2\ln n}} = \infty \quad \nu - a.e.$$

Proposition 2 shows that ν is concentrated on a slim set of $(\mathbb{R}^\infty, \mu)$. It can be easily proved in the manner of [6, Th.A(ii)], [7, Th.5].

Proposition 2. *Let (X, H, μ) be an abstract Wiener space, and let $(l_n)_{n=1}^\infty$ be an orthonormal sequence in $L_2(\mu)$ consisting of elements of X^*. Then the set*

$$A = \left\{ \overline{\lim_{n\to\infty}} \frac{l_n(x)}{\sqrt{2\ln n}} \neq 1 \right\}$$

is a slim set in (X, H, μ).

The described construction can be translated to the space of continuous functions with the Wiener measure μ. To formulate the respective result, introduce another notations.

As (X, μ), we take the space $C_0([0,1])$ of continuous functions vanishing at zero with the Wiener measure. Denote $\Omega = \mathbb{R}^\infty, \mu_0 = \gamma_1^{\otimes\infty}$. Choose a sequence $(l_n)_{n\geq 1}$ in X^* such that $\{R_\mu(l_n)\}_{n\geq 1}$ forms an orthonormal basis (ONB) in $L_2([0,1])$. Define

$$S_N : \Omega \to X,\ S_N(\omega) = \sum_{n=1}^N \omega_n R_\mu(l_n).$$

Proposition 3. *Suppose that the measure ν_0 on Ω is finitely absolutely continuous with respect to μ_0. Then the series $\sum_{n=1}^\infty \omega_n R_\mu(l_n)$ converges in X in the measure ν_0.*

Further, if we denote the law of $\sum_{n=1}^\infty \omega_n R_\mu(l_n)$ in X as ν, then

1) ν has all weak moments of an arbitrary order;

2) $\nu \ll_0 \mu$.

Proof. 1) Fix $l \in X^*, \varepsilon > 0$.

$$\nu_0(|l\circ S_{N+p} - l\circ S_N| \geq \varepsilon) =$$

$$= \nu_0\left(\Big|\sum_{n=N+1}^{N+p} \omega_n (l, l_n)_{L_2(\mu)}\Big| \geq \varepsilon\right) \leq \varepsilon^{-2} \int_\Omega \left(\sum_{n=N+1}^{N+p} \omega_n (l, l_n)_{L_2(\mu)}\right)^2 \nu_0(d\omega) \leq$$

$$\leq \varepsilon^{-2}\,\mathrm{const} \left(\int_\Omega \left(\sum_{n=N+1}^{N+p} \omega_n (l, l_n)_{L_2(\mu)}\right)^4 \mu_0(d\omega)\right)^{1/2} \leq$$

$$\leq \varepsilon^{-2}\,\mathrm{const} \int_\Omega \left(\sum_{n=N+1}^{N+p} \omega_n (l, l_n)_{L_2(\mu)}\right)^2 \mu_0(d\omega) =$$

$$= \varepsilon^{-2}\,\mathrm{const} \sum_{n=N+1}^{N+p} (l, l_n)^2_{L_2(\mu)} \to 0,\ N\to\infty.$$

The second and third inequalities are implied by $\nu_0 \ll_0 \mu_0$ and the equivalence of $L_p(\mu_0)$-norms on $\mathcal{P}_d(\Omega)$.

Hence, for any $l \in X^*$, $(l\circ S_N)_{N\geq 1}$ converges in the measure ν_0.

Now we state the relative compactness of the distributions of S_N in X

$$E_{\nu_0}(S_N(t) - S_N(r))^4 = \int_\Omega \left(\sum_{n=1}^N \omega_n (l_n, \delta_t - \delta_r)_{L_2(\mu)}\right)^4 \nu_0(d\omega) \leq$$

$$\leq \mathrm{const} \left(\int_\Omega \left(\sum_{n=1}^N \omega_n (l_n, \delta_t - \delta_r)_{L_2(\mu)}\right)^8 \mu_0(d\omega)\right)^{1/2} \leq$$

$$\leq \mathrm{const} \left(\int_\Omega \left(\sum_{n=1}^N \omega_n (l_n, \delta_t - \delta_r)_{L_2(\mu)}\right)^2 \mu_0(d\omega)\right)^2 =$$

$$= \text{const}\left(\sum_{n=1}^{N}(l_n, \delta_t - \delta_r)^2_{L_2(\mu)}\right)^2 \le$$

$$\le \text{const}\left(\sum_{n=1}^{\infty}(l_n, \delta_t - \delta_r)^2_{L_2(\mu)}\right)^2 =$$

$$= \text{const}\,\|\delta_t - \delta_r\|^4_{L_2(\mu)} = \text{const}(t-r)^2.$$

Next, we prove the convergence of $(S_N)_{N\ge1}$ in ν_0. Fix $\varepsilon > 0$. Choose a compact $K \subset X$ such that

$$\inf_{N\ge1} \nu_0(S_N^{-1}(K)) \ge 1 - \frac{\varepsilon}{3}.$$

There exists $\delta > 0$ such that $|t-s| \le \delta$ implies $|h(t) - h(s)| \le \frac{\varepsilon}{3}$ for all $h \in K$. Choose an integer $L > \frac{1}{\delta}$, and put $t_l = \frac{l}{L}, 0 \le l \le L$. There exists an integer N_0 such that, for all $N > N_0$ and all $p \ge 1$,

$$\nu_0(|S_{N+p}(t_l) - S_N(t_l)| > \frac{\varepsilon}{3}) \le \frac{\varepsilon}{3L}.$$

$$\nu_0(\|S_{N+p} - S_N\| > \varepsilon) \le \frac{2\varepsilon}{3} + \nu_0(\|S_{N+p} - S_N\| > \varepsilon; S_N, S_{N+p} \in K) \le$$

$$\le \frac{2\varepsilon}{3} + \sum_{l=1}^{L} \nu(|S_{N+p}(t_l) - S_N(t_l)| > \frac{\varepsilon}{3}) \le \varepsilon.$$

2) Let ν be the distribution of the sum in the measure ν_0 of the series $\sum_{n=1}^{\infty} \omega_n R_\mu(l_n)$. Choose $l \in X^*, k \ge 1$.

$$E_{\nu_0}(l \circ S_N)^{2k} \le \text{const}(E_{\mu_0}(l \circ S_N)^{4k})^{1/2} \le \text{const}(E_{\mu_0}(l \circ S_N)^2)^k \le \text{const}\,\|l\|^{2k}_{L_2(\mu)}.$$

Hence, ν has all weak moments of an arbitrary order. The same arguments of uniform integrability imply that, for any polynomial Q,

$$\left|\int_X Q d\nu\right| = \lim_{N\to\infty}\left|\int_\Omega Q(S_N(\omega))\nu_0(d\omega)\right| \le$$

$$\le \text{const} \lim_{N\to\infty}\left(\int_\Omega Q^2(S_N(\omega))\mu_0(d\omega)\right)^{1/2} = \text{const}\left(\int_X Q^2 d\mu\right)^{1/2}.$$

As for the last equality, see [5, Th.3.4.4]. □

Corollary. *On the space $X = C_0([0,1])$, there exists the probability ν that is finitely absolutely continuous in the Wiener measure μ and is concentrated on a slim set of (X, H, μ).*

Proof. Consider the probability ν_0 on Ω that is finitely absolutely continuous in μ_0 and is concentrated on the set

$$A = \left\{\overline{\lim_{n\to\infty}} \frac{\omega_n}{\sqrt{2\ln n}} \ne 1\right\}.$$

Let ν be the distribution of the sum in the measure ν_0 of the series $\sum_{n=1}^{\infty} \omega_n R_\mu(l_n)$. Then $\nu \ll_0 \mu$. It is easy to see that the image of the measure ν under the mapping $F : X \to \Omega$, $F(x) = (l_n(x))_{n=1}^{\infty}$ coincides with ν_0. Proposition 2 implies that the measure ν is concentrated on a slim set $F^{-1}(A)$ in (X, μ). □

4. Shifts of finitely absolutely continuous measures

In the previous section, we saw that the moment condition of finite absolute continuity allows measures to have "heavy tails". It is natural to investigate the behavior of the tails of finitely absolutely continuous measures. Return to the case $X = \mathbb{R}^\infty, \mu = \gamma_1^{\otimes\infty}$. By π_k denote the shift of the space X:

$$\pi_k : \ X \to X, \ (\pi_k x)_n = x_{n+k}.$$

The following theorem clarifies the behavior of shifts of a finitely absolutely continuous measure.

Theorem 2. *Suppose that ν is a probability on X that has all weak moments of an arbitrary order, and $\nu \ll_0 \mu$. Then, for all $A, B \in \mathcal{B}(X), \mu(\partial B) = 0$:,*

$$\lim_{k\to\infty} \nu(A \cap \pi_k^{-1}(B)) = \nu(A)\mu(B).$$

Proof. 1) Consider the polynomials

$$Q(x) = \prod_{n=1}^N H_{i_n}(x_n), \ i_1 + \ldots + i_N \geq 1; \ R(x) = \prod_{n=1}^N H_{j_n}(x_n).$$

Fix $r \in \{0, \ldots, N-1\}, a_1, \ldots, a_L \in \mathbb{R}$ and define $S_r(x) = \sum_{l=1}^L a_l R(x) Q(\pi_{lN+r} x)$. As above, we get

$$\left| \int_X S_r d\nu \right| \leq C_{i_1+\ldots+i_N+j_1+\ldots+j_N} \left(\sum_{l=1}^L a_l^2 \right)^{1/2}.$$

Hence,

$$\sum_{l=1}^\infty \left(\int_X R(x) Q(\pi_{lN+r} x) \nu(dx) \right)^2 < \infty.$$

This yields

$$\sum_{k=1}^\infty \left(\int_X R(x) Q(\pi_k x) \nu(dx) \right)^2 < \infty. \tag{2}$$

Evidently, relation (2) can be proved for any polynomial R.

2) Consider arbitrary polynomials R and Q on X. Q has the form

$$Q(x) = \sum_{i_1+\ldots+i_N \leq d} b_{i_1\ldots i_N} \prod_{n=1}^N H_{i_n}(x_n)$$

for some $N \geq 1, \ b_{i_1\ldots i_N} \in \mathbb{R}$.

Note that $b_{0\ldots0} = \int_X Q d\mu$. Using (2), we get

$$\sum_{k=1}^\infty \left(\int_X R(x) Q(\pi_k x) \nu(dx) - \int_X R d\nu \int_X Q d\mu \right)^2 < \infty. \tag{3}$$

3) Consider $Q \in \mathcal{P}(X), f \in L_2(X, \nu)$. We write $f = f_1 + f_2$, where $f_1 \in \overline{\mathcal{P}(X)}_{L_2(\nu)}$ and $f_2 \perp \mathcal{P}(X)$. Choose

$$P \in \mathcal{P}(X) : \ \|f_1 - P\|_{L_2(\nu)} < \varepsilon.$$

Then

$$\left| \int_X f(x) Q(\pi_k x) \nu(dx) - \int_X f d\nu \int_X Q d\mu \right| =$$

$$= \left| \int_X f_1(x) Q(\pi_k x) \nu(dx) - \int_X f_1 d\nu \int_X Q d\mu \right| \leq$$

$$\le \varepsilon\|Q\circ\pi_k x\|_{L_2(\nu)} + \varepsilon\|Q\|_{L_1(\mu)} + \left|\int_X P(x)Q(\pi_k x)\nu(dx) - \int_X P d\nu \int_X Q d\mu\right|.$$

Using the inequality

$$\|Q\circ\pi_k x\|^2_{L_2(\nu)} \le C_{(\deg Q)^2}\|Q\|^2_{L_2(\mu)},$$

we obtain

$$\lim_{k\to\infty}\int_X f(x)Q(\pi_k x)\nu(dx) = \int_X f d\nu \int_X Q d\mu.$$

4) Choose $A \in B(X), \nu(A) > 0$. Define $\nu_A(\cdot) = \nu(A)^{-1}\nu(\cdot\cap A)$. We now prove that $(\nu_A\circ\pi_k^{-1})_{k\ge1}$ converges weakly to μ. It is enough to prove the weak convergence of the finite-dimensional distributions $(\nu_A\circ\pi_k^{-1})_N$ of $(\nu_A\circ\pi_k^{-1})$ for all $N\ge1$ [8, Example 2.6]. We have

$$\int_{\mathbb{R}^N}\sum_{n=1}^N u_n^2(\nu_A\circ\pi_k^{-1})_N(du) = \int_X\sum_{n=1}^N x_n^2(\nu_A\circ\pi_k^{-1})(dx) =$$

$$= \int_X\sum_{n=k+1}^{k+N} x_n^2\nu_A(dx) = \nu(A)^{-1}\int_A\sum_{n=k+1}^{k+N}x_n^2\nu(dx) \underset{k\to\infty}{\to} \int_X\sum_{n=1}^N x_n^2\mu(dx).$$

Particularly,

$$\sup_{k\ge1}\int_{\mathbb{R}^N}\sum_{n=1}^N u_n^2(\nu_A\circ\pi_k^{-1})_N(du) < \infty,$$

and the family $\{(\nu_A\circ\pi_k^{-1})_N\}_{k\ge1}$ is relatively compact. Repeating the uniform integrability arguments from Proposition 3, we obtain that each limit point of the sequence $\{(\nu_A\circ\pi_k^{-1})_N\}$ has all moments, and they coincide with moments of $\gamma_1^{\otimes N}$. Hence,

$$(\nu_A\circ\pi_k^{-1})_N \Rightarrow \gamma_1^{\otimes N} = \mu_N$$

and

$$\nu_A\circ\pi_k^{-1}\Rightarrow\mu.$$

□

Corollary 1. *Let (X,H,μ) be an abstract Wiener space, and let* $\dim X = \infty$, *ν be a probability on $X, \nu \ll_0 \mu$. Then $\nu(H) = 0$.*

Proof. Choose $l_n \in X^*, n\ge1$, such that $\{R_\mu(l_n)\}_{n=1}^\infty$ forms an ONB in H. We introduce the mapping $\varphi : X\to\mathbb{R}^\infty$ $\varphi(x) = (l_n(x))_{n=1}^\infty$. Denote $\nu_0 := \nu\circ\varphi^{-1}$, $\mu_0 := \mu\circ\varphi^{-1} = \gamma_1^{\otimes\infty}$. Then $\nu_0 \ll_0 \mu_0$. Consider the function $f(\omega) = \exp\{-\sum_{n=1}^\infty\omega_n^2\}, \omega\in\mathbb{R}^\infty$ which is continuous in μ_0 a.e. Hence,

$$\int_\Omega f(\pi_k\omega)\nu_0(d\omega) \underset{k\to\infty}{\to} \int_\Omega f(\omega)\mu_0(d\omega) = 0.$$

On the other hand, $f(\pi_k\omega) \underset{k\to\infty}{\to} 1\!\!1_{l_2}(\omega), \omega\in\mathbb{R}^\infty$. Hence, $\nu_0(l_2) = \nu(H) = 0$. □

Remark. S.Fang in [9] proved that H is a slim set if $\dim X = \infty$.

Corollary 2. [10, Th.2.2] *In the setting $(X,\mu) = (\mathbb{R}^\infty, \gamma_1^{\otimes\infty})$, if $\nu\ll_0\mu$, then* $\overline{(\pi_k x)_{k\ge1}} =$ *X $\nu-a.s.$*

Proposition 4. *Suppose that ν satisfies the condition of Theorem 2. By $(A_d)_{d=0}^\infty$, we denote the orthogonal polynomial expansion of ν with respect to μ. Let $\{e_n\}_{n=1}^\infty$ be the standard basis in l_2. Then, for any $k\ge1$, $\nu\circ\pi_k^{-1}\ll_0\mu$ with constants independent of k, and the orthogonal polynomial expansion of $\nu\circ\pi_k^{-1}$ with respect to μ is of the form*

$$\left(A_d^k = \sum_{n_1,\dots,n_d>k}(A_d, e_{n_1}\otimes\ldots\otimes e_{n_d})e_{n_1-k}\otimes\ldots\otimes e_{n_d-k}\right)_{d=0}^\infty.$$

Remark. If each measure ν_α of the family $\{\nu_\alpha, \alpha \in \mathfrak{A}\}$ is finitely absolutely continuous with respect to μ and constants in Definition 1 can be chosen independently of α, then we say that $\nu_\alpha \ll_0 \mu$ uniformly in α.

Denote $(X, H, \mu) = (C_0([0,1]), L_2([0,1]), \mu)$. Choose $l_n \in X^*, n \geq 1$ such that $(R_\mu(l_n))_{n\geq 1}$ is an ONB in $L_2([0,1])$. Introduce the mapping $\varphi : X \to \Omega = \mathbb{R}^\infty, \varphi(x) = (l_n(x))_{n\geq 1}$. Denote $\nu_0 := \nu \circ \varphi^{-1}$, $\mu_0 := \mu \circ \varphi^{-1} = \gamma_1^{\otimes\infty}$.

Using Proposition 3, we now construct measures $\nu \circ \pi_k^{-1}$ on X as the distributions of series $\sum_{n=1}^\infty \omega_n R_\mu(l_n)$ with respect to $\nu_0 \circ \pi_k^{-1}$. It is easy to see that $\nu \circ \pi_k^{-1} \ll_0 \mu$ uniformly in k.

Theorem 3. $\nu \circ \pi_k^{-1} \Rightarrow \mu, k \to \infty$.

Proof. As $\nu \circ \pi_k^{-1} \ll_0 \mu$ uniformly in k, the sequence $(\nu \circ \pi_k^{-1})_{k\geq 1}$ is relatively compact (see the proof of Proposition 3). On the other hand, the proof of Theorem 2 shows that the unique limit point of this sequence is μ. $\square$

The orthogonal polynomial expansion $(A_d^k)_{d=0}^\infty$ of $\nu \circ \pi_k^{-1}$ with respect to μ has the same form as that in Proposition 4. The connection between the Itô–Wiener expansion and the weak convergence is shown by the following result.

Theorem 4. *Consider an abstract Wiener space* $(X, H, \mu) = (C_0([0,1]), L_2([0,1]), \mu)$ *and a sequence* $(\nu_k)_{k=1}^\infty$ *of probabilities on* $(X, \mathcal{B}(X))$ *such that* $\nu_k \ll_0 \mu$ *with the orthogonal polynomial expansion* $(A_d^k)_{d=0}^\infty$, $k \geq 1$. *If the convergence* $A_d^k \to 0, k \to \infty$ *in* $Z_d(\mu)$ *holds for every* $d \geq 1$, *then* $\nu_k \Rightarrow \mu, k \to \infty$.

Proof. For any $d \geq 0$, the sequence $(A_d^k)_{k=0}^\infty$ is bounded in $Z_d(\mu)$. Hence, $\nu_k \ll_0 \mu$ uniformly in k, and the sequence $(\nu_k)_{k=1}^\infty$ is relatively compact. The rest of the proof follows the pattern of the proof of Theorem 2. $\square$

Acknowledgment. The author is grateful to Professor A.A. Dorogovtsev for the statement of the problem and a constant attention to the work and thanks the referee for numerous comments concerning the presentation of the paper.

References

1. A. A. Dorogovtsev, *Measurable functionals and finitely absolutely continuous measures on Banach spaces*, Ukr. Math. J. **52** (2000), no. 9, 1366–1379.
2. H. Sugita, *Positive generalized Wiener functions and potential theory over abstract Wiener space*, Osaka J. Math. **25** (1988), no. 3, 665–696.
3. G. V. Riabov, *Finite absolute continuity of Gaussian measures on infinite-dimensional spaces*, Ukr. Math. J. **60** (2008), no. 10, 1367–1377.
4. G. V. Riabov, *On finite absolute continuity of measures related to Wiener measure*, Inf. Dim. An. and Quant. Pr. **12**, no. 2 (2009), 341–352.
5. V. I. Bogachev, *Gaussian Measures*, Moscow: Nauka, 1997 (in Russian).
6. M. Takeda, *(r,p)-capacity on the Wiener space and properties of Brownian motion*, Z. Wahrsch. Verw. Gebiete **68** (1984), no. 2, 149–162.
7. M. Fukushima, *Basic properties of Brownian motion and a capacity on the Wiener space*, J. Math. Soc. Japan **36** (1984), no. 1, 161–176.
8. P. Billingsley, *Convergence of Probability Measures*, New York: Wiley, 1999.
9. S. Fang, *Sur un résultat d'entropie de l'espace de Wiener*, C. R. Acad. Sci. Paris. Sér I **312** (1991), 995–998.
10. H. Rootzen, *Fluctuations of sequences which converge in distribution*, Ann. Probab. **4** (1976), no. 3, 456–463.

Institute of Mathematics of the Ukrainian Academy of Sciences, Kiev, Ukraine
E-mail address: ryabov.george@gmail.com