Theory of Stochastic Processes
Vol. 17 (33), no. 1, 2011, pp. 5–11

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ON ASYMPTOTIC BEHAVIOR OF CONDITIONAL PROBABILITY OF CROSSING THE NONLINEAR BOUNDARY BY A PERTURBED RANDOM WALK

We prove a theorem on the limit behavior of the conditional probability of crossing the nonlinear boundary by a perturbed random walk with a distribution which belongs to the domain of attraction of the stable law with index $\alpha \in (1, 2]$.

1. Introduction.

Let a sequence ξ_n $n \geq 1$, of independent identically distributed random variables with $E|\xi_1| < \infty$ be given on the probability space $(\Omega, \mathcal{F}, P)$, and let the distribution F of the random variable ξ_1 have an interval-support $X \subseteq R = (-\infty, \infty)$, for which $F(X) = 1$ and $\nu = E\xi_1 \in X$.

Assume that the function $\Delta(x), x \in X$, is determined on X and is continuous. Moreover, $\mu = \Delta(\nu) > 0$. We set

$$S_n = \sum_{k=1}^{n} \xi_k, \overline{S}_n = \frac{S_n}{n} \text{ and } T_n = n\Delta\left(\overline{S}_n\right) \ n \geq 1.$$

Consider the first passage time

$$\tau_a = \inf\{n \geq 1 : T_n > f_a(n)\}, \tag{1}$$

where $f_a(t)$, $t > 0$, $a > 0$, is some family of nonlinear boundaries, and we set $\inf\{\oslash\} = \infty$.

Many important stopping times, arising in nonlinear renewal theory and in sequential analysis are of the form (1). In this case,, T_n is the statistics of likelihood ratio test, and τ_a is the number of necessary observations ([7], [8], [9]).

Asymptotic properties and limit theorems for τ_a were studied in papers [1]-[4] (see also monographs [5], [7], [8]).

In the present paper for a sufficiently wide class of functions $\Delta(x)$ and boundaries $f_a(t)$, we will study the limit behavior of the conditional probability $P\left(\tau_a \geq n | \overline{S}_n = x\right)$ of crossing the nonlinear boundary by a perturbed random walk T_n, when $n = n(a) \to \infty$ and $x = x(a) \to \nu$ as $a \to \infty$. This problem was studied in the case of a finite variance $D\xi_1 < \infty$ for a linear boundary $f_a(t) = a$ in [8] and for a nonlinear boundary $f_a(t) \neq a$ in [2].

For $\Delta(x) = x$, the limit behavior of the indicated conditional probability of crossing a nonlinear boundary was studied in paper [1], where it was supposed that the distribution of the step of a random walk belongs to the domain of attraction of a stable distribution with a parameter $\alpha \in (1, 2]$.

Notice that the conditional probabilities of crossing the boundary arise in the problems on the asymptotic behavior of local probabilities of crossing the boundary by a random walk ([3]).

2000 *Mathematics Subject Classification.* Primary 60G50.

Key words and phrases. Perturbed random walk, first passage time, conditional probability of crossing the boundary, overshoot of a random walk.

2. Conditions and formulation of the main result.

We assume that the function $\Delta(x)$ is continuously differentiable in a neighborhood of the point $x=\nu$ with $\Delta(\nu)>0$ and $\Delta'(\nu)\neq 0$.

For the boundary $f_a(t)$, we assume that it satisfies the following regularity conditions:

1) For each a, the function $f_a(t)$ increases monotonically, is continuously differentiable for $t>0$, and $f_a(1)\uparrow\infty$ as $a\to\infty$;

2) For any function $n=n(a)\to\infty$ satisfying the condition $\frac{1}{n}f_a(n)\to\mu=\Delta(\nu)>0$ as $a\to\infty$, the relation $f_a'(n)\to\theta\in[0,\mu)$ holds as $a\to\infty$;

3) For each a, the function $f_a'(t)$ weakly oscillates at infinity, i.e. $\frac{f_a'(m)}{f_a'(n)}\to 1$ as $\frac{n}{m}\to 1$, $n\to\infty$.

We note that the family of functions of the form $f_a(t)=at^{\beta}$, $0\le\beta<1$, satisfies conditions 1)-3). It is easy to show that condition 2) is valid for this family with $\theta=\beta\mu$. Other examples of such functions are given in papers [3], [4].

We assume that the distribution F of a random variable ξ_1 belongs to the domain of attraction of a stable law $G_\alpha(x)$ with characteristic index $\alpha\in(1,2]$, i.e.

$$P\left(\frac{S_n-n\nu}{A(n)}\le x\right)\to G_a(x),\ \text{ as } n\to\infty, \tag{2}$$

where $x\in R$, $A(t)=t^{1/\alpha}L(t)$, and $L(t)$, $t>0$, is a slowly varying function at infinity [6].

The assumptions on the function $\Delta(x)$ yield

$$T_n=Z_n+\varepsilon_n, \tag{3}$$

where

$$Z_n=\sum_{k=1}^{n}X_k,\ \ X_k=\Delta(\nu)+\Delta'(\nu)(\xi_k-\nu)$$

and

$$\varepsilon_n=n\left[\Delta\left(\overline{S}_n\right)-\Delta(\nu)-\Delta'(\nu)\left(\overline{S}_n-\nu\right).\right]$$

From the strong law of large numbers, it follows that

$$\frac{\varepsilon_n}{n}\overset{a.s.}{\to}0\ \text{ and }\ \frac{T_n}{n}\overset{a.s.}{\to}\Delta(\nu)=EX_1>0\quad\text{as } n\to\infty.$$

Representation (3) shows that the sequence $T_n, n\ge 1$, is a perturbed random walk, i.e. it is the sum of an ordinary random walk (Z_n) and a random perturbation (ε_n).

Introduce the following notation:

$$J=\inf_{n\ge 1}(Z_n-n\theta),$$

$$\Psi(r)=P(J\ge r), r\in R;$$

$$\varphi(t)=Me^{it\xi_1};$$

$$\delta_a(n,x)=n\Delta(x)-f_a(n);$$

$$l_a(n,x)=P\left(\tau_a\ge n|\,\overline{S}_n=x\right)$$

and

$$L(n,x,r)=P\left(J_n>r|\,\overline{S}_n=x\right),\quad r\in R,$$

where

$$J_n=\min_{1\le i<n}(T_n-T_{n-i}-i\theta).$$

We note that, for each $x\in(-\infty,\infty)$, the function $L(n,x,r)$ doesn't increase and is continuous from the left at each point $r\in(-\infty,\infty)$.

The following proposition holds.

Theorem. *Assume that the conditions enumerated above are satisfied and, for some integer* $m \geq 1$,

$$\int_{-\infty}^{\infty} |\varphi(t)|^m \, dt < \infty. \tag{4}$$

Let $x = x(a) \to \nu$ *and* $n = n(a) \to \infty$ *as* $a \to \infty$ *so that* $x - \nu = O(A(n)/n)$ *and* $\delta_a(n, x) = O(1)$.

Then

$$L(n, x, r) \to \Psi(r) \ \text{ as } a \to \infty$$

for all $r \geq 0$.

Corollary. Let the conditions of the theorem be fulfilled and $\delta_a(n, x) \to r \geq 0$. Then

$$l_a(n, x) \to \Psi(r) \ \text{ as } a \to \infty.$$

It follows from condition (3) that the sum S_n has a bounded continuous density $P_n(x)$ for all $n \geq m$.

We also note that relation (4) implies that the function $\Psi(r)$ is continuous at each point $r \geq 0$, and Theorem 2.7 in [8] yields

$$\Psi(r) = (\mu - \theta) h(r),$$

where

$$h(r) = \frac{P(Z_\tau - \tau\theta > r)}{E(Z_\tau - \tau\theta)}, r \geq 0$$

and

$$\tau = \inf\{n \geq 1 : Z_n - \theta n > 0\}.$$

The function $h(r), r \geq 0$, is the limit distribution density for the overshoot of a random walk $Z_n - n\theta$, $n \geq 1$ for the level [8].

3. Auxiliary facts.

To prove the theorem, we need the following facts formulated in the form of lemmas.

For $1 \leq k \leq n-1$ and $n \geq m$, we set

$$Q_{nk} = Q_{nk}(B|x) = \int_B q_{nk}(x_1, \ldots, x_k|x) \, F(dx_1) \ldots F(dx_k),$$

where

$$q_{nk}(x_1, \ldots, x_k/x) = \begin{cases} \frac{P_{n-k}\left(nx - \sum\limits_{k=1}^{n} x_k\right)}{P_n(nx)}, & \text{if } P_n(nx) > 0 \\ 1 & \text{, if } P_n(nx) = 0 \end{cases},$$

$B \in \beta(R^k)$ is the σ-algebra of Borel sets in R^k and $F(x) = P(\xi_1 \leq x)$.

We note that Q_{nk} is the conditional probability distribution of a random vector $(\xi_1, \ldots, \xi_k)$ under condition that $\overline{S}_n = x$.

Lemma 1. *Let conditions (2) and (4) be satisfied. Then*

1) For each k, *the conditional distribution* Q_{nk} *weakly converges as* $n \to \infty$ *to an unconditional distribution of a random vector* $(\xi_1, \ldots, \xi_k)$, *and the convergence is uniform in* $x : x - \nu = O(A(n)/n)$;

2) For any $\delta \in (0, 1)$, *there exists a constant* $M = M(\delta)$ *such that*

$$q_{nk}(x_1, \ldots, x_k|x) \leq M$$

for all $x_1, \ldots, x_n$, $k \leq (1-\delta)n$, $n \geq m$ *and* $x : x - \nu = O(A(n)/n)$.

The statement of this lemma is proved in paper [1] (see also [8]).

Lemma 2. *Let conditions (2), (4) be satisfied. Let* $x = x(a) \to \nu$ *and* $n = n(a) \to \infty$ *as* $a \to \infty$ *so that* $x - \nu = O(A(n)/n)$. *Then the joint conditional distribution of random variables*

$$J_{nk} = T_n - T_{n-i}, \quad i = \overline{1, k},$$

under condition that $\overline{S}_n = x$ weakly converges to an unconditional joint distribution of random variables $Z_1, \ldots, Z_k$.

Proof. Assume

$$\eta_{ni} = \xi_i - \overline{S}_n, \quad 1 \le i \le n$$

and

$$\Gamma_{nk} = \sum_{i=1}^{k} \eta_{ni}, \ 1 \le k \le n.$$

It follows from the first part of Lemma 1 that, for each fixed k, the conditional distribution $(\eta_{n1}, \ldots, \eta_{nk})$ weakly converges to an unconditional distribution $(\xi_1 - \nu, \ldots, \xi_k - \nu)$.

It is clear that, for $\overline{S}_n = x$ and $1 \le k \le n$,

$$J_{nk} = (n-k)\left(\Delta\left(\overline{S}_n\right) - \Delta\left(\overline{S}_{n-k}\right)\right) + k\Delta(x). \tag{5}$$

It is easy to see that

$$(n-k)\left(\overline{S}_n - \overline{S}_{n-k}\right) = \sum_{i=n-k+1}^{n} \eta_{ni} \stackrel{d}{=} \Gamma_{nk}, \tag{6}$$

where the symbol $\xi \stackrel{d}{=} \eta$ means the equality in distribution.

It follows from (5) and (6) that the joint conditional distribution of random variables $J_{nk}, 1 \le k \le n-1$ under condition that $\overline{S}_n = x$ coincides with the joint conditional distribution of random variables

$$W_{nk} = (n-k)\left[\Delta(x) - \Delta\left(x - \frac{1}{n-k}\Gamma_{nk}\right)\right] + k\Delta(x), \ 1 \le k = n-1.$$

Assume

$$U_{nk}(t) = (n-k)\left[\Delta(x) - \Delta\left(x - \frac{1}{n-k}t\right)\right] + k\Delta(x).$$

Taking into account that $x = x(a) \to \nu$ as $a \to \infty$, the mean-value theorem for each fixed k yields

$$U_{nk}(t) \to \Delta'(\nu)\, t + k\Delta(\nu) \ \text{ as } a \to \infty \tag{7}$$

uniformly with respect to t from the bounded set in $(-\infty, \infty)$.

Then it follows from (7) that, for each k, the conditional distribution of the vector $(W_{n1}, \ldots, W_{nk})$ under condition that $\overline{S}_n = x$ weakly converges to an unconditional distribution $(Z_1, \ldots, Z_k)$, where $Z_k = \Delta'(\nu)(S_k - k\nu) + k\Delta(\nu)$, since the conditional distribution Γ_{nk} under condition that $\overline{S}_n = x$ weakly converges to an unconditional distribution $S_k - k\nu$ for each k.

Lemma 3. *Let $x = x(a) \to \nu$ and $n = n(a) \to \infty$ as $a \to \infty$ so that $x - \nu = O(A(n)/n)$. Then, for $\theta \in [0, \Delta(\nu))$,*

1) $\varepsilon_1 = \varepsilon_1(a, \delta, y) = P\left(J_{ni} - i\theta < y, \exists i \in (n\delta, n-1] \mid \overline{S}_n = x\right) \to 0$ as $a \to \infty$ uniformly in y from a bounded set of R and $x : x - \nu = O(A(n)/n)$, f

2) $\varepsilon_2 = \varepsilon_2(a, k, \delta, y) = P\left(J_{ni} - i\theta < y, \exists i \in (k, \ n\delta] \mid \overline{S}_n = x\right) \to 0$ as $k \to \infty$ uniformly in y from a bounded set of R and $x : x - \nu = O(A(n)/n)$ for sufficiently large a.

Proof. Assuming $T'_n = T_n - n\theta$ and $b = n(\Delta(x) - \theta) - y$, we have

$$\begin{aligned} \varepsilon_1 &= P\left(T'_{n-i} > b, \exists i \in (\ n\delta, \ n-1] \mid \overline{S}_n = x\right) = \\ &= P\left(T'_j > b, \exists j \in [1, n(1-\delta)) \mid \overline{S}_n = x\right). \end{aligned} \tag{8}$$

By the second part of Lemma 1, relation (8) yields

$$\begin{aligned} \varepsilon_1 &\le MP\left(T'_j > b, \exists i \in [1, n(1-\delta))\right) = \\ &= MP\left(t_b < n(1-\delta)\right), \end{aligned} \tag{9}$$

where

$$t_b = \inf \{n \geq 1 : T'_n > b\}$$

is the first passage time of a random walk for the level b.

By (3), it follows from Lemma 2.4 in [8] that

$$\frac{t_b}{b} \overset{a.n}{\to} \frac{1}{\mu - \theta} \text{ as } a \to \infty. \tag{10}$$

Taking into account that $b \sim n (\Delta (\nu) - \theta)$ as $a \to \infty$, it follows from (10) that

$$\frac{t_b}{n} \overset{a.n}{\to} 1 \text{ as } a \to \infty.$$

Hence, we obtain easily that, for any $\delta \in (0, 1)$,

$$P (t_b \leq n (1 - \delta)) \to 0 \text{ as } a \to \infty.$$

Statement 1) of the proved lemma follows from (9).

We now prove statement 2). It suffices to show that

$$\varepsilon_2 = P \left(W_{ni} - i\theta < y, \exists i \in (k, n\delta] \mid \overline{S}_n = x\right) \to 0, k \to \infty.$$

From the differentiability of the function $\Delta (x)$ in a neighborhood of the point $x = \nu$, it follows that there exist an integer N and a positive number $\gamma > 0$ such that, for $i \leq n\delta$ and $n \geq N$ on the set $\left\{\omega : \frac{1}{n-i} |\Gamma_{ni}| \leq \gamma\right\}$,

$$\left|(n - i) \left[\Delta (x) - \Delta \left(x - \frac{1}{n - i}\Gamma_{ni}\right)\right]\right| \leq 2 |\Delta' (x)| |\Gamma_{ni}|$$

or

$$|W_{ni} - i (\mu - \theta)| \leq 2 |\Delta' (\nu)| |\Gamma_{ni}| . \tag{11}$$

It follows from inequality (11) that the event $C = \{\omega : W_{ni} < y\}$ implies the event $A = \{\omega : |\Gamma_{ni}| > \gamma (1 - \delta) n\}$ or the event

$$B = \left\{\omega : |\Gamma_{ni}| > \frac{i (\mu - \theta) - y}{2 |\Delta' (\nu)|}\right\} \quad (C \subseteq A \cup B) .$$

It is easy to understand that if $\delta > 0$ is a sufficiently small number, then, for each $i \leq n\delta$, the event A implies the event $B : A \subseteq B$.

Further, the equality

$$\Gamma_{ni} = i \left(\overline{S}_i - \overline{S}_n\right)$$

implies that, on the set B,

$$\left|\overline{S}_i - \overline{S}_n\right| > \frac{i (\mu - \theta) - y}{2 |\Delta' (\nu)| i}.$$

Hence, we find

$$\left|\overline{S}_i - \nu\right| > \frac{i (\mu - \theta) - y}{2 |\Delta' (\nu)| i} - \left|\overline{S}_n - \nu\right| . \tag{12}$$

It follows from the convergence $x = x (a) \to \nu$ as $a \to \infty$ that there exist the numbers a_0, k_0, and γ_0 such that, for all $i > k_0$ and $a > a_0$,

$$\frac{i (\mu - \theta) - y}{2 |\Delta' (\nu)| i} - |x - \nu| > \gamma_0. \tag{13}$$

Then it follows from (12) and (13) that, for $i > k_0$ and $a > a_0$, the event B implies the event $D = \left\{\omega : \left|\overline{S}_i - \nu\right| > \gamma_0\right\} : \ B \subseteq D$.

Thus, it follows from the above arguments that, for sufficiently large a and k and small $\delta > 0$, we have

$$\varepsilon_2 = P \left(C, \exists i \in (k,\ n\delta] \mid \overline{S}_n = x\right) \leq$$
$$\leq P \left(B, \exists i \in (k,\ n\delta] \mid \overline{S}_n = x\right) \leq$$

$$\leq P\left(D, \exists i \in (\ k, n\delta] \ \mid S_n = x\right). \tag{14}$$

From the second part of Lemma 1, we obtain

$$P\left(D, \exists i \in (k,\ n\delta] \ \mid \overline{S}_n = x\right) \leq$$
$$\leq P\left(D, \exists i \in (k,\ n\delta]\right) \leq$$
$$\leq MP\left(\left|\overline{S}_i - \nu\right| > \gamma_0, \exists i > k\right). \tag{15}$$

It follows from the strong law of large numbers that

$$P\left(\left|\overline{S}_i - \nu\right|\right) > \gamma_0,\ \exists i > k\right) \to 0 \text{ as } k \to \infty. \tag{16}$$

From (14), (15), and (16), we get statement 2) of Lemma 3.

4. Proof of the theorem.

Assume

$$L_k\left(n, x, r\right) = P\left(J_{ni} - i\theta \geq r, 1 \leq i \leq k \mid \overline{S}_n = x\right),\ J_{ni} = T_n - T_{n-i},$$
$$J_k = \min_{1 \leq i \leq k}\left(Z_i - i\theta\right)$$

and

$$\Psi_k\left(r\right) = P\left(J_k \geq r\right) = P\left(Z_i - i\theta \geq r,\ 1 \leq i \leq k\right).$$

It follows from Lemma 2 that, for each k and $r \geq 0$,

$$L_k\left(n, x, r\right) \to \Psi_k\left(r\right) \text{ as } a \to \infty. \tag{17}$$

Since $\Psi_k\left(r\right) \to \Psi\left(r\right)$ as $k \to \infty$, it remains to show that, for sufficiently large k,

$$\varepsilon_3 = \varepsilon_3\left(n, x, r\right) = L_k\left(n, x, r\right) - L\left(n, x, r\right) \to 0 \text{ as } a \to \infty. \tag{18}$$

For any $\delta \in (0, 1)$, we have

$$0 \leq \varepsilon_3 \leq P\left(J_{ni} - i\theta < r,\ \exists i \in (k,\ n-1] \ \mid \overline{S}_n = x\right) \leq$$
$$\leq P\left(J_{ni} - i\theta < r, \exists i \in (k,\ n\delta] \ \mid \overline{S}_n = x\right) +$$
$$+P\left(J_{ni} - i\theta < r, \exists i \in (n\delta, n-1] \ \mid \overline{S}_n = x\right) = \varepsilon_2 + \varepsilon_1,$$

where ε_1 and ε_2 are from Lemma 3.

Therefore, Lemma 3 yields (18).

The statement of the theorem follows from (17) and (18).

Proof of the Corollary. Following [1], we have

$$l_a\left(n, x\right) = P\left(T_k \leq f_a\left(k\right),\ 1 \leq k \leq n-1 \mid \overline{S}_n = x\right) =$$
$$= P\left(T_n - T_{n-k} \geq T_n - f_a\left(n-k\right),\ 1 \leq k \leq n-1 \mid \overline{S}_n = x\right) =$$
$$= P\left(J_{nk} \geq n\Delta\left(x\right) - f_a n + \left(f_a\left(n\right) - f_a\left(n-k\right)\right), 1 \leq k \leq n-1 \mid \overline{S}_n = x\right).$$

Hence, recalling the notation $\delta_a\left(n, x\right) = n\Delta\left(x\right) - f_a\left(n\right)$ and taking into account that, for some intermediate point $m = m\left(n, k\right)$ from the segment $[n-k, n]$,

$$f_a\left(n\right) - f_a\left(n-k\right) = kf'_a\left(m\right),$$

we get

$$l_a\left(n, x\right) = P\left(J_{nk} \geq \delta_a\left(n, x\right) + kf'_a\left(m\right),\ 1 \leq k \leq n-1 \mid \overline{S}_n = x\right).$$

Denote

$$J'_n = \min_{1 \leq k \leq n-1}\left(J_{nk} - kf'_a\left(m\right)\right)$$

and

$$L'_a\left(n, x, r\right) = P\left(J'_n > r \mid \overline{S}_n = x\right).$$

It is clear that

$$l_a\left(n, x\right) = L'\left(n, x, \delta_a\left(n, x\right)\right).$$

By the scheme of the proof of relation (18), it is easy to show that, for each fixed $k \geq 1$,

$$L'_a(n, x, r) - L(n, x, r) \to 0 \text{ as } a \to \infty.$$

The statement of the corollary follows from the theorem.

Remark. The theorem and the corollary were established for the case $\Delta(x) = x$ in [1] and for the case of $f_a(t) = a$ and $D\xi_1 < \infty$ in [8].

The authors thank the reviewer for useful remarks.

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