Theory of Stochastic Processes
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THE DISTRIBUTION OF RANDOM EVOLUTION IN ERLANG SEMI-MARKOV MEDIA

We study a one-dimensional random motion by using a general Erlang distribution for the sojourn times of a switching process and obtain the solution of a four-order hyperbolic PDE in the 2-Erlang case.

1. Introduction

In paper [1], we studied a one-dimensional random motion with the m-Erlang distribution between consequent epochs of velocity alternations. Let $f(t, x)$ be the probability density function (pdf) of a particle position at time t, provided that it exists. We obtained the following higher order hyperbolic equations for $f(t, x)$:

$$\left(\frac{\partial}{\partial t} - v\frac{\partial}{\partial x} + \lambda\right)^m \left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x} + \lambda\right)^m f(t,x) - \lambda^{2m} f(t,x) = 0, \tag{1}$$

where $v > 0$ is the velocity of a particle, and λ is the parameter of the m-Erlang distribution. It is assumed that the particle started at $x = 0$, and, hence, $f(0, x) = \delta(x)$.

The pdf $f(t, x)$ can be represented in the form $f(t, x) = f_c(t, x) + f_s(t, x)$, where $f_c(t, x)$ is the absolute continuous part, and $f_s(t, x)$ is the singular part w.r.t. Lebesgue measure on the line.

Lemma 1.1. *The singular part $f_s(t, x)$ of the pdf $f(t, x)$ is of the following form:*

$$f_s(t, x/v) = \delta(t - x/v)e^{-\lambda t} \sum_{i=0}^{m-1} (\lambda t)^i / i!. \tag{2}$$

Proof. It is evident that, for $t = x/v$, the pdf $f(t, x)$ has the singularity given by Eq. (2). Let us show that, for $t > |x/v|$, the pdf $f(t, x)$ has no singularity w.r.t. Lebesgue measure on $\mathbb{R}$. By v_k, we denote the random event "k velocity alternations occurred". For $\Delta x = [x, x + \Delta]$, $\Delta > 0$, let us consider

$$P_{\bar{\nu}_0}(x(t) \in \Delta x) = \sum_{k\geq 1} P(x(t) \in \Delta x, \nu_k),$$

which is the probability of the event where at least one alternation occurred and $x(t) \in \Delta x$. Let us show that, for each $t > 0$, there exists a constant $C_t < \infty$ such that

$$\sup_x \frac{P_{\bar{\nu}_0}(x(t) \in \Delta x)}{\Delta x} < C_t.$$

By $\theta_k, k \geq 1$, we denote the time between $(k-1)$-th and k-th velocity alternations. Recall that θ_k, $k \geq 1$, are independent m-Erlang distributed random variables. It is easy to verify that

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$$
\begin{aligned}
P_{\bar\nu_0}(x(t)\in\Delta x) &= \sum_{k\ge1} P\left(\sum_{i=1}^{k}(-1)^{i+1}\theta_i v + (-1)^k\left(t-\sum_{i=1}^{k}\theta_i v\right)\in\Delta x, \sum_{i=1}^{k}\theta_i<t\right)\\
&= \sum_{k\ge1} P\left(\left(\sum_{i=1}^{k}(-1)^{i+1}\theta_i - (-1)^k\sum_{i=1}^{k}\theta_i\right)v\in\Delta x-(-1)^k vt, \sum_{i=1}^{k}\theta_i<t\right)\\
&= \sum_{l\ge0} P\left(2v(\theta_1+\theta_3+\ldots+\theta_{2l+1})\in\Delta x - vt, \sum_{i=1}^{2l+1}\theta_i<t\right)\\
&= \sum_{l\ge0} P\left(-2v(\theta_2+\theta_4+\ldots+\theta_{2l+2})\in\Delta x + vt, \sum_{i=1}^{2l+2}\theta_i<t\right)\\
&\le \sup_x\sum_{l\ge0} P\left(2v\sum_{i=1}^{l}\theta_{2i-1}\in\Delta x, 2v\sum_{i=1}^{l}\theta_{2i}<vt-x\right)\\
&+ \sup_x\sum_{l\ge0} P\left(-2v\sum_{i=1}^{l}\theta_{2i}\in\Delta x, 2v\sum_{i=1}^{l}\theta_{2i+1}<vt+x\right).
\end{aligned}
$$

Since $|x|\le vt$ and, for every $m\ge1$, the pdf $p_m(x,\lambda)$ of the m-Erlang distribution with the parameter λ satisfies $p_m(x,\lambda)\le\lambda$, we have

$$
\begin{aligned}
&\sum_{l\ge1} P(2v(\theta_1+\theta_3+\ldots+\theta_{2l-1})\in\Delta x, 2v(\theta_2+\theta_4+\ldots+\theta_{2l})<vt-x)\\
&\qquad\le \frac{\lambda\Delta}{2v}\sum_{l\ge1}P(\theta_2+\theta_4+\ldots+\theta_{2l}<t). \qquad (3)
\end{aligned}
$$

Since θ_i is m-Erlang distributed, we have, for $2lm+1>t$,

$$
P(\theta_2+\theta_4+\ldots+\theta_{2l}<t)\le\left(e^{\lambda t}-\sum_{i=0}^{2lm}\frac{(\lambda t)^i}{i!}\right)e^{-\lambda t}\le\frac{(\lambda t)^{2lm+1}e^{-\lambda t}}{2lm!(2lm+1-\lambda t)}.
$$

Therefore, taking (3) into account, there exists a constant A_t such that

$$
\sup_x\sum_{l\ge1}P\left(2v\sum_{i=1}^{l}\theta_{2i-1}\in\Delta x, 2v\sum_{i=1}^{l}\theta_{2i}<vt-x\right)\le A_t\Delta.
$$

In the same way, we can show that there exists a constant B_t such that

$$
\sup_x\sum_{l\ge1}P\left(-2v\sum_{i=1}^{l}\theta_{2i}\in\Delta x, 2v\sum_{i=1}^{2l-1}\theta_{2i}<vt+x\right)\le B_t\Delta.
$$

Putting $C_t=A_t+B_t$, we conclude the proof.

Corollary 1.1. *The absolute continuous part $f_c(t,x)$ of the pdf $f(t,x)$ satisfies Eq. (1) for $t<|\frac{x}{v}|$.*

We now study the behavior of the continuous part $f_c(t,x)$ close to lines $t=\pm\frac{x}{v}$.

Lemma 1.2. *For $m\ge2$, we have*

$$
\lim_{\varepsilon\downarrow0}\frac{P\{0<t-x(t)<\varepsilon\}}{\varepsilon}=\frac{\lambda^m t^{m-1}e^{-\lambda t}}{2(m-1)!},
$$

$$
\lim_{\varepsilon\to0}\frac{P\{t+x(t)<\varepsilon\}}{\varepsilon}=0.
$$

Proof. It is easy to verify that

$$P\{0<t-x(t)\le\varepsilon\}=P\left\{t-\frac{\varepsilon}{2}\le\theta_1<t\right\}+\int_0^t P\left\{\theta_3\ge t-u,\theta_2\le\frac{\varepsilon}{2},\theta_1\in du\right\}+o(\varepsilon),$$

where $\theta_i, i=1,2,3$ are independent m-Erlang distributed random variables with the parameter λ. Since $\int_0^t P(\theta_3\ge t-u,\theta_2\le\frac{\varepsilon}{2},\theta_1\in du)=o(\varepsilon)$, we pass to the limit and obtain

$$\begin{aligned}\lim_{\varepsilon\downarrow 0}&\frac{P\{0<t-x(t)<\varepsilon\}}{\varepsilon}\\&=\lim_{\varepsilon\downarrow 0}\frac{e^{-\lambda t}}{\varepsilon}\left(\left(\sum_{i=0}^{m-1}\frac{(\lambda t)^i}{i!}\right)-e^{\lambda\frac{\varepsilon}{2}}\left(\sum_{i=0}^{m-1}\frac{\left(\lambda\left(t-\frac{\varepsilon}{2}\right)\right)^i}{i!}\right)\right)\\&=\frac{\lambda^m t^{m-1}e^{-\lambda t}}{2(m-1)!}.\end{aligned}$$

Similarly, $P\{t+x(t)\le\varepsilon\}=P\left\{\theta_2\ge t-\frac{\varepsilon}{2},\theta_1\le\frac{\varepsilon}{2}\right\}+o(\varepsilon)$, and it is easily seen that

$$\lim_{\varepsilon\downarrow 0}\frac{P\{t+x(t)<\varepsilon\}}{\varepsilon}=0.$$

Remark 1.1. The case where $m=1$ will be considered in what follows as an example.

Remark 1.2. We will seek solutions of Eq. (1) among functions whose continuous part $f_c(t,x)$ satisfies the conditions

$$\lim_{x\uparrow t}f_c(t,x)=\lim_{\varepsilon\downarrow 0}\frac{P\{0<t-x(t)<\varepsilon\}}{\varepsilon},\ \lim_{x\downarrow -t}f_c(t,x)\ \lim_{\varepsilon\downarrow 0}\frac{P\{t+x(t)<\varepsilon\}}{\varepsilon}. \tag{4}$$

By applying the transformation $f(t,x)=e^{\lambda t}g(t,x)$ and changing the variable $y=\frac{x}{v}$, we reduce Eq. (1) to

$$\left(\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial y^2}\right)^m g_c(t,y)-\lambda^{2m}g_c(t,y)=0 \tag{5}$$

with the singular part $g_s(t,y)=\left(\sum_{i=0}^{m-1}\frac{(\lambda t)^i}{i!}\right)\delta(t-y)$.

In the sequel, we assume, without loss of generality, that $\lambda=1$. By introducing the function $\mathbf{f}(t,y,z)=e^z g_c(t,y)$, we reduce Eq. (5) to the equation

$$\left(\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial y^2}\right)^m\mathbf{f}(t,y,z)-\frac{\partial^{2m}}{\partial z^{2m}}\mathbf{f}(t,y,z)=0. \tag{6}$$

We will seek solutions of this equation by using the theory of differentiable functions on commutative algebras [2].

2. Main results

Let A_0 be a $2m$-dimensional commutative algebra over $\mathbb{R}$. We assume that the set e_0, e_1, …, e_{2m-1} is a basis of A_0 with the Cayley table:

$$e_ie_j=e_{i\oplus j},$$

where $i\oplus j=i+j \pmod{2m}$.
The algebra A_0 has a matrix representation

$$e_k\to P_k=P_1^k,$$

where $P_1=[p_{ij}]_{2m\times 2m}$, $p_{ii+1}=1$ for $0\le i\le 2m-1$, $p_{2m0}=1$, and $p_{ij}=0$ for the rest of $i,\ j$.

We put

$$\tau_0^l=e_l,\quad l=0,1,\dots,2m-1,$$

$$\tau_1^l = e_l i \sin s, \quad l = 0, 1, \ldots, 2m-1,$$
$$\tau_2^l = e_l \cos s, \quad l = 0, 1, \ldots, 2m-1,$$
$$\tau_{2k}^l = e_l \cos ks, \quad \tau_{2k+1}^l = e_l i \sin(k+1)s, \quad l = 0, 1, \ldots, 2m-1,$$
$$k = 0, 1, 2, \ldots.$$

It is easily seen that $\tau_{2n}^0 \tau_{2k}^0 = \frac{1}{2}\left(\tau_{2(n-k)}^0 + \tau_{2(n+k)}^0\right), \quad n \geq k,$

$$\tau_{2n+1}^{l_1} \tau_{2k+1}^{l_2} = \frac{1}{2}\left(\tau_{2(n-k)}^{l_1 \bigoplus l_2} - \tau_{2(n+k)}^{l_1 \bigoplus l_2}\right), \quad n \geq k,$$

$$\tau_{2n+1}^{l_1} \tau_{2k}^{l_2} = \frac{1}{2}\left(\tau_{2(n-k)+1}^{l_1 \bigoplus l_2} + \tau_{2(n+k)+1}^{l_1 \bigoplus l_2}\right), \quad n \geq k.$$

We now introduce the algebra

$$A = \left\{ \sum_{k=0}^{+\infty} \sum_{l=0}^{2m-1} \left(a_{2k}^l \tau_{2k}^l + a_{2k+1}^l \tau_{2k+1}^l\right) \;\middle|\; a_j^l \in \mathbb{R} \right\},$$

where $\sum_{k=0}^{+\infty} \sum_{l=0}^{2m-1} \left(\left|a_{2k}^l\right|^2 + \left|a_{2k+1}^l\right|^2\right) < +\infty.$

It is easy to verify that A is commutative.

We consider the subspace $B = \left\{a_0 \tau_1^1 + a_1 \tau_2^1 + a_2 \tau_0^0 \;\middle|\; a_i \in \mathbb{R}\right\}$ of the algebra A.

Let us introduce the function $\mathbf{f} : B \to A$ ($\mathbf{f}(t, y, z) = f\left(e_1 (t\cos s + yi\sin s) + z\right)$) as follows:

$$\mathbf{f}(t, y, z) = \sum_{k=0}^{+\infty} \sum_{l=0}^{2m-1} \left(v_{2k}^l (t, y, z) \tau_{2k}^l + v_{2k+1}^l (t, y, z) \tau_{2k+1}^l\right).$$

The function $\mathbf{f}$ is called B/A differentiable at $\mathbf{x}_0 \in B$ if there exists $\mathbf{f}'(\mathbf{x}_0) \in A$ such that, for any $\mathbf{h} \in B$,

$$\mathbf{f}'(\mathbf{x}_0)\, \mathbf{h} = \lim_{\varepsilon \to 0} \frac{\mathbf{f}(\mathbf{x}_0 + \varepsilon \mathbf{h}) - \mathbf{f}(\mathbf{x}_0)}{\varepsilon}$$

In [2], it was proved that if $\mathbf{f}$ is B/A differentiable, then

$$\frac{\partial}{\partial t}\mathbf{f} = e_1 \cos s \, \frac{\partial}{\partial z}\mathbf{f} \tag{7}$$

and

$$\frac{\partial}{\partial y}\mathbf{f} = e_1 i \sin s \, \frac{\partial}{\partial z}\mathbf{f}. \tag{8}$$

In this case, all $v_{2k}^l (t, y, z)$ are solutions of Eq. (6). Indeed,

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial y^2}\right)^m \mathbf{f} - \frac{\partial^{2m}}{\partial z^{2m}}\mathbf{f} = e_1^{2m}\left(\cos^2 s - (i\sin s)^2\right)^m - 1 = 0.$$

In the sequel, we denote the element e_1 by $\mathbf{e}$.

We will seek a solution of Eq. (5) in the form

$$g_c\left(\mathbf{e}(t\cos s + yi\sin s)\right) = e^{\mathbf{e}(t\cos s + yi\sin s)}$$

. Since $f\left(\mathbf{e}(t\cos s + yi\sin s) + z\right) = g_c(\mathbf{e}(t\cos s + yi\sin s))e^z$, we have

$$v_k^l (t, y, z) = u_k^l (t, y)\, e^z, l = 0, 1, \ldots, 2m-1, \; k = 0, 1, 2, \ldots,$$

where $g_c(t, y) = \sum_{k=0}^{+\infty} \sum_{l=0}^{2m-1} \left(u_{2k}^l (t, y) \tau_{2k}^l + u_{2k+1}^l (t, y) \tau_{2k+1}^l\right).$

Therefore, we obtain the functions $u_0^l (t, y)$ for $t \geq |y|$ from the equation

$$\sum_{l=0}^{2m-1} u_0^l (t, y)\, \tau_0^l = \sum_{l=0}^{2m-1} u_0^l (t, y)\, \mathbf{e}^l$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{\mathbf{e}(t\cos s + yi\sin s)} ds = J_0\left(\mathbf{e} i \sqrt{y^2 - t^2}\right) = I_0\left(\mathbf{e}\sqrt{t^2 - y^2}\right),$$

where I_k (resp. J_k) is the modified Bessel (resp. Bessel) function of the first kind and k-th order [4].

Equations (7) and (8) yield the following Cauchy–Riemann-type conditions:

$$\frac{\partial}{\partial t}u_0^{l\oplus 1} = \frac{1}{2}u_2^l,$$
$$\frac{\partial}{\partial t}u_1^{l\oplus 1} = \frac{1}{2}u_3^l,$$
$$\frac{\partial}{\partial t}u_2^{l\oplus 1} = u_0^l + \frac{1}{2}u_4^l,$$
$$\frac{\partial}{\partial t}u_{2k-1}^{l\oplus 1} = \frac{1}{2}\left(u_{2k-3}^l + u_{2k+1}^l\right),$$
$$\frac{\partial u_{2k}^{l\oplus 1}}{\partial t} = \frac{1}{2}\left(u_{2(k-1)}^l + u_{2(k+1)}^l\right), \tag{9}$$
$$k = 2, 3, \ldots;$$

and

$$\frac{\partial}{\partial y}u_0^{l\oplus 1} = -\frac{1}{2}u_1^l,$$
$$\frac{\partial u_1^{l\oplus 1}}{\partial y} = u_0^l - \frac{1}{2}u_4^l,$$
$$\frac{\partial}{\partial y}u_2^{l\oplus 1} = -\frac{1}{2}u_3^l,$$
$$\frac{\partial u_{2k+1}^{l\oplus 1}}{\partial y} = \frac{1}{2}\left(u_{2k}^l - u_{2(k+2)}^l\right),$$
$$\frac{\partial u_{2k+2}^{l\oplus 1}}{\partial y} = \frac{1}{2}\left(u_{2k-1}^l - u_{2k+3}^l\right), \tag{10}$$
$$k = 1, 2, \ldots.$$

By using Eqs. (9) and (10) and the functions $u_0^l(t,y)$, we can obtain recurrently the function $u_k^l(t,y)$ for any $k \geq 1$, which will be used to solve Eq. (1).

In the sequel, unless otherwise specified, the case where $m = 2$ is studied. In this case, $f_s(t,y)$ is of the form $f_s(t,x) = e^{-t}(1+t)\,\delta(t-x)$, and, hence,

$$g_s(t,y) = (1+t)\,\delta(t-y).$$

The algebra A_0 is as follows:

$$A_0 = \{a + e_1 b + e_2 c + e_3 d \mid \quad a,b,c,d \in \mathbb{R}\}.$$

Here, the basis $e_l = \mathbf{e}^l$, $l = 0,1,2,3$, and $\mathbf{e}$ has the matrix representation

$$\mathbf{e} \to P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

Therefore, we have $\tau_0^0 = 1$, $\tau_{2k}^0 = \cos ks$, $\tau_{2k}^l = \mathbf{e}^l \cos ks$, $\tau_{2k+1}^l = \mathbf{e}^l i \sin(k+1)s$, $l = 0,1,2,3,\ k = 0,1,2,\ldots$

Taking into account that $g_c(\mathbf{e}(t\cos s + yi\sin s)) = e^{\mathbf{e}(t\cos s + yi\sin s)}$, we have

$$u_0^0(t,y) + \mathbf{e}u_0^1(t,y) + \mathbf{e}^2 u_0^2(t,y) + \mathbf{e}^3 u_0^3(t,y) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{\mathbf{e}(t\cos s + yi\sin s)}ds$$
$$= I_0\left(\mathbf{e}\sqrt{t^2 - y^2}\right).$$

It is easily seen that

$$I_0\left(\mathbf{e}\sqrt{t^2-y^2}\right)$$
$$=\frac{I_0\left(\sqrt{t^2-y^2}\right)+I_0\left(i\sqrt{t^2-y^2}\right)}{2}+\mathbf{e}^2\left(\frac{I_0\left(\sqrt{t^2-y^2}\right)-I_0\left(i\sqrt{t^2-y^2}\right)}{2}\right)$$
$$=\frac{I_0\left(\sqrt{t^2-y^2}\right)+J_0\left(\sqrt{t^2-y^2}\right)}{2}+\mathbf{e}^2\frac{I_0\left(\sqrt{t^2-y^2}\right)-J_0\left(\sqrt{t^2-y^2}\right)}{2}.$$

Therefore, for $t\geq|y|$, we have $u_0^1(t,y)=u_0^3(t,y)=0$ and

$$u_0^0(t,y)=\frac{I_0\left(\sqrt{t^2-y^2}\right)+J_0\left(\sqrt{t^2-y^2}\right)}{2},$$
$$u_0^2(t,y)=\frac{I_0\left(\sqrt{t^2-y^2}\right)-J_0\left(\sqrt{t^2-y^2}\right)}{2}.$$

It follows from the first two equations of (10) that

$$u_1^1=-2\frac{\partial}{\partial y}u_0^2=-\frac{\partial\left[I_0\left(\sqrt{t^2-y^2}\right)-J_0\left(\sqrt{t^2-y^2}\right)\right]}{\partial y}$$
$$=\frac{y}{\sqrt{t^2-y^2}}\left(I_1\left(\sqrt{t^2-y^2}\right)+J_1\left(\sqrt{t^2-y^2}\right)\right),$$

$$u_1^3=-2\frac{\partial}{\partial y}u_0^0=-\frac{\partial\left[I_0\left(\sqrt{t^2-y^2}\right)+J_0\left(\sqrt{t^2-y^2}\right)\right]}{\partial y}$$
$$=\frac{y}{\sqrt{t^2-y^2}}\left(I_1\left(\sqrt{t^2-y^2}\right)-J_1\left(\sqrt{t^2-y^2}\right)\right),$$

$$u_1^0=-2\frac{\partial}{\partial y}u_0^1=0,$$

$$u_1^2=-2\frac{\partial}{\partial y}u_0^3=0.$$

Then the Cauchy–Riemann-type conditions (9) yield

$$u_2^0(t,y)=2\frac{\partial u_0^1(t,y)}{\partial t}=0;$$

$$u_2^1(t,y)=2\frac{\partial u_0^2(t,y)}{\partial t}=\frac{\partial\left(I_0\left(\sqrt{t^2-y^2}\right)-J_0\left(\sqrt{t^2-y^2}\right)\right)}{\partial t}$$
$$=\frac{t}{\sqrt{t^2-y^2}}\left(I_1\left(\sqrt{t^2-y^2}\right)+J_1\left(\sqrt{t^2-y^2}\right)\right);$$

$$u_2^2(t,y)=2\frac{\partial u_0^3(t,y)}{\partial y}=0;$$

$$u_2^3(t,y)=2\frac{\partial u_0^0(t,y)}{\partial t}=\frac{\partial\left[I_0\left(\sqrt{t^2-y^2}\right)+J_0\left(\sqrt{t^2-y^2}\right)\right]}{\partial t}$$
$$=\frac{t}{\sqrt{t^2-y^2}}\left(I_1\left(\sqrt{t^2-y^2}\right)-J_1\left(\sqrt{t^2-y^2}\right)\right).$$

Similarly, for u_3^l, we have

$$\begin{aligned} u_3^0 =& 2\frac{\partial}{\partial t}u_1^1 = 2\frac{\partial}{\partial t}\left[\frac{y}{\sqrt{t^2-y^2}}\left(I_1\left(\sqrt{t^2-y^2}\right)+J_1\left(\sqrt{t^2-y^2}\right)\right)\right] \\ =& -\frac{2ty}{\sqrt{(t^2-y^2)^3}}\left(I_1\left(\sqrt{t^2-y^2}\right)+J_1\left(\sqrt{t^2-y^2}\right)\right) \\ &+\frac{ty}{t^2-y^2}\left(I_0\left(\sqrt{t^2-y^2}\right)+I_2\left(\sqrt{t^2-y^2}\right)+J_0\left(\sqrt{t^2-y^2}\right)-J_2\left(\sqrt{t^2-y^2}\right)\right); \end{aligned}$$

$$\begin{aligned} u_3^2 =& 2\frac{\partial}{\partial t}u_1^3 = 2\frac{\partial}{\partial t}\left[\frac{y}{\sqrt{t^2-y^2}}\left(I_1\left(\sqrt{t^2-y^2}\right)-J_1\left(\sqrt{t^2-y^2}\right)\right)\right] \\ =& -\frac{2ty}{\sqrt{(t^2-y^2)^3}}\left(I_1\left(\sqrt{t^2-y^2}\right)-J_1\left(\sqrt{t^2-y^2}\right)\right) \\ &+\frac{2ty}{t^2-y^2}\left(I_0\left(\sqrt{t^2-y^2}\right)+I_2\left(\sqrt{t^2-y^2}\right)-J_0\left(\sqrt{t^2-y^2}\right)+J_2\left(\sqrt{t^2-y^2}\right)\right). \end{aligned}$$

It is easily seen that $u_3^1 = u_3^3 = 0$.

Next, it follows from (9) that

$$\begin{aligned} u_4^0 =& 2\frac{\partial u_2^1}{\partial t} - 2u_0^0 = 2\frac{\partial}{\partial t}\frac{t}{\sqrt{t^2-y^2}}\left(I_1\left(\sqrt{t^2-y^2}\right)+J_1\left(\sqrt{t^2-y^2}\right)\right) - 2u_0^0 \\ =& \frac{-2y^2}{\sqrt{(t^2-y^2)^3}}\left(I_1\left(\sqrt{t^2-y^2}\right)+J_1\left(\sqrt{t^2-y^2}\right)\right) \\ &+\frac{t^2}{t^2-y^2}\left(I_0\left(\sqrt{t^2-y^2}\right)+I_2\left(\sqrt{t^2-y^2}\right)+J_0\left(\sqrt{t^2-y^2}\right)-J_2\left(\sqrt{t^2-y^2}\right)\right) \\ &-I_0\left(\sqrt{t^2-y^2}\right)-J_0\left(\sqrt{t^2-y^2}\right); \end{aligned}$$

$$\begin{aligned} u_4^2 =& 2\left(\frac{\partial u_2^3}{\partial t} - u_0^2\right) = \frac{-2y^2}{\sqrt{(t^2-y^2)^3}}\left(I_1\left(\sqrt{t^2-y^2}\right)-J_1\left(\sqrt{t^2-y^2}\right)\right) \\ &+\frac{t^2}{t^2-y^2}\left(I_0\left(\sqrt{t^2-y^2}\right)+I_2\left(\sqrt{t^2-y^2}\right)-J_0\left(\sqrt{t^2-y^2}\right)+J_2\left(\sqrt{t^2-y^2}\right)\right) \\ &-I_0\left(\sqrt{t^2-y^2}\right)+J_0\left(\sqrt{t^2-y^2}\right). \end{aligned}$$

It is also easy to verify that $u_4^1 = u_4^3 = 0$.

By using the well-known integrals for Bessel functions [3-5], we have

$$\int_{-t}^{t} u_0^0 dy = \sinh t + \sin t, \int_{-t}^{t} u_0^2 dy = \sinh t - \sin t, \int_{-t}^{t} u_1^1 dy = \int_{-t}^{t} u_1^3 dy = 0,$$

$$\begin{aligned} \int_{-t}^{t} u_2^1 dy =& 2\int_{-t}^{t}\frac{\partial u_0^2}{\partial t}dy = 2\left(\frac{\partial}{\partial t}\int_{-t}^{t} u_0^2 dy - u_0^2(t,t) - u_0^2(t,-t)\right) \\ =& 2\cosh t - 2\cos t, \end{aligned}$$

$$\begin{aligned} \int_{-t}^{t} u_2^3 dy =& 2\int_{-t}^{t}\frac{\partial u_0^0}{\partial t}dy = 2\left(\frac{\partial}{\partial t}\int_{-t}^{t} u_0^0 dy - u_0^0(t,t) - u_0^0(t,-t)\right) \\ =& 2\cosh t + 2\cos t - 4. \end{aligned}$$

As an example, we obtain the pdf in the case where $m = 1$. In this case, $e_1 = 1$, and, hence, we can consider the functions $\sum_{l=0}^{4} u_k^l(t,y)$, $k = 0,1,2,\dots$ as solutions of Eq. (5) for $m = 1$.
For $t \le |y|$, consider the function $g(t,y) = g_c(t,y) + g_s(t,y)$ of the form

$$\begin{aligned} g_c(t,y) =& \frac{1}{2}\left(u_0^0(t,y) + u_0^2(t,y)\right) + \frac{1}{4}\left(u_1^1(t,y) + u_1^3(t,y) + u_2^1(t,y) + u_2^3(t,y)\right) \\ =& \frac{I_0\left(\sqrt{t^2-y^2}\right)}{2} + \frac{t+y}{2\sqrt{t^2-y^2}} I_1\left(\sqrt{t^2-y^2}\right) \end{aligned}$$

and $g_s(t,y) = \delta(t-y)$.
It is easily seen that the function $g_c(t,y)$ is a solution of the equation for $t < y$:

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial y^2}\right) g(t,y) - g(t,y) = 0. \tag{11}$$

In addition, we have $\lim_{y\uparrow t} g_c(t,y) = \frac{1}{2}(1+t)$ and $\lim_{y\downarrow -t} g_c(t,y) = \frac{1}{2}$.
To avoid cumbersome calculations, we put $v = 1$.
Therefore, $f(t,x) = e^{-t} g(t,x)$ is a solution of the equation

$$\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x} + 1\right)\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x} + 1\right) f_c(t,x) - f_c(t,x) = 0, \tag{12}$$

$$f_s(t,x) = \delta(t-x)\, e^{-t}.$$

In addition, $f_c(t,x)$ satisfies the conditions

$$\lim_{x\uparrow t} f_c(t,x) = \frac{1}{2}\left(e^{-t} + te^{-t}\right), \quad \lim_{x\downarrow -t} f_c(t,x) = \frac{1}{2}e^{-t}.$$

For all $t > 0$, we have $\int_{-t}^{t} f(t,x)dx = 1$.
For a small $\varepsilon > 0$, consider the probability $P\{0 < t - x(t) < \varepsilon\}$.
Let us verify that $\lim_{x\uparrow t} f_c(t,x) = \lim_{\varepsilon\downarrow 0} \frac{P\{0<t-x(t)<\varepsilon\}}{\varepsilon}$, i.e.,

$$\lim_{\varepsilon\downarrow 0} \frac{P\{0 < t - x(t) < \varepsilon\}}{\varepsilon} = \frac{1}{2}\left(e^{-t} + te^{-t}\right).$$

Indeed, it is easily seen that

$$P\{0 < t - x(t) \le \varepsilon\} = P\left\{t - \frac{\varepsilon}{2} \le \theta_1 < t\right\} + \int_0^t P\left\{\theta_3 \ge t-u, \theta_2 \le \frac{\varepsilon}{2}, \theta_1 \in du\right\} + o(\varepsilon),$$

where θ_i, $i = 1,2,3$ are independent exponentially distributed random variables.
The random variable θ_1 is the time of the first velocity alternation, θ_2 is the time interval between the first and second velocity alternations, and θ_3 is the time interval between the second and third velocity alternations.
We have $P\left\{t - \frac{\varepsilon}{2} \le \theta_1 < t\right\} = e^{-t+\frac{\varepsilon}{2}} - e^{-t}$. Moreover, it is easy to calculate

$$\begin{aligned} \int_0^t P\left\{\theta_3 \ge t-u,\ \theta_2 \le \frac{\varepsilon}{2}, \theta_1 \in du\right\} &= \left(1 - e^{-\frac{\varepsilon}{2}}\right)\int_0^t e^{-t+u}e^{-u}du \\ &= \left(1 - e^{-\frac{\varepsilon}{2}}\right) te^{-t}. \end{aligned}$$

Whence, it is easy to verify that $\lim_{\varepsilon\downarrow 0} \frac{P\{0<t-x(t)<\varepsilon\}}{\varepsilon} = \frac{1}{2}\left(e^{-t} + te^{-t}\right)$.
Similarly, $P\{t + x(t) \le \varepsilon\} = P\left\{\theta_2 \ge t - \frac{\varepsilon}{2}, \theta_1 \le \frac{\varepsilon}{2}\right\} + o(\varepsilon)$. This implies that

$$\lim_{\varepsilon\downarrow 0} \frac{P\{t + x(t) < \varepsilon\}}{\varepsilon} = \frac{1}{2}e^{-t} = \lim_{x\downarrow -t} f_c(t,x).$$

Therefore, $f_c(t,x)$ is a solution of the Goursat problem for the linear second-order hyperbolic equation that ensures the uniqueness of the solution of Eq. (12) with conditions (4). This means that $f(t,x)$ is the pdf of the particle's position for $m=1$.
It is worth to remark that the function $f(t,x)$ coincides with the result obtained in [5].

We now turn to the case $m=2$ and continue to calculate the integrals of u_k^l.
It follows from $u_4^0 = 2\frac{\partial u_2^1}{\partial t} - 2u_0^0$ that

$$\begin{aligned}\int_{-t}^{t} u_4^0 dy =& 2\left(\frac{\partial}{\partial t}\int_{-t}^{t} u_2^1 dy - u_2^1(t,t) - u_2^1(t,-t)\right) - 2\sinh t - 2\sin t\\ =& 4(\sinh t + \sin t - t) - 2\sinh t - 2\sin t = 2\sinh t + 2\sin t - 4t.\end{aligned}$$

Next, it follows from $u_4^2 = 2\frac{\partial u_2^3}{\partial t} - 2u_0^2$ that

$$\begin{aligned}\int_{-t}^{t} u_4^2 dy =& 2\left(\frac{\partial}{\partial t}\int_{-t}^{t} u_2^3 dy - u_2^3(t,t) - u_2^3(t,-t)\right) - 2\sinh t + 2\sin t\\ =& 4\sinh t - 4\sin t - 2\sinh t + 2\sin t = 2\sinh t - 2\sin t.\end{aligned}$$

For $t \le |y|$, we introduce the function $g(t,y) = g_c(t,y) + g_s(t,y)$, where

$$g_c(t,y) = \frac{1}{2}u_0^2(t,y) + \frac{1}{4}\left(u_1^1(t,y) + u_1^3(t,y) + u_2^1(t,y) + u_2^3(t,y) + u_4^0(t,y)\right),$$

$$g_s(t,y) = \delta(t-y) + t\delta(t-y). \tag{13}$$

By construction, the function $g_c(t,y)$ is a solution of the equation

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial y^2}\right)^2 g(t,y) - g(t,y) = 0. \tag{14}$$

Therefore, the function $f_c(t,x) = e^{-t}g_c(t,x)$ is a solution of Eq. (1) for $m=2$ $(\lambda = v = 1)$.

We put $f(t,x) = f_c(t,x) + e^{-t}g_s(t,x)$. Taking into account the values of the integrals of functions, which are involved in the expression for $g_c(t,y)$, we have $\int_{-t}^{t} f(t,x)dx = 1$ for all $t \ge 0$.

Let us prove that $\lim_{x\uparrow t} f_c(t,x) = \lim_{\varepsilon\downarrow 0} \frac{P\{0<t-x(t)<\varepsilon\}}{\varepsilon}$ and $\lim_{x\downarrow -t} f_c(t,x) = \lim_{\varepsilon\downarrow 0} \frac{P\{t+x(t)<\varepsilon\}}{\varepsilon}$.
It follows from Lemma 2 that, for $m=2$,

$$\lim_{\varepsilon\downarrow 0} \frac{P\{0<t-x(t)<\varepsilon\}}{\varepsilon} = \frac{1}{2}te^{-t}$$

and

$$\lim_{\varepsilon\downarrow 0} \frac{P\{t+x(t)<\varepsilon\}}{\varepsilon} = 0.$$

It is easy to verify that $\lim_{y\uparrow t} u_4^0(t,y) = 0$, $\lim_{y\uparrow t} u_0^2(t,y) = 0$, and, consequently,

$$\begin{aligned}\lim_{y\uparrow t} g_c(t,y) &= \lim_{y\uparrow t} \frac{t+y}{2\sqrt{t^2-y^2}} I_1\left(\sqrt{t^2-y^2}\right) = \frac{t}{2},\\ \lim_{y\downarrow -t} g_c(t,y) &= \lim_{y\downarrow -t} \frac{t+y}{2\sqrt{t^2-y^2}} I_1\left(\sqrt{t^2-y^2}\right) = 0.\end{aligned} \tag{15}$$

Thus,

$$\begin{aligned}\lim_{x\uparrow t} f_c(t,x) &= \frac{1}{2}te^{-t} = \lim_{\varepsilon\downarrow 0} \frac{P\{t-x(t)<\varepsilon\}}{\varepsilon},\\ \lim_{x\downarrow -t} f_c(t,x) &= 0 = \lim_{\varepsilon\downarrow 0} \frac{P\{t+x(t)<\varepsilon\}}{\varepsilon}.\end{aligned} \tag{16}$$

Let us show that conditions (15) with the condition $\int_{-t}^{t} g\,(t,y)\,e^{-t}dx = 1$ insure the uniqueness of the solution $g_c\,(t,y)$ for Eq. (14) and consequently, the uniqueness of the solution $f_c\,(t,x)$ of Eq. (12).
It is easily seen that each solution of Eq. (11) is a solution of Eq. (14). By changing the variables $s = t + y$, $p = t - y$, we reduce Eq. (14) to

$$\frac{\partial^4}{\partial s^2 \partial p^2} G\,(s,p) - G\,(s,p) = 0. \tag{17}$$

Passing to the Fourier transform $\hat{G}\,(s,\alpha) = \int_0^\infty G\,(s,p)\,e^{i\alpha p}dp$ in Eq. (17), we get the ordinary differential equation of order 4. Taking into account that $\lim_{y\downarrow -t} g_c\,(t,y) = 0$, we have

$$\hat{G}\,(0,\alpha) = 0. \tag{18}$$

Hence, at most four independent solutions of the ordinary differential equation satisfy the initial condition (18) for each α. Passing to the inverse Fourier transform, we have four independent solutions of Eq. (14) under the condition $\lim_{x\downarrow -t} g_c\,(t,x)\ = 0$, and just two of them satisfy Eq. (14) but not Eq. (11). By construction, one of these solutions, $g_c\,(t,y)$, is given by Eq. (13). As another solution, we can take

$$g_2(t,y) = \ u_0^2\,(t,y) + \ u_4^0\,(t,y)\,.$$

It is easy to verify that no linear combination $c(t,y)$ of the functions $g_c\,(t,y)$ and $g_2(t,x)$ satisfies conditions (16) and $\int_{-t}^{t} (c\,(t,x) + g_s\,(t,y))\,e^{-t}dx = 1$ for all $t > 0$, but solution $g_c\,(t,y)$.
Therefore, the function $f\,(t,x)$ is the pdf of the particle position at time t for $m = 2$, $v = \lambda = 1$, and has the form

$$\begin{aligned}
f\,(t,x) = &-\frac{J_0\left(\sqrt{t^2-x^2}\right)}{2}e^{-t} + \frac{(t+x)\,e^{-t}}{2\sqrt{t^2-x^2}} I_1\left(\sqrt{t^2-x^2}\right)\\
&-\frac{x^2e^{-t}}{2\sqrt{(t^2-x^2)^3}}\left(I_1\left(\sqrt{t^2-x^2}\right) + J_1\left(\sqrt{t^2-x^2}\right)\right)\\
&+\frac{t^2e^{-t}}{4\,(t^2-x^2)}\left(I_0\left(\sqrt{t^2-x^2}\right) + I_2\left(\sqrt{t^2-x^2}\right) + J_0\left(\sqrt{t^2-x^2}\right) - J_2\left(\sqrt{t^2-x^2}\right)\right)\\
&+\delta\,(t-x)\,e^{-t} + t\delta\,(t-x)\,e^{-t}.
\end{aligned}$$

In the same way as the pdf $f\,(t,x)$ of the particle position for $m = 2$ was obtained, we can also get solutions of Eq. (1) with conditions (2) and (4) for each $m > 2$.

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