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INTERSECTION LOCAL TIMES IN L_2 FOR MARKOV PROCESSES

We provide sufficient conditions for the existence of intersection and self-intersection local times with additional weight in the space of square integrable random variables for Markov processes under specific local upper bounds for their transition density. We determine when this condition is satisfied for standard Brownian motion, symmetric stable processes and Brownian motions on Carnot group.

1. Introduction

The goal of this paper is to study the existence of functionals similar to intersection and self-intersection local times as elements of L_2, the space of square integrable random variables, for a special class of Markov processes. The functionals that we want to study are meant to describe intersections and self-intersections of a process. It is well-known that for 2-dimensional Brownian motion the existence of self-intersection local times as limits in L_2 is only possible after renormalization [5, 11]. The renormalization makes it possible to associate a functional with self-intersections, and therefore to have a random variable that describes the self-intersection properties of the trajectory of the process (see also [9] for an application of renormalized self-intersection local times). In this paper we propose a different approach for defining such functionals, by introducing an additional "weight", a function that assigns a value for each selection of points on the trajectory (see (1) for exact definition). For example in the case of self-intersections we can avoid blow-up in the limit by assigning sufficiently small weight to two points on trajectory, which are close to each other on time scale (see Remark 7 for more detailed explanation).

However our main motivation for this investigation lies in the ability to generalize our results to a class of processes, which we call Brownian motions on Carnot group. The name is an analogy of Brownian motions on Lie group, introduced by Ito [7], since Carnot group is a stratified Lie group, with $\mathbb{R}^d$ as a base space and a specific choice of coordinate system (see [2]). Since $\mathbb{R}^d$ with usual addition is also a Carnot group the case of standard Brownian motion is included. There are many properties of such processes, that distinguish them from standard Brownian motion, but for our context (studying the second moment of its functionals) it is important that its transition density behaviour can be described by a natural distance on Carnot group (see [2] or [13] for details). The local behaviour of such distance differs from the behaviour of Euclidean distance (in any Carnot group with non-trivial addition they are not locally equivalent), and it is also different in different points. We choose our assumptions to describe the local behavior of transition density, using specific upper bounds of the transition density inside a compact, such that in the case of Brownian motion on Carnot group they follow from the well-known upper bounds with the corresponding distance. Under such assumptions we study the existence of self-intersection and intersection local times with weight. Since our assumptions involve only transition density it is irrelevant that the process behind is a Brownian motion on Carnot group, and we may consider a class of Markov processes.

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To our knowledge there are no known results about intersection or self-intersection local times for Brownian motion on Carnot group (but as it was already mentioned the partial case of standard Brownian motion is well-studied), except [12]. However in two papers by M. Chaleyat-Maurel and J.-F. Le Gall [4, 3] the existence of intersections and self-intersections (among other things) were studied for a class of processes generalizing Brownian motion on Heisenberg group. The conditions used in [4] are very similar to the assumptions of Theorem 13, and the conclusion is also similar, leading to the existence of intersections. In our case the existence of the intersections or self-intersections follows from the existence of non-zero intersection or self-intersection local time, as it happens in our Theorems 3 and 12, since without intersections our approximations (1) always become zero for small ε. It is worth emphasizing that the application of our results to the case of Brownian motion on Carnot group becomes possible only because of the estimates for transition densities proven in [13].

The paper is structured as follows: the main part is divided into two sections, first devoted for self-intersection local times and second to intersection local times of independent processes. Even though the general definitions of the first of these section formally include the objects considered in the second, the approach we use for the case of independent processes is different, and so this case demands separate treatment. To simplify the exposition we reuse some of the notation from one section to denote the different objects in the other, hopefully to the benefit of the reader. The structure of both sections is similar: we present a general result about local time existence in terms of the finiteness of some integral with transition density, then key technical result, which allows us to use geometrical arguments to study finiteness of the integrals with transition density, then prove the existence results for self-intersection local time (in one section) and for intersection local time of independent processes (in other section) based on the geometrical properties of the process, and finally provide applications for particular well-known classes of processes, such as elliptic diffusions, Brownian motions on Carnot group and symmetric stable processes.

2. Self-intersection local times

2.1. n-fold local time on the surface. Suppose that $X(t)$ is a Markov process on $\mathbb{R}^d$ and there is a non-negative function $p(t,x,y)$ continuous for $t>0$, which is a density of the distribution of $X(t)$ w.r.t. y, given that $X(0)=x$. Let m and n be a positive integers, $F:\mathbb{R}^{nd}\to\mathbb{R}^m$ be an infinitely differentiable function, $\psi:\mathbb{R}^{nd}\to\mathbb{R}$ be a non-negative bounded continuous function. We define approximations of local time as follows

$$\gamma_\varepsilon(\psi,F,A)=\int_A \psi(X(t_1),\ldots,X(t_n))f_{m,\varepsilon}(F(X(t_1),\ldots,X(t_n)))dt_1\ldots dt_n \tag{1}$$

where $A\subset[0,1]^n$ is a Borel set and $f_{m,\varepsilon}$ is an approximation of δ-measure at $0\in\mathbb{R}^m$ in a sense of weak convergence of measures, which we define as follows:

$$f_{m,\varepsilon}(u)=\varepsilon^{-d}\phi_m(\frac{u}{\varepsilon})$$

where ϕ_m is any non-negative function on $\mathbb{R}^m$, which has compact support and satisfies $\int_{\mathbb{R}^m}\phi_m(u)du=1$.

We denote $H=\{z=(z_1,\ldots,z_n)\in\mathbb{R}^{nd}|F(z_1,\ldots,z_n)=0\}$ and assume that the matrix of derivatives F' of F at z has maximal rank for all $z\in H$. Here and below we use the derivatives sign to denote matrices of derivatives of the functions between subsets of Euclidean spaces with the additional convention that a column index is the index of direction of derivation, i.e. it indicates the coordinate in the original set (in particular F' is $m\times nd$ matrix). In the following we will use a fixed coordinate system in $\mathbb{R}^d$ where it is needed, in particular to define matrices of derivatives. The choice of such system does not matter, since both statements and proofs of our results do not depend on such choice. We also fix an open bounded set $M \subset \mathbb{R}^{nd}$ and always assume that $supp\, \psi \subset M$.

Later we will consider self-intersections by taking

$$F(x_1, \ldots, x_n) = (f(x_1) - f(x_2), \ldots, f(x_{n-1}) - f(x_n)),$$

where $n \geqslant 2$, $f : \mathbb{R}^d \to \mathbb{R}^k$ for some fixed $k \leqslant d$, but Theorem 1 below does not use this assumption.

The following construction is standard (its different variants can be found in literature), but the idea behind it is an important part of our approach to existence of local times, so we describe it in details.

Proposition 1. *There exists a unique family of finite measures $\nu(u, dy)$ on Borel sets of $\mathbb{R}^{nd}$, such that for all non-negative bounded continuous functions f on $\mathbb{R}^m$ with support in some neighbourhood N of zero and g on $\mathbb{R}^{nd}$ with support inside M:*

$$\int_{\mathbb{R}^{nd}} g(y) f(F(y)) dy = \int_{\mathbb{R}^m} f(u) \int_{\mathbb{R}^{nd}} g(y) \nu(u, dy) du$$

and additionally $\nu(u, \cdot) = 0$ for $u \notin N$ and the support $\nu(u, \cdot)$ is in M for all $u \in N$.

Proof. For any $z \in H$ there is an open neighbourhood $U_z \subset M$ of z in $\mathbb{R}^{nd}$, and an open neighbourhood V_z of 0 in $\mathbb{R}^{nd}$, such that there is an infinitely differentiable function $\theta_z : V_z \to U_z$, which is an isomorphism between V_z and U_z satisfying $\theta_z(0) = z$ and $\theta_z(y) \in H$ if and only if $y_1 = y_2 = \ldots = y_m = 0$. Indeed if we define θ_z^{-1} as F for first m coordinates and as a linear function for the rest, such that $(\theta_z^{-1})'$ is non-degenerate at z (it is always possible since F' has maximal rank), then θ_z also exist in some neighbourhood of zero by the theorem of inverse function, and satisfies the conditions.

Using θ_z as a change of variables we obtain for all non-negative bounded continuous functions f on $\mathbb{R}^m$ with support in some neighbourhood N of zero (N is chosen such that $F(y) \in N$ if $y \in U_z$) and g on $\mathbb{R}^{nd}$ with support inside U_z:

$$\int_{\mathbb{R}^{nd}} g(y) f(F(y)) dy = \int_{\mathbb{R}^{nd}} g(\theta_z(a)) f(a_1, \ldots, a_m) |\det \theta_z'(a)| da$$

Denote $\nu_z(u, da) = |\det \theta_z'(a)| \delta_{u_1}(da_1) \ldots \delta_{u_m}(da_m) da_{m+1} \ldots da_{nd}$ then we have

$$\int_{\mathbb{R}^{nd}} g(y) f(F(y)) dy = \int_{\mathbb{R}^m} f(u) \int_{\mathbb{R}^{nd}} g(\theta_z(a)) \nu_z(u, da) du$$

For each $u \in N$ there exists a unique measure $\nu(u, \cdot)$ which restriction to U_z coincides with $\theta_z^{-1} \circ \nu_z(u, \cdot)$ ($\nu(u, A) = \nu_z(u, \theta_z^{-1}(A))$ for all measurable $A \subset U_z$). The existence follows since for each g with support in M we can write $\int_{\mathbb{R}^{nd}} g(y) \nu(u, dy)$ as sum of the integrals over finite number of U_z by representing g as sum of g_z with supports in U, and it is easy to check that the result does not depend on the particular choice of such decomposition. On the other hand the restriction of such measure on M is determined uniquely, so ν is also unique. But such ν also satisfies the integral condition, by construction (it follows from the definition of ν and ν_z), so the Proposition is proved. □

Some of the additional statements that was shown in the proof will be useful later so we gather them as a separate result.

Corollary 1. *For any $z \in H \cap M$ there is an open neighbourhood $U_z \subset M$ of z in $\mathbb{R}^{nd}$, and an open neighbourhood V_z of 0 in $\mathbb{R}^{nd}$, such that there is an infinitely differentiable function $\theta_z : V_z \to U_z$, which is an isomorphism between V_z and U_z satisfying $\theta_z(0) = z$*

and $\theta_z(y) \in H$ if and only if $y_1 = y_2 = \ldots = y_m = 0$. For all g on $\mathbb{R}^{nd}$ with support inside U_z and u in some neighbourhood of zero in $\mathbb{R}^m$ we have:

$$\int_{\mathbb{R}^{nd}} g(y)\nu(u, dy) = \int_{\mathbb{R}^{nd}} (g(\theta_z(a))|\det \theta_z'(a)|)|_{a_1=u_1,\ldots,a_m=u_m} da_{m+1} \ldots da_{nd}$$

Define

$$q_l(x, z_1, \ldots, z_l, B) = \int_B 1_{t_1<\ldots<t_l} p(t_1, x, z_1)p(t_2-t_1, z_1, z_2) \ldots p(t_l-t_{l-1}, z_{l-1}, z_l)dt_1 \ldots dt_l$$

for any Borel set $B \subset [0,1]^l$. This integral can be infinite, and in this case we write $q_l(x, z_1, \ldots, z_l, B) = +\infty$.

Theorem 1. *Fix $X(0) = x \in \mathbb{R}^d$ and non-negative bounded continuous function ψ on $\mathbb{R}^{nd}$ with $\operatorname{supp} \psi \subset M$. If for all $\sigma \in S_{2n}$ we have*

$$q_{2n}(x, y_{\sigma(1)}, \ldots, y_{\sigma(2n)}, A_\sigma)) < +\infty$$

for almost all $y \in M^2$ w.r.t. Lebesgue measure, where

$$A_\sigma = \{t : (t_{\sigma(1)}, \ldots, t_{\sigma(n)}) \in A; (t_{\sigma(n+1)}, \ldots, t_{\sigma(2n)}) \in A\},$$

and there is a function $h_\sigma(x, y)$, which satisfies the equality

$$h_\sigma(x, y) = \psi(y_1, \ldots, y_n)\psi(y_{n+1}, \ldots, y_{2n})q_{2n}(x, y_{\sigma(1)}, \ldots, y_{\sigma(2n)}, A_\sigma)$$

for almost all $y \in M^2$ w.r.t. Lebesgue measure such that $h_\sigma(x, y)$ is continuous for almost all $y \in M^2$ w.r.t. $\nu(0, dy_1 \ldots dy_n)\nu(0, dy_{n+1} \ldots dy_{2n})$ and there are such positive numbers δ and β that

$$\sup_{|u|<\delta,|v|<\delta} \sum_{\sigma\in S_{2n}} \int_{M^2} (h_\sigma(x, y))^{1+\beta}\nu(u, dy_1 \ldots dy_n)\nu(v, dy_{n+1} \ldots dy_{2n}) < +\infty \tag{2}$$

then there is a limit of $\gamma_\varepsilon(\psi, F, A)$ in L_2. Additionally the L_2 norm of the limit is equal to

$$\sum_{\sigma\in S_{2n}} \int_{M^2} h_\sigma(x, y)\nu(0, dy_1 \ldots dy_n)\nu(0, dy_{n+1} \ldots dy_{2n})$$

and if $h_\sigma(x, y) > 0$ for all y in a neighbourhood of some $z \in (H \cap M)^2$, then the L_2 norm of the limit is not zero.

Proof. First we find (the expectation is conditional on $X(0) = x$)

$$E\gamma_{\varepsilon_1}(\psi, F, A)\gamma_{\varepsilon_2}(\psi, F, A) =$$
$$= E\int_A \psi(X(t_1), \ldots, X(t_n))f_{m,\varepsilon_1}(F(X(t_1), \ldots, X(t_n)))dt_1 \ldots dt_n$$
$$\int_A \psi(X(t_{n+1}), \ldots, X(t_{2n}))f_{m,\varepsilon_1}(F(X(t_{n+1}), \ldots, X(t_{2n})))dt_{n+1} \ldots dt_{2n} =$$
$$\sum_{\sigma\in S_{2n}} \int_{A\times A} 1_{t_{\sigma(1)}<\ldots<t_{\sigma(2n)}} E\psi(X(t_1), \ldots, X(t_n))f_{m,\varepsilon_1}(F(X(t_1), \ldots, X(t_n)))$$
$$\psi(X(t_{n+1}), \ldots, X(t_{2n}))f_{m,\varepsilon_1}(F(X(t_{n+1}), \ldots, X(t_{2n})))dt =$$
$$= \sum_{\sigma\in S_{2n}} \int_{A_\sigma} 1_{t_1<\ldots<t_{2n}} \int_{\mathbb{R}^{2nd}} p(t_1, x, y_{\sigma(1)})p(t_2-t_1, y_{\sigma(1)}, y_{\sigma(2)}) \ldots p(t_{2n}-t_{2n-1}, y_{\sigma(2n-1)}, y_{\sigma(2n)})$$
$$\psi(y_1, \ldots, y_n)\psi(y_{n+1}, \ldots, y_{2n})$$

$$f_{m,\varepsilon_1}(F(y_1,\dots,y_n))f_{m,\varepsilon_2}(F(y_{n+1},\dots,y_{2n}))dydt =$$
$$= \sum_{\sigma\in S_{2n}}\int_{\mathbb{R}^m}\int_{\mathbb{R}^m}\int_{\mathbb{R}^{2nd}} h_\sigma(x,y)f_{m,\varepsilon_1}(u)f_{m,\varepsilon_2}(v)\nu(u,dy_1\dots dy_n)\nu(v,dy_{n+1}\dots dy_{2n})dudv$$

Here changing of the order of integration is always valid since we are integrating non-negative functions and taking ψ outside of the integral w.r.t t gives us exactly h_σ. In order to check the convergence of this expression as $(\varepsilon_1,\varepsilon_2)\to 0+$ it is enough to show that all functions

$$\int_{\mathbb{R}^{2nd}} h_\sigma(x,y)\nu(u,dy_1\dots dy_n)\nu(v,dy_{n+1}\dots dy_{2n})$$

are continuous by (u,v) in some neighbourhood of zero.

We can represent the integral as the sum of finite number of integrals over $K_{z_i}\times K_{z_j}$, where $K_{z_i}\subset U_{z_i}$ and U_z are open neighbourhoods of $z\in H$, such that on each of such sets we can define differentiable one-to-one maps $\theta_z : V_z\to U_z$, and θ_z^{-1} transforming $\nu(u,\cdot)$ on U_z into the measure (see Proposition 1)

$$\nu_z(u,da) = |\det\theta_z'(a)|\delta_{u_1}(da_1)\dots\delta_{u_m}(da_m)da_{m+1}\dots da_{nd}.$$

Then each such integral has form

$$\int_{K_{z_i}\times K_{z_j}} h_\sigma(x,\theta_{z_i}(a),\theta_{z_j}(b))|\det\theta_{z_i}'(a)||\det\theta_{z_j}'(b)|$$
$$\delta_{u_1}(da_1)\dots\delta_{u_m}(da_m)da_{m+1}\dots da_{nd}\delta_{v_1}(db_1)\dots\delta_{v_m}(db_m)db_{m+1}\dots db_{nd}$$

To show the convergence of such integral it is enough to verify that the function under integral is continuous w.r.t. u, v a.s., which immediately follows from continuity assumption on h_σ, and to check the following condition (then the convergence follows by uniform integrability) :

$$\sup_{|u|<\delta,|v|<\delta}\int_{K_{z_i}\times K_{z_j}} (h_\sigma(x,\theta_{z_i}(a),\theta_{z_j}(b))|\det\theta_{z_i}'(a)||\det\theta_{z_j}'(b)|)^{1+\beta}$$
$$\delta_{u_1}(da_1)\dots\delta_{u_m}(da_m)da_{m+1}\dots da_{nd}\delta_{v_1}(db_1)\dots\delta_{v_m}(db_m)db_{m+1}\dots db_{nd} < +\infty$$

But if we rewrite this again as an integral with ν we obtain that it holds under our integrability condition (since all θ_z' are bounded).

We note that we also proved that the L_2-norm of the limit is

$$\sum_{\sigma\in S_{2n}}\int_{\mathbb{R}^{2nd}} h_\sigma(x,y)\nu(0,dy_1\dots dy_n)\nu(0,dy_{n+1}\dots dy_{2n}).$$

Using again that in some neighbourhood of such z each ν is a Lebesgue measure on a linear manifold after a change of variable, we obtain that this integral is not zero, if $h_\sigma(x,y)>0$ in a neighbourhood of $z\in(H\cap M)^2$, and so the Theorem is proved. □

2.2. **Density estimates and finiteness of integrals.** We make some additional assumptions on p, with the aim to provide a way of checking the conditions of the Theorem 1. Let M_1 be a bounded open set such that $M\subset M_1^n$.

(1) Suppose that $p(t,x,y)$ is continuous at $(0,x,y)$ for all $x\neq y$.
(2) Let a differentiable function $S:\mathbb{R}^{2d}\to\mathbb{R}^d$ be such that $S(x,x)=0$ for all x, $S(x,y)\neq 0$ for $x\neq y$ and the derivatives of $S(x,y)$ w.r.t. x and w.r.t. y are non-degenerate (as two separate $d\times d$ matrices) for all x, y. We suppose that

there are positive numbers $p_i > 0$, $i = 1, \dots, d$, $Q > 2$ and $C_1 > 0$, such that for all $x \in M_1$, $y \in M_1$ with $x \neq y$:

$$\int_0^1 p(t, x, y)dt \leqslant C_1(\rho(x, y))^{2-Q}. \tag{3}$$

where $\rho(x, y) = \max_{i=1,\dots,d} |S_i(x, y)|^{1/p_i}$ also satisfies pseudo-triangle inequality: there is $C > 0$ such that for all $x \in M_1$, $y \in M_1$:

$$\rho(x, y) \leqslant C(\rho(x, a) + \rho(a, y))$$

Remark 1. Note that we may have the same estimate in the second condition with different Q and p_i, more specifically the condition stays the same with $2+c(Q-2)$ instead of Q, and cp_i instead of p_i (pseudo-triangle inequality is also true for any positive power of ρ). Our setup comes from Carnot groups where Q is the homogeneous dimension of the group and p_i is a homogeneous dimension of coordinate i, and so we have $Q = \sum_{i=1}^{d} p_i$. There is no need to make that additional assumption for the general case.

The continuity assumption is motivated by the following proposition.

Proposition 2. *If $p(t, x, y)$ is continuous at $(0, x, y)$ for all $x \neq y$ and also at (t, x, y) for all $t > 0$, x, y, then for all positive integers l, $x_0 \in \mathbb{R}^n$ and Borel sets $B \subset [0, 1]^l$ the function $q_l(x_0, y_1, \dots, y_l, B)$ is continuous at any $y = (y_1, \dots, y_l) \in \mathbb{R}^{ld}$, satisfying $y_{i+1} \neq y_i$ for $i = 1, \dots, l-1$ and $y_1 \neq x_0$.*

Proof. The statement follows from Lebesgue bounded convergence theorem and the estimate

$$\sup_{|x-x_0|<\delta, |z_i-y_i|<\delta} 1_{t_1<\dots<t_l} p(t_1, x_0, z_1)p(t_2 - t_1, z_1, z_2)\dots$$
$$\dots p(t_l - t_{l-1}, z_{l-1}, z_l) \leqslant \sup_{t\in(0,1), |x-x_0|<\delta, |y_1-z_1|<\delta} p(t, x, z_1)$$
$$\sup_{t\in(0,1), |y_1-z_1|<\delta, |y_2-z_2|<\delta} p(t, z_1, z_2)\dots \sup_{t\in(0,1), |y_{l-1}-z_{l-1}|<\delta, |y_l-z_l|<\delta} p(t, z_{l-1}, z_l)$$

□

The following Theorem is the main technical result, which shows the integrability of powers of ρ w.r.t. ν under some geometrical condition on the structure of ρ and ν (allowing us to check the condition (2) under our assumptions). Denote

$$R_{\sigma,x}(y_1, \dots, y_{2n}) = (S(x, y_{\sigma(1)}), S(y_{\sigma(1)}, y_{\sigma(2)}), \dots, S(y_{\sigma(2n-1)}, y_{\sigma(2n)})).$$

and let $T(z)$ be any $2nd \times (2nd - 2m)$ matrix composed of the vectors, forming a basis of the tangent space at z of $H \times H$, written in columns (the nature of the dependence on z is irrelevant as it is only used for a fixed z). We also denote as $N(D)$ the set of all such multiindices $I = (i_1, \dots, i_q)$, $i_1 < \dots < i_q$ that the rows $i_1, \dots, i_q$ of matrix D are linearly independent.

Theorem 2. *Fix $\sigma \in S_{2n}$, $z = (z_1, z_2, \dots, z_{2n}) \in (H \cap M)^2$ ($z_i \in \mathbb{R}^d$) and real numbers $k_1, \dots, k_{2n}$. If for all $\lambda_1 \geqslant 0, \dots, \lambda_{2n} \geqslant 0$ not all zero, but with $\lambda_j = 0$ if $z_{\sigma(j-1)} \neq z_{\sigma(j)}$ for $j = 2, \dots, 2n$, $\lambda_1 = 0$ if $x \neq z_{\sigma(1)}$ and $\lambda_j = 0$ if $k_j \geqslant 0$ for $j = 1, \dots, 2n$ (if all λ are forced to be zero the condition is trivially fulfilled), there exists $I = (i_1, \dots, i_{2(nd-m)}) \in N(R'_{\sigma,x}(z)T(z))$, such that*

$$\sum_{j=1}^{2n} \lambda_j \Big(\sum_{s:[i_s/d]=j} (p_{i_s \, mod \, d}) + k_j \Big) > 0 \tag{4}$$

then there is an open neighbourhood U_z of z, $\delta > 0$ and $\beta > 0$ such that

$$\sup_{|u|<\delta,|v|<\delta} \int_{U_z} (\rho(x, y_{\sigma(1)})^{k_1} \rho(y_{\sigma(1)}, y_{\sigma(2)})^{k_2} \dots \tag{5}$$
$$\dots \rho(y_{\sigma(2n-1)}, y_{\sigma(2n)})^{k_{2n}})^{1+\beta} \nu(u, dy_1 \dots dy_n) \nu(v, dy_{n+1} \dots dy_{2n}) < +\infty$$

Before we prove this theorem we need two lemmas. The first one describes the finitness of the integral from the minimum of the set of power functions in several variables in terms of values of the powers (it explains the origin of the form of the condition in the Theorem 2).

Lemma 1. *Let $b_{ij}, i = 1, \dots, p, j = 1, \dots, q$ be a set of real numbers. Then the integral*

$$\int_{[0,1]^p} \min_{j=1,\dots,q} (\prod_{i=1}^{p} u_i^{b_{ij}}) du$$

is finite if and only if for any non-negative real numbers $\lambda_1, \dots, \lambda_p$, not all zero, there exists $j \in \{1, \dots, q\}$, such that $\sum_{i=1}^{p} \lambda_i (b_{ij} + 1) > 0$.

Proof. After change of variables $v_i = -\ln u_i$ we obtain

$$\int_{[0,\infty)^p} \exp(-\max_{j=1,\dots,q} (\sum_{i=1}^{p} v_i (b_{ij} + 1))) dv$$

Let us prove the "if" part first. Notice that under the assumption we have

$$\max_{j=1,\dots,q} (\sum_{i=1}^{p} v_i (b_{ij} + 1)) > 0$$

for all $v \in [0, \infty)^p$ satisfying $|v| = 1$. But since the function on the left side is continuous on v we also have the same inequality with some $\delta > 0$ on the right side. It means that our integral can be bounded above with the integral

$$\int_{[0,\infty)^p} e^{-\delta|v|} dv$$

which is obviously finite.

To prove the "only if" part we assume that for some $\lambda_1, \dots, \lambda_p$ with $|\lambda| = \sqrt{\sum_{i=1}^{p} \lambda_i^2} = 1$ we have $\sum_{i=1}^{p} \lambda_i (b_{ij} + 1) \leqslant 0$ for all $j = 1, \dots, q$. Then, the following estimate, valid for all $v \in [0, \infty)^p$ and $r > 0$,

$$|\max_{j=1,\dots,q} (\sum_{i=1}^{p} v_i (b_{ij} + 1)) - \max_{j=1,\dots,q} (\sum_{i=1}^{p} \lambda_i r (b_{ij} + 1))| \leqslant$$
$$\leqslant \max_{j=1,\dots,q} \sum_{i=1}^{p} |v_i - r\lambda_i| |b_{ij} + 1| \leqslant |v - r\lambda| \max_{j=1,\dots,q} \sqrt{\sum_{i=1}^{p} (b_{ij} + 1)^2} = C|v - r\lambda|$$

allows us to find, that for any $v \in [0, \infty)^p$, satisfying for some $r > 0$, $\varepsilon > 0$ the inequality $|v - r\lambda| < \varepsilon$ we have

$$\max_{j=1,\dots,q} (\sum_{i=1}^{p} v_i (b_{ij} + 1)) \leqslant C\varepsilon$$

Consequently

$$\int\limits_{[0,\infty)^p} \exp(-\max_{j=1,\dots,q}(\sum_{i=1}^{p} v_i(b_{ij}+1)))dv \geqslant e^{-C\varepsilon} \int\limits_{[0,\infty)^p} 1_{|v-(v,\lambda)\lambda|<\varepsilon} dv = +\infty$$

as the last integral is clearly infinite since the domain of integration contains infinitely many disjoint balls from the family $\{v : |v - r\lambda| < \varepsilon\}$. □

The next lemma provides an upper bound that allows us to involve only the tangent space in the condition of Theorem 2.

Lemma 2. *Suppose that $f : \mathbb{R}^{d+k} \to \mathbb{R}^d$ has continuous derivative in some neighbourhood of 0 and $A = (\frac{\partial}{\partial x_j} f_i(0))_{i=1,\dots,d;j=1,\dots,d}$ is non-degenerate. There exist such constants $\delta > 0$ and $C > 0$, that for all $\varepsilon_i \in [0,\delta], i = 1,\dots,n$, we have*

$$\sup_{|u|<\delta} \lambda(\{x \in \mathbb{R}^d : |x| < \delta, |f_i(x,u)| < \varepsilon_i, i = 1,\dots,d\}) \leqslant C \prod_{i=1}^{d} \varepsilon_i$$

Proof. First of all by the inverse function theorem we can choose δ small enough, so that for each $|u| < \delta$ there is a differentiable function g_u, which is the inverse of $f(x,u)$ w.r.t. $x \in \mathbb{R}^d$ on $|x| < \delta$. In particular we have that $f(g_u(a),u) = a$ for all a with $|g_u(a)| < \delta$, that $A(x,u) = (\frac{\partial}{\partial x_j} f_i(x,u))_{i=1,\dots,d;j=1,\dots,d}$, $|x| < \delta$, $|u| < \delta$ is non-degenerate and $g_u'(f(x,u)) = A(x,u)^{-1}$ for $|x| < \delta$ and $|u| < \delta$. Then we get for all $|u| < \delta$:

$$\lambda(\{x \in \mathbb{R}^d : |x| < \delta, |f_i(x,u)| < \varepsilon_i, i = 1,\dots,d\}) = \int\limits_{|x|<\delta} 1_{|f_i(x,u)|<\varepsilon_i, i=1,\dots,d} dx =$$

$$= \int\limits_{|g_u(a)|<\delta} 1_{|a_i|<\varepsilon_i, i=1,\dots,d} |\det g_u'(a)| da \leqslant \sup_{|x|<\delta} |\det g_u'(f(x,u))| 2^d \prod_{i=1}^{d} \varepsilon_i =$$

$$= \sup_{|x|<\delta} |\det A(x,u)|^{-1} 2^d \prod_{i=1}^{d} \varepsilon_i$$

After taking supremum over $|u| < \delta$ Lemma is proved. □

Remark 2. This lemma provides the following interesting result (which we will not use in this paper, but it explains the idea behind the lemma).

Proposition 3. *Suppose that $d \leqslant n$, $f : \mathbb{R}^d \to \mathbb{R}^n$ has continuous derivative in some neighbourhood of 0, f' has maximal rank d at 0 and $f(0) = 0$. Denote*

$$g(x) = f'(0)x = (\sum_{j=1}^{d} \frac{\partial f_i}{\partial x_j}(0) x_j)_{i=1,\dots,n}$$

There exist such constants $\delta > 0$ and $C > 0$, that for all $\varepsilon_i \in [0,\delta], i = 1,\dots,n$, we have

$$\lambda(\{x : |x| < \delta, |f_i(x)| < \varepsilon_i, i = 1,\dots,n\}) \leqslant C\lambda(\{x : |g_i(x)| < \varepsilon_i, i = 1,\dots,n\})$$

Proof. By Lemma 2 for any $I = (i_1,\dots,i_d)$ with $i_1 < \dots < i_d$, such that $A_I(x) = (\sum_{j=1}^{d} \frac{\partial f_{i_k}}{\partial x_j}(x))_{k=1,\dots,d;j=1,\dots,d}$ is non-degenerate at $x = 0$, we have

$$\lambda(\{x : |x| < \delta, |f_{i_k}(x)| < \varepsilon_{i_k}, k = 1,\dots,d\}) \leqslant C \prod_{k=1}^{d} \varepsilon_{i_k} \leqslant$$

$$\leqslant C_1 \lambda(\{x : |g_{i_k}(x)| < \varepsilon_{i_k}, k = 1,\dots,d\})$$

It remains to show that

$$\min_{I:\det A_I(0)\neq 0} \lambda(\{x : |g_{i_k}(x)| < \varepsilon_{i_k}, k = 1, \dots, d\}) \leqslant C\lambda(\{x : |g_i(x)| < \varepsilon_i, i = 1, \dots, n\})$$

But this follows from the known expressions for the volume of such sets. In particular we have that the measure on the left hand side is equal to $2^d \varepsilon_{i_1} \dots \varepsilon_{i_d} |\det A_I(0)|^{-1}$, and the measure on the right hand side is bounded below with $C_1(\det B^T B)^{-1/2}$, where $B = diag(\varepsilon_i^{-1}; i = 1, \dots, n) f'(0)$. Moreover

$$\det B^T B = \sum_I \varepsilon_{i_1}^{-2} \dots \varepsilon_{i_d}^{-2} (\det A_I(0))^2$$

and we obtain the final inequality after bounding this sum with maximum of its members. □

What we obtained is something not entirely obvious: the intersection of a surface with arbitrary small boxes (or ellipsoids) with box center being at the surface, can not have significantly larger surface measure then their intersection with the tangent linear manifold at box center. It may seem that, since any C^1 surface is close to linear on small scale, this should follow immediately, but arbitrary box can have arbitrary ratios of its sizes along different coordinates, and therefore non-linearity can still be significant on small scale. Note that the same inequality with the opposite sign is not true, for example if we take $d = 1$, $n = 2$ and $f(x) = (x, x^2)$, then the additional bound from the second coordinate can make the intersection with boxes much smaller compared to $g(x) = (x, 0)$.

In the proof of Theorem 2 we will use the same idea of bounding the intersection of boxes with the surface in terms of tangent linear manifold (unfortunately we can not use the above result directly, since we need additional uniformity over some parameter and also the ability to move the centers of the boxes). Therefore there may be cases, where the sufficient condition in Theorem 2 is not necessary (it does not seem to be easy to provide an example, so we will not pursue this here).

Proof of Theorem 2. In the case $k_j < 0$ for all j we may use the formula

$$\delta^{k_j} = -k_j \int_0^{+\infty} 1_{\delta<\varepsilon} \varepsilon^{k_j - 1} d\varepsilon$$

with $\rho(y_{\sigma(j)}, y_{\sigma(j-1)})$ (or $\rho(y_{\sigma(1)}, x)$ for $j = 1$) in place of δ and rewrite the integral in (5) as follows:

$$\int_{[0,\infty)^{2n}} \varepsilon_1^{k_1(1+\beta)-1} \dots \varepsilon_{2n}^{k_{2n}(1+\beta)-1}$$
$$\nu(u, \cdot) \times \nu(v, \cdot)(U_z \cap \{|S_i(x, y_\sigma(1))| < \varepsilon_1^{p_i}, \dots$$
$$\dots, |S_i(y_{\sigma(2n-1)}, y_{\sigma(2n)})| < \varepsilon_{2n}^{p_i}, i = 1, \dots, d\}) d\varepsilon_1 \dots d\varepsilon_{2n}.$$

If any of k_j is non-negative we may simply bound the corresponding multiplier with a constant and obtain the same formula without any components related to ε_j for such j. For simplicity we assume in the following that $k_j < 0$ for all j, but the general case can be proved similarly.

Recall that in some neighbourhood of z the measure ν is the image under θ_z^{-1} of Lebesgue measure with bounded density on the linear manifold. Therefore we can choose U_z so that

$$\nu(u, \cdot) \times \nu(v, \cdot)(U_z \cap \{|S_i(x, y_\sigma(1))| < \varepsilon_1^{p_i}, \dots$$
$$\dots, |S_i(y_{\sigma(2n-1)}, y_{\sigma(2n)})| < \varepsilon_{2n}^{p_i}, i = 1, \dots, d\}) \leqslant$$
$$C\lambda(\{(a, b) : (u, a) = \theta_{(z_1,\dots,z_n)}(y_1, \dots, y_n), (v, b) = \theta_{(z_{n+1},\dots,z_{2n})}(y_{n+1}, \dots, y_{2n}),$$

$$y \in U_z, |R_{\sigma,x}(y)_{i+(j-1)d}| < \varepsilon_j^{p_i}, j = 1, \ldots, 2n, i = 1, \ldots, d\})$$

Note that if at least one of the coordinates $i+(j-1)d$,$i = 1, \ldots, d$ of $R_{\sigma,x}(z)$ is not zero (which is equivalent to $S(z_{\sigma(j-1)}, z_{\sigma(j)}) \neq 0$), then for small ε_j and small u,v the value of measure is zero. It means that the integral w.r.t ε_j always finite for small u, v as long as the rest of the integral is finite and therefore may be ignored, i.e. in that case we look for finiteness of the remaining integral for each fixed ε_j (it is enough to consider one value of ε_j since we have an increasing function of ε_j multiplied by power of ε_j under integral).

We may assume that θ_z and $T(z)$ satisfy:

$$T(z) = (\theta'_{(z_1,\ldots,z_n)}(\theta^{-1}_{(z_1,\ldots,z_n)}((z_1, \ldots, z_n)))^T_{ij}|_{i=1,\ldots,nd;j=m+1,\ldots,nd},$$
$$\theta'_{(z_{n+1},\ldots,z_{2n})}(\theta^{-1}_{(z_{n+1},\ldots,z_{2n})}((z_{n+1}, \ldots, z_{2n})))^T_{ij}|_{i=1,\ldots,nd;j=m+1,\ldots,nd})$$

i.e. if we select columns $m+1, \ldots, nd$ from both θ' and join them we obtain $T(z)$. Indeed such selection is also a basis of tangent space of $H \times H$ at z, and because the choice of $T(z)$ does not matter for the condition in the Theorem, we can always assume such identity. Then the derivative of $R_{\sigma,x}(\theta^{-1}_{(z_1,\ldots,z_n)}(u, a), \theta^{-1}_{(z_{n+1},\ldots,z_{2n})}(v, b))$ at $a = b = 0$, is equal to $R'_{\sigma,x}(z)T(z)$ if $u = 0$, $v = 0$ according to our definitions.

Consider all possible ways to choose linearly independent rows $I = (i_1, \ldots, i_{2(nd-m)})$ in $R'_{\sigma,x}(z)T(z)$. For each such choice we apply Lemma 2 to the selection according to I of the coordinates of the function $R_{\sigma,x}(\theta^{-1}_{(z_1,\ldots,z_n)}(u, a), \theta^{-1}_{(z_{n+1},\ldots,z_{2n})}(v, b))$ (with a, b as the main variables, denoted as x in Lemma 2, and u, v as parameters). Therefore for all small u,v and fixed I we obtain the bound (after shrinking U_z if necessary):

$$\lambda(\{(a, b) : (u, a) = \theta_{(z_1,\ldots,z_n)}(y_1, \ldots, y_n), (v, b) = \theta_{(z_{n+1},\ldots,z_{2n})}(y_{n+1}, \ldots, y_{2n}),$$
$$y \in U_z, |R_{\sigma,x}(y)_{i+(j-1)d}| < \varepsilon_j^{p_i}, j = 1, \ldots, 2n, i = 1, \ldots, d\}) \leqslant$$
$$\leqslant C \prod_{j=1}^{2n} \prod_{s:[i_s/d]=j} \varepsilon_j^{p_{i_s \, mod \, d}}$$

After taking minimum in the right hand side over all possible choices of I we can apply Lemma 1 and find the condition for the finiteness of our integral: for all $\lambda_j \geqslant 0$, not all zero, there exists multiindex I, which corresponds to a set of linearly independent rows $I = (i_1, \ldots, i_{2(nd-m)})$ in $R'_{\sigma,x}(z)T(z)$, such that

$$\sum_{j=1}^{2n} \lambda_j (\sum_{s:[i_s/d]=j} (p_{i_s \, mod \, d}) + k_j(1 + \beta)) > 0$$

For $\beta = 0$ this condition coincides with the condition (4), after taking into account that some ε_j are fixed or absent entirely as noted above, and we may set $\lambda_j = 0$ for such j. Note that if the condition above is fullfilled with $\beta = 0$ then it is also fullfilled with some $\beta > 0$. Additionally, since our bounds was uniform over u, v in some neighbourhood of 0, we also have the finiteness of the supremum of the same integral over small u, v for some $\beta > 0$, and Theorem is proved. □

2.3. **Self-intersection local times.** For the following theorem we take, assuming $n \geqslant 2$,

$$F(x_1, \ldots, x_n) = (f(x_1) - f(x_2), \ldots, f(x_{n-1}) - f(x_n))$$

where $f : \mathbb{R}^d \to \mathbb{R}^k$, for some fixed $k \leqslant d$, is a continuously differentiable function with derivative of maximal rank for all x. This corresponds to self-intersections, since the set of zeros of F is exactly $\{f(x_1) = f(x_2) = \ldots = f(x_n)\}$ and therefore $\gamma_\varepsilon(1, F, [0, 1]^n)$ gives approximations of well-known n-fold self-intersection local time of $f(X(t))$ (for planar Brownian motion and $\psi = 1$ it converges in L_2 only after certain renormalization, see [11]). Let $L(x)$ be any $d \times (d - k)$ matrix consisting from vectors, giving basis of tangent space of $\{u \in \mathbb{R}^d : f(u) = f(x)\}$ at x, written in columns (such basis can be constructed for example as a basis in the orthogonal complement of the linear space of gradients of $f(x)$ at x). We write a derivative of S as $S'(x,y) = (S'_1(x,y), S'_2(x,y))$, where S'_1 is the matrix of derivatives of $S(x,y)$ w.r.t. $x \in \mathbb{R}^d$ and S'_2 is the matrix of derivatives of $S(x,y)$ w.r.t. $y \in \mathbb{R}^d$. Define

$$m_f(x) = \min\{\sum_{j \notin I} p_j | I = (i_1, \ldots, i_{d-k}) \in N(S'_2(x,x)L(x))\}$$

Theorem 3. *Let*

$$\psi(y_1, \ldots, y_n) = \prod_{i \neq j} \rho(y_i, y_j)^{Q-2} \phi_M(y)$$

where ϕ_M is any continious non-negative bounded function with support inside M. Suppose that $k < d$. If

$$m_f(y_i) < \frac{2n}{n-1} + \frac{n}{n-1}(\sum_{l=1}^{d} p_l - Q), \tag{6}$$

for all y_i, $i = 1, \ldots, n$ such that $(y_1, \ldots, y_n) \in H \cap M$, then there is a limit of $\gamma_\varepsilon(\psi, F, A)$ in L_2 for any Borel $A \subset [0,1]^n$. If additionally $\phi_M(w) > 0$ at some $w \in H \cap M$ and for some $\sigma \in S_{2n}$ we can find $z \in (H \cap M)^2$ in any neighbourhood of (w,w), such that $q_{2n}(x, z_{\sigma(1)}, \ldots, z_{\sigma(2n)}, A_\sigma)) > 0$, and $x \neq z_{\sigma(1)}$, $z_{\sigma(i)} \neq z_{\sigma(i-1)}, i = 2, \ldots, 2n$, then the limit is not zero ($x = X(0)$). If $\psi(y_1, \ldots, y_n) = \phi_M(y)$, then the same is true, if we replace the inequality (6) *with*

$$m_f(y_i) < 2 + \sum_{l=1}^{d} p_l - Q. \tag{7}$$

Proof. We can define $h_\sigma(x,y)$ from Theorem 1 exactly by the equality in Theorem 1, if

$$q_{2n}(x, y_{\sigma(1)}, \ldots, y_{\sigma(2n)}, A_\sigma)) < +\infty$$

and zero otherwise. We will show that

$$\int_{M^2} (1_{x = y_{\sigma(1)}} + 1_{\{y | \exists i,j \in \{1,\ldots,2n\}, i \neq j : y_i = y_j\}}) \nu(0, dy_1 \ldots dy_n) \nu(0, dy_{n+1} \ldots dy_{2n}) = 0.$$

Then such h_σ satisfies the continuity condition of Theorem 1, since by Proposition 2 the function $h_\sigma(x, \cdot)$ is continuous outside the set $\{y | \exists i,j \in \{1,\ldots,2n\}, i \neq j : y_i = y_j\} \cup \{x = y_{\sigma(1)}\}$.

The measure $\nu(0, dy_1 \ldots dy_n)$ is zero on any set $\{y | y_j = a\}$, since it is easy to see, that even $\{y | f(y_j) = a\}$ has zero measure. It remains to show that for $k < d$ the set

$$\{y | \exists i,j \in \{1,\ldots,n\}, i \neq j : y_i = y_j\}$$

also has zero measure w.r.t $\nu(0, dy_1 \ldots dy_n)$. We fix $z = (z_1, \ldots, z_n) \in H \cap M$, $i \neq j$, take θ_z constructed in the proof of Proposition 1 and find $a_z \in \mathbb{R}^d$ such that a_z is not zero and orthogonal to all rows of $f'(z_i)$. Then it is clear that vector $h = (h_1, \ldots, h_n) \in \mathbb{R}^{nd}$ with $h_q = 0$, $q \neq i$ and $h_i = a_z$ is a tangent vector for H at z. On the other hand it is not tangent for $\{y = (y_1, \ldots, y_n) : y_i = y_j\}$ at z. Because of this, we can see that the image of $\{y = (y_1, \ldots, y_n) : y_i = y_j\}$ w.r.t θ_z in a linear subspace $\theta_z(H)$ of $\mathbb{R}^{nd}$ has zero Lebesgue measure inside $\theta_z(H)$. But the image of $\nu(0, dy)$ is absolutely continuous w.r.t. Lebesgue measure on $\theta_z(H)$ by definition, which proves our assertion.

Such definition of h_σ works for both choices of ψ. To prove the integrability condition (2) of Theorem 1 it is enough to prove the condition (5) using Theorem 2 for a fixed $z \in (H \cap M)^2$ and some specific choice of k_i, provided by an estimate of ψ. Note that if the limit exists it is non-zero, since by our construction there is a point $z \in (H \cap M)^2$

(in a neigbourhood of (w, w)), such that $h_\sigma(x, z) > 0$ and $h_\sigma(x, \cdot)$ is continuous in a neighbourhood of z. Therefore in the following we can fix z and $\sigma \in S_{2n}$ and focus on proving (5) using Theorem 2. We assume that the starting point of the process $X(0) = x$ is also fixed (the bound we are about to derive does not depend on x).

Let us describe the structure of $R'_{\sigma,x}(z)T(z)$ in our special case. The matrix $R'_{\sigma,x}(z)$ can be seen to have the following structure in the column basis where $y_1, \dots, y_{2n}$ has coordinates $y_{\sigma(1)}, y_{\sigma(2)}, \dots, y_{\sigma(2n)}$ (i.e. $2n$ sets of d columns are permuted according to σ) in terms of $d \times d$ blocks:

$$R'_{\sigma,x}(z) = \begin{pmatrix} S'_2(x, y_{\sigma(1)}) & 0 & \dots & 0 & 0 \\ S'_1(y_{\sigma(1)}, y_{\sigma(2)}) & S'_2(y_{\sigma(1)}, y_{\sigma(2)}) & \dots & 0 & 0 \\ \dots & & & & \\ 0 & 0 & \dots & S'_2(y_{\sigma(2n-2)}, y_{\sigma(2n-1)}) & 0 \\ 0 & 0 & \dots & S'_1(y_{\sigma(2n-1)}, y_{\sigma(2n)}) & S'_2(y_{\sigma(2n-1)}, y_{\sigma(2n)}) \end{pmatrix}$$

To describe $T(z)$ we split rows in $2n$ blocks of size d, and we split columns into two blocks of size k and $2n$ blocks of size $d-k$. Denote $G(z) = f'(z)^T(f'(z)f'(z)^T)^{-1}$. In the first block of columns row blocks $1, 2 \dots, n$ are equal to $G(z_1), \dots, G(z_n)$ correspondingly. In the second block of columns row blocks $n+1, n+2, \dots, 2n$ are equal to $G(z_{n+1}), \dots, G(z_{2n})$ correspondingly. In the column block $i+2$, $i = 1, \dots, 2n$ the row block i is equal to $L(z_i)$. The rest of the blocks contain only zeros. It is easy to see that all column vectors are such that directional derivatives of $(F(z_1, \dots, z_n), F(z_{n+1}, \dots, z_{2n}))$ along them are zero and they are linearly independent, so we have a basis in the tangent space of $H \times H$ at z. For example the directional derivatives of $f(z_1) - f(z_2)$ along the columns in the first column block can be calculated as follows:

$$\begin{aligned} &(f(z_1) - f(z_2))'_{z_1} G(z_1) + (f(z_1) - f(z_2))'_{z_2} G(z_2) = \\ &\qquad = f'(z_1)f'(z_1)^T(f'(z_1)f'(z_1)^T)^{-1} - f'(z_2)f'(z_2)^T(f'(z_2)f'(z_2)^T)^{-1} = 0 \end{aligned}$$

and the linear independence of all columns follows from the linear independence of joined columns of $f'(z_i)^T$ and $L(z_i)$ (they are linearly independent since gradients are orthogonal to tangent vectors).

We split the set of indices $\{1, \dots, 2n\}$ into five disjoint sets $N_1, \dots, N_5$ according to σ: $N_1 = \{1\}$,

$$N_2 = \{i = 2, \dots, 2n : \sigma(i) \in \{1, 2, \dots, n\}, \sigma(i-1) \in \{n+1, n+2, \dots, 2n\}\},$$

$$N_3 = \{i = 2, \dots, 2n : \sigma(i) \in \{1, 2, \dots, n\}, \sigma(i-1) \in \{1, 2, \dots, n\}\},$$

$$N_4 = \{i = 2, \dots, 2n : \sigma(i) \in \{n+1, n+2, \dots, 2n\}, \sigma(i-1) \in \{1, 2, \dots, n\}\},$$

$$N_5 = \{i = 2, \dots, 2n : \sigma(i) \in \{n+1, n+2, \dots, 2n\}, \sigma(i-1) \in \{n+1, n+2, \dots, 2n\}\}.$$

Denote for $i = 1, \dots, 2n$ and all $a \in \mathbb{R}^d$, $b \in \mathbb{R}^d$

$$A_i(a, b) = \begin{cases} 1_{\sigma(1) \in \{1,2,\dots,n\}} S'_2(a, b)G(b), i \in N_1; \\ S'_2(a, b)G(b), i \in N_2; \\ S'_1(a, b)G(a) + S'_2(a, b)G(b), i \in N_3; \\ S'_1(a, b)G(a), i \in N_4; \\ 0, i \in N_5; \end{cases}$$

$$B_i(a, b) = \begin{cases} 1_{\sigma(1) \in \{n+1,n+2,\dots,2n\}} S'_2(a, b)G(b), i \in N_1; \\ S'_1(a, b)G(a), i \in N_2; \\ 0, i \in N_3; \\ S'_2(a, b)G(b), i \in N_4; \\ S'_1(a, b)G(a) + S'_2(a, b)G(b), i \in N_5; \end{cases}$$

$$C(a, b) = S'_2(a, b)L(b)$$

$$D(a,b) = S_1'(a,b)L(a)$$

Then it easy to see that $R'_{\sigma,x}(z)T(z)$ is equal to (we changed the order of column blocks starting from the third according to σ for convenience)

$$\begin{pmatrix} A_1(x,z_{\sigma(1)}) & B_1(x,z_{\sigma(1)}) & C(x,z_{\sigma(1)}) & 0 & \dots & 0 \\ A_2(z_{\sigma(1)},z_{\sigma(2)}) & B_2(z_{\sigma(1)},z_{\sigma(2)}) & D(z_{\sigma(1)},z_{\sigma(2)}) & C(z_{\sigma(1)},z_{\sigma(2)}) & \dots & 0 \\ \dots \\ A_{2n}(z_{\sigma(2n-1)},z_{\sigma(2n)}) & B_{2n}(z_{\sigma(2n-1)},z_{\sigma(2n)}) & 0 & 0 & \dots & C(z_{\sigma(2n-1)},z_{\sigma(2n)}) \end{pmatrix}$$

We choose $2k+2n(d-k)$ rows of $R'_{\sigma,x}(z)T(z)$ as follows: take all rows at block 1, all rows corresponding to all coordinates except some fixed set $I_j = (i_1^j, \dots, i_k^j)$ in every block $j > 1$, and finally rows corresponding to coordinates $i_1^s, \dots, i_k^s$ at some block s, where $s \in N_2 \cup N_4$ and $z_{\sigma(s)} = z_{\sigma(s-1)}$ (so that all rows are taken in such block). If such s does not exist, then all λ_j for $j > 2$ are zero, except for $j \in N_3 \cup N_5$ with $z_{\sigma(j)} = z_{\sigma(j-1)}$. In this case we can make the following choice: for $j \in N_3 \cup N_5$ with $z_{\sigma(j)} = z_{\sigma(j-1)}$ we select all rows corresponding to all coordinates except some fixed set $I_j = (i_1^j, \dots, i_k^j)$. We also select all rows at block 1 and the rest of selection is from row blocks $j > 1$ with $\lambda_j = 0$ and can be chosen arbitrary. Note that if $z_{\sigma(j)} \neq z_{\sigma(j-1)}$ for all j then the desired finitness of the integral in (5) is clearly fullfiled. The precise choice of I_j, s and the last unspecified part is discussed below.

Let us find the condition that such choice produce linearly independent rows. Assume that $\sigma(1) \in \{1, 2, \dots, n\}$ (the other case $\sigma(1) \in \{n+1, n+2, \dots, 2n\}$ can be considered similarly), then first d rows are linearly independent since they contain non-degenerate $d\times d$ matrix $S_2'(x, z_{\sigma(1)})(G(z_{\sigma(1)}), L(z_{\sigma(1)}))$, in column blocks 1 and 3 combined. Moreover the linear independence of the rest of rows can be considered separately after removing column blocks 1 and 3 from the matrix, since the rest of the column blocks in the first row block are zero. We can say that we now select from

$$\begin{pmatrix} B_2(z_{\sigma(1)},z_{\sigma(2)}) & C(z_{\sigma(1)},z_{\sigma(2)}) & 0 & \dots & 0 \\ B_3(z_{\sigma(2)},z_{\sigma(3)}) & D(z_{\sigma(2)},z_{\sigma(3)}) & C(z_{\sigma(2)},z_{\sigma(3)}) & \dots & 0 \\ \dots \\ B_{2n}(z_{\sigma(2n-1)},z_{\sigma(2n)}) & 0 & 0 & \dots & C(z_{\sigma(2n-1)},z_{\sigma(2n)}) \end{pmatrix}$$

Suppose that $s \in N_2 \cup N_4$ with $z_{\sigma(s)} = z_{\sigma(s-1)}$ exists and we choose all rows in row block s (we always use original row block numbers to avoid confusion, even after we removed the first block). We can test linear independence of all chosen rows before row block $s+1$ and chosen rows in each block row starting from $s+1$ separately, since the matrix for such choice can be seen having square block structure with zero blocks above diagonal, if we define the same set of new blocks sizes on both columns and rows $(d-k)(s-1) + k, d-k, \dots, d-k$ ($2n-s+1$ blocks althogether) for the selection matrix. Notice that linear independence for blocks of sizes $d-k$ is equivalent to the linear independence of the selection of all rows except I_j from $C(z_{\sigma(j-1)}, z_{\sigma(j)}) = S_2'(z_{\sigma(j-1)}, z_{\sigma(j)})L(z_{\sigma(j)})$. For the rest we can imagine that we make selection from

$$\begin{pmatrix} B_2(z_{\sigma(1)},z_{\sigma(2)}) & C(z_{\sigma(1)},z_{\sigma(2)}) & 0 & \dots & 0 & 0 \\ B_3(z_{\sigma(2)},z_{\sigma(3)}) & D(z_{\sigma(2)},z_{\sigma(3)}) & C(z_{\sigma(2)},z_{\sigma(3)}) & \dots & 0 & 0 \\ \dots \\ B_s(z_{\sigma(s-1)},z_{\sigma(s)}) & 0 & 0 & \dots & D(z_{\sigma(s-1)},z_{\sigma(s)}) & C(z_{\sigma(s-1)},z_{\sigma(s)}) \end{pmatrix}$$

Since $z_{\sigma(s)} = z_{\sigma(s-1)}$ we have that $C(z_{\sigma(s-1)}, z_{\sigma(s)}) = -D(z_{\sigma(s-1)}, z_{\sigma(s)})$, because $S_2'(x,x) = -S_1'(x,x)$, which follows from the identity $S(x,x) = 0$. But then we can replace $D(z_{\sigma(s-1)}, z_{\sigma(s)})$ with zeros using linear transformation (multiplication by the non-degenerate matrix from the right), and separate d last chosen rows (which is the whole last row block s) and first and last column blocks in the new selection, which contain $S_2'(z_{\sigma(s-1)}, z_{\sigma(s)})(G(z_{\sigma(s)}), L(z_{\sigma(s)}))$ if $i \in N_4$ or

$$(S_1'(z_{\sigma(s-1)}, z_{\sigma(s)})G(z_{\sigma(s-1)}), S_2'(z_{\sigma(s-1)}, z_{\sigma(s)})L(z_{\sigma(s)})) =$$

$$= S_2'(z_{\sigma(s-1)}, z_{\sigma(s)})(-G(z_{\sigma(s)}), L(z_{\sigma(s)}))$$

if $i \in N_2$, both matrices being non-degenerate. The rest of the selection again has block structure with zero blocks above diagonal, now with all block sizes $d-k$. Block selected from row block j for $1 < j < s$ of initial matrix is non-degenerate if the selection of all rows except I_j from $C(z_{\sigma(j-1)}, z_{\sigma(j)}) = S_2'(z_{\sigma(j-1)}, z_{\sigma(j)})L(z_{\sigma(j)})$ are linearly independent. Therefore we obtain that linear independence of the selection is equivalent to the linear independence of each of the selections of all rows except I_j from $C(z_{\sigma(j-1)}, z_{\sigma(j)}) = S_2'(z_{\sigma(j-1)}, z_{\sigma(j)})L(z_{\sigma(j)})$ for all $j > 1$, $j \neq s$. This also proves that such choice of linearly independent rows is always possible, as long as there is $s \in N_2 \cup N_4$ with $z_{\sigma(s)} = z_{\sigma(s-1)}$, since the rank of $C(z_{\sigma(j-1)}, z_{\sigma(j)})$ is always $d-k$.

Let us now consider the case where there is no such s. We recall that for row blocks $j \in N_3 \cup N_5$ with $z_{\sigma(j)} = z_{\sigma(j-1)}$ we select all rows except I_j. We can deal with first row block as before, and then assuming that, again, we have the linear independence of the selection of all rows except I_j from $C(z_{\sigma(j-1)}, z_{\sigma(j)})$ for all $j \in N_3 \cup N_5$ with $z_{\sigma(j)} = z_{\sigma(j-1)}$, we can transform the matrix $R'_{\sigma,x}(z)T(z)$ with multiplication by non-degenerate matrix from the right, so that in the chosen rows from row block $j \in N_3 \cup N_5$ with $z_{\sigma(j)} = z_{\sigma(j-1)}$ all other elements except corresponding to $C(z_{\sigma(j-1)}, z_{\sigma(j)})$ (column block $j+2$) become zero. Indeed for $j \in N_3 \cup N_5$ with $z_{\sigma(j)} = z_{\sigma(j-1)}$ we have zeros in two first column blocks by construction. Moreover $D(z_{\sigma(j-1)}, z_{\sigma(j)}) = -C(z_{\sigma(j-1)}, z_{\sigma(j)})$, so column block corresponding to D (column block $j+1$) in this row block can be made zero with the transformation, without changing other elements of the matrix, except that the intersection of row block $j+1$ and column block $j+1$ (block numbers are as in the original matrix) becomes $D(z_{\sigma(j)}, z_{\sigma(j+1)}) = D(z_{\sigma(j-1)}, z_{\sigma(j+1)})$ (it was zero). It is easy to see this new matrix is similar to the original matrix after we remove row block j, so after all such transformations we obtain the matrix of the form

$$\begin{pmatrix} B_2(z_{j_1}, z_{j_2})) & C(z_{j_1}, z_{j_2}) & 0 & \dots & 0 & 0 \\ B_3(z_{j_2}, z_{j_3}) & D(z_{j_2}, z_{j_3}) & C(z_{j_2}, z_{j_3}) & \dots & 0 & 0 \\ \dots & & & & & \\ B_l(z_{j_{l-1}}, z_{j_l}) & 0 & 0 & \dots & D(z_{j_{l-1}}, z_{j_l}) & C(z_{j_{l-1}}, z_{j_l}) \end{pmatrix}$$

where $j_1, \dots, j_l$ is $\sigma(1), \dots, \sigma(2n)$ after all $j \in N_3 \cup N_5$ with $z_{\sigma(j)} = z_{\sigma(j-1)}$ removed. The rest of the selection, which were specified as arbitrary, is to be made from this matrix. But we can always choose the needed number of linearly independent rows from this matrix, since its rank is maximal $(d + (d-k)(l-2))$ by construction.

Now we can see that the sets of indices I_j can be chosen separately such that $\sum_{q=1}^{k} p_{i_q^j}$ is minimal possible (or $\sum_{q \notin I_j} p_q$ is maximal possible), while keeping the linear independence of the choice, at least for all those $j \geqslant 2$, where $\lambda_j \neq 0$. For all such j we obtain that this minimal value of $\sum_{q=1}^{k} p_{i_q^j}$ is equal to $m_f(z_{\sigma(j)})$ (and for the rest the choice of I_j does not matter). We also choose the index $s \in N_2 \cup N_4$ with $z_{\sigma(s)} = z_{\sigma(s-1)}$ such that the corresponding λ_s is maximal possible. The value of λ_s is also the maximal among all λ_j for $j \in N_2 \cup N_4$, since $\lambda_j = 0$ if $z_{\sigma(j)} \neq z_{\sigma(j-1)}$.

In order to finish the proof we need to specify k_j from our choice of ψ. Suppose that $\psi(y_1, \dots, y_n) = \phi_M(y)$, so there is no impact from ψ on k_j. The sum in the condition can bounded below with

$$\sum_{j \neq s} \lambda_j (2 - Q + \sum_{l=1}^{d} p_l - m_f(z_{\sigma(j)})) + (2 - Q + \sum_{l=1}^{d} p_l)(\lambda_s + \lambda_1)$$

which is positive since $\max_{j=1,\dots,2n} m_f(z_{\sigma(j)}) < 2 - Q + \sum_{l=1}^{d} p_l$ by our assumptions. When $s \in N_2 \cup N_4$ with $z_{\sigma(s)} = z_{\sigma(s-1)}$ does not exist the bound still holds by the same argument.

For the other choice of ψ, if we just use additional bound on ψ where we can, we obtain $k_j = 0$ if $j \in N_3 \cup N_5$ (the bound for ψ adding $Q-2$), and $k_j = 2-Q$ for $j \in N_1 \cup N_2 \cup N_4$ (no bound coming from ψ). However since there might be no $j \in N_3 \cup N_5$, we need to find a better bound on ψ for a given σ. Note that if $s \in N_2 \cup N_4$ with $z_{\sigma(s)} = z_{\sigma(s-1)}$ does not exist then all λ_j, for $j \geqslant 2$ are zero, and since the coefficient near λ_1 is always positive the condition is fulfilled, so we may assume $s \in N_2 \cup N_4$ with $z_{\sigma(s)} = z_{\sigma(s-1)}$ exists.

We claim that there is a set of $\alpha_j^r, j = 1, \dots, 2n, r = 1, \dots, R$ with values 0 or $Q-2$, such that for each $j = 3, \dots, 2n$ and $r = 1, \dots, R$ we have either $\alpha_j^r = Q - 2$ or $\alpha_{j-1}^r = Q - 2$ (no two consecutive 0 starting from $j = 2$) and

$$\psi(z_1, \dots, z_n)\psi(z_{n+1}, \dots, z_{2n}) \leqslant C \sum_{r=1}^{R} \prod_{j=2,\dots,2n} \rho(z_{\sigma(j-1)}, z_{\sigma(j)})^{\alpha_j^r}.$$

To prove that we start with a trivial bound of such form with $R = 1$, but without an extra condition of not having two consecutive 0 after $j = 2$. Let us improve this bound by additionally keeping all $\rho(z_p, z_q)^{Q-2}$ for all available pairs p, q (not only such that (p, q) coincides with $(\sigma(j-1), \sigma(j))$ for some j). Then if we have two consecutive $\alpha_j = 0$ and $\alpha_{j+1} = 0$, we have either $j \in N_2$, $j+1 \in N_4$ or $j \in N_4$, $j+1 \in N_2$. In the first case we conclude that both $\sigma(j)$ and $\sigma(j+2)$ are in $\{1, 2, \dots, n\}$, and in the second case both $\sigma(j)$ and $\sigma(j+2)$ are in $\{n+1, n+2, \dots, 2n\}$. It means that in the both cases we have a multiplier $\rho(z_{\sigma(j)}, z_{\sigma(j+2)})^{Q-2}$, which can be bounded with $C(\rho(z_{\sigma(j)}, z_{\sigma(j+1)})^{Q-2} + \rho(z_{\sigma(j+1)}, z_{\sigma(j+2)})^{Q-2})$ using pseudo-triangle inequality. Therefore we obtain an inequality of the same type, but with $R = 2$ and $\alpha_j^1 = Q - 2$, $\alpha_{j+1}^2 = Q - 2$, i.e. we eliminated a pair of consecutive 0 in α_j^r and α_{j+1}^r. By repeating this operation for all members of the sum in the new bound for the other j we can remove all pairs of consecutive 0 for this j. Then we repeat this again until all consecutive 0 are removed. The total number of operations required is not larger than 2^{2n} (the number of such pairs is not larger than $2n - 1$ and to remove each we need no more than 2^k operations, if we have already removed k pairs), so in the end we obtain our claimed inequality.

Now we can use this inequality and deal with each member of the sum separately, meaning that we fix r. We can ignore the coefficient near λ_1 since its always $2-Q+\sum_{l=1}^{d} p_l$, which should be positive by the assumptions in the Theorem. For $j \in N_3 \cap N_5$ we have $\alpha_j^r = Q - 2$ and for all j such that $\alpha_j^r = Q - 2$ we have $\lambda_j = 0$. We can also ignore the case $\alpha_s^r = Q - 2$, since then $\lambda_s = 0$, but it is also the maximal among all λ_j, $j > 2$, so only non-zero λ is λ_1, which means in this case the sum is always positive. The sum in the condition can bounded below with

$$\sum_{j \in N_2 \cup N_4, \alpha_j^r = 0, j \neq s} \lambda_j (2 - Q + \sum_{l=1}^{d} p_l - m_f(z_{\sigma(j)})) + (2 - Q + \sum_{l=1}^{d} p_l)\lambda_s$$

If $m_f(z_{\sigma(j)}) < 2 - Q + \sum_{l=1}^{d} p_l$ then the corresponding member of the sum is non-negative and we may bound it with zero. Therefore we can bound this sum from below as follows, replacing λ_j with larger λ_s, and using that the maximal number of $j \in N_2 \cap N_4$ such that $\alpha_j^r = 0$ is n:

$$\lambda_s(n(2 - Q + \sum_{l=1}^{d} p_l) - (n-1) \max_{j \in N_2 \cup N_4} m_f(z_{\sigma(j)}))$$

which is positive if

$$\max_{j=1,\dots,2n} m_f(z_j) < \frac{2n}{n-1} + \frac{n}{n-1}(\sum_{l=1}^{d} p_l - Q).$$

But this follows from our assumptions and the Theorem is proved. □

Remark 3. Note that the statement of the Theorem about the existence does not depend on the starting point x. In other words we always assume the worst case $z_{\sigma(1)} = x$ in the proof. It is interesting that we also do not need an additional multiplier $\rho(x, z_{\sigma(1)})$ in ψ (such multiplier does not improve the bound for our choice of rows), which can only mean that under our assumptions the corresponding singularity is always integrable. The question remains whether it is possible to find weaker assumptions, so that we obtain a stronger general statement for some x. Note that to find such assumptions we need to find choices of rows in the proof, that does not always include the whole first row blocks, but there are cases when such choice is surely not possible, for example for self-intersections of Brownian motion.

Remark 4. Note that the absence of additional multipliers in ψ makes the condition significantly worse. This is of course related to the known fact, that self-intersection local time for Brownian motion in a standard sense (i.e. without any renormalizations or weight, which in our definition means without multiplier ψ) exists only in one-dimensional case (see [11] for 2-dimensional case). The reason to include this case in the Theorem is to compare it to the case with additional multiplier, and also to relate with the known results. Also note that the choice of additional multiplier in ψ is definitely not unique and can be improved in some cases, for example in a sense of taking lower powers of $\rho(y_i, y_j)$. We did not try to find the best form for the multiplier (in any sense), since we do not have the use for such result. Also there is nothing in the proof suggesting that taking different ψ (from the one which already has multipliers) allows us to improve other conditions.

Remark 5. The assumption that $k < d$ seems too restrictive, since in classical self-intersection situations we always have $k = d$. Unfortunately to handle such cases we need some additional properties of $p(t, x, y)$, so it is better to do this separately (see below). On the other hand we can deal with classical self-intersections of standard Brownian motion by artificial dimension increment: we may simply add one independent Brownian motion as extra coordinate of the process, which is not involved in self-intersections. In this way we obtain an interesting effect: an independent random object is used in the mutiplier ψ to make the singularities in self-intersection local times integrable.

Let us drop the restriction $k < d$ and replace it with a condition on p.

Theorem 4. *Suppose that for all $a \in M_1$, $b \in M_1$, $a \neq b$ we have $\int_0^1 p(s, a, b) ds > 0$, the function $(\int_0^1 p(s, a, b) ds)^{-1}$ is bounded on $a \in M_1$, $b \in M_1$, $a \neq b$ and converges to zero as*

$(a, b) \to (c, c)$ for any $c \in M_1$. Also suppose that the function

$$g(t,a,b) = \frac{\int\limits_0^t p(s,a,b)ds}{\int\limits_0^1 p(s,a,b)ds}, a \neq b$$

can be extended to a continuous function for all (t, a, b) with $t \in (0, 1]$, $a \in M_1$, $b \in M_1$, and this extension (also denoted as g) satisfies $g(t, a, a) = 1$ for all $t \in (0, 1]$, $a \in M_1$. Let $n = 2$ and

$$\psi(y_1, y_2) = (\int\limits_0^1 p(s, y_1, y_2)ds)^{-1}\phi_M(y), y_1 \neq y_2; \psi(y_1, y_1) = 0$$

where ϕ_M is any continious non-negative bounded function with support inside M. If

$$m_f(y_i) < 4 + 2(\sum_{l=1}^{d} p_l - Q), \tag{8}$$

for all y_i, $i = 1, 2$ such that $(y_1, y_2) \in H \cap M$, then there is a limit of $\gamma_\varepsilon(\psi, F, [0, 1]^2)$ in L_2. If $\phi_M(w) > 0$ at some $w \in H \cap M$ then the limit is not zero.

Proof. We can define $h_\sigma(x, y)$ from Theorem 1 as

$$h_\sigma(x,y) = \begin{cases} 0, \exists i \in \{1,2\}, j \in \{3,4\} : y_i = y_j; \\ 0, y_{\sigma(1)} = x; \\ 0, q_4(x, y_{\sigma(1)}, \dots, y_{\sigma(4)}, [0,1]^2)) = +\infty, y_1 \neq y_2, y_3 \neq y_4; \\ \psi(y_1,y_2)\psi(y_3,y_4)q_4(x, y_{\sigma(1)}, \dots, y_{\sigma(4)}, [0,1]^2)), \\ q_4(x, y_{\sigma(1)}, \dots, y_{\sigma(4)}, [0,1]^2)) < +\infty, y_1 \neq y_2, y_3 \neq y_4; \end{cases}$$

for all $y \in M^2$ w.r.t. Lebesgue measure, except the set

$$E = (\{y_1 = y_2\} \cup \{y_3 = y_4\}) \cap \{\forall i \in \{1,2\}, j \in \{3,4\} : y_i \neq y_j\} \cap \{y_{\sigma(1)} \neq x\}.$$

We will show that it is possible to extend h_σ on this set by continuation. Consider a case $\sigma(i) = i$. Then

$$\psi(y_1,y_2)\psi(y_3,y_4)q_4(x, y_{\sigma(1)}, \dots, y_{\sigma(4)}, [0,1]^2)) =$$
$$= \phi_M(y_1,y_2)\phi_M(y_3,y_4)(\int\limits_0^1 p(s,y_1,y_2)ds \int\limits_0^1 p(s,y_3,y_4)ds)^{-1}$$
$$\int\limits_{[0,1]^4} 1_{s_1+s_2+s_3+s_4<1} p(s_1,x,y_1)p(s_2,y_1,y_2)p(s_3,y_2,y_3)p(s_4,y_3,y_4)ds_1ds_2ds_3ds_4 \leqslant$$
$$\leqslant \phi_M(y_1,y_2)\phi_M(y_3,y_4)g(1,y_1,y_2)g(1,y_3,y_4) \int\limits_{[0,1]^2} 1_{s_1+s_3<1} p(s_1,x,y_1)p(s_3,y_2,y_3)ds_1ds_3$$

and on the other hand

$$\psi(y_1,y_2)\psi(y_3,y_4)q_4(x, y_{\sigma(1)}, \dots, y_{\sigma(4)}, [0,1]^2)) \geqslant$$
$$\geqslant \phi_M(y_1,y_2)\phi_M(y_3,y_4)(\int\limits_0^1 p(s,y_1,y_2)ds \int\limits_0^1 p(s,y_3,y_4)ds)^{-1}$$

$$\int\limits_{[0,1]^4} 1_{s_2<\varepsilon}1_{s_4<\varepsilon}1_{s_1+s_3<1-2\varepsilon}p(s_1,x,y_1)p(s_2,y_1,y_2)p(s_3,y_2,y_3)p(s_4,y_3,y_4)ds_1ds_2ds_3ds_4$$
$$= \phi_M(y_1,y_2)\phi_M(y_3,y_4)g(\varepsilon,y_1,y_2)g(\varepsilon,y_3,y_4) \int\limits_{[0,1]^2} 1_{s_1+s_3<1-2\varepsilon}p(s_1,x,y_1)p(s_3,y_2,y_3)ds_1ds_3$$

which by our assumptions and using arguments similar to those in Proposition 2 allows us to extend this function continuosly to each $y \in E$ with both $y_1 = y_2$ and $y_3 = y_4$ satisfied at the same time.

Other cases can be treated similarly and in this way we obtain h_σ which satisfies the continuity condition from Theorem 1, since sets $\{\exists i \in \{1,2\}, j \in \{3,4\} : y_i = y_j\}$ and $\{y_{\sigma(1)} = x\}$ have zero measure w.r.t. $\nu(0,dy_1dy_2)\nu(0,dy_3dy_4)$. The rest of the proof follows the lines of the proof of Theorem 3. Note that we can show that the limit is not zero, since our assumptions guarantee that $h_\sigma(x,y)$ is positive a.e. w.r.t. $\nu(0,dy_1dy_2)\nu(0,dy_3dy_4)$ in all points y where $\phi_M(y_1,y_2)\phi_M(y_3,y_4)$ is positive. □

Remark 6. Note that using the same arguments we can also prove the existence of the limit also for $n = 3,4,\ldots$ for a choice of ψ as in Theorem 3. However in this case we have no guarantee that the limit is not zero. In fact it will always be zero for $f(x) = x$, which makes this result essentially useless. To get a non-zero limit of γ we need to choose another ψ. One such choice is shown below in Theorem 7 for self-intersection local time for 2-dimensional Brownian motion. But since it leads to the absence of continuity for h_σ, we can not rely on Theorem 1. The proof of Theorem 7 shows how to weaken this assumption. We will not pursue a general result analogous to Theorem 7 here (see also the Remark 8 after Theorem 7).

2.4. **Applications.** Now we are set to find the corollaries of Theorems 3 and 4 for specific processes (we keep all assumptions on F and specify X).

Theorem 5. *Let $d \geqslant 2$ and $X(t)$ be a solution of SDE:*

$$dX(t) = a(X(t))dt + \sigma(X(t))dW(t)$$

where $W(t)$ is a d-dimensional standard Brownian motion, a, σ are bounded Lipshitz functions and $\sigma\sigma^T(x) \geqslant cI$ for all $x \in \mathbb{R}^d$ and some $c > 0$. Suppose that $k < d$. If $d > 2$ let

$$\psi(y_1,\ldots,y_n) = \prod_{i\neq j}|y_i - y_j|^{d-2}\phi_M(y),$$

where ϕ_M is any continious non-negative bounded function with support inside M. If $d = 2$ let

$$\psi(y_1,\ldots,y_n) = \prod_{i\neq j}|y_i - y_j|^{\beta}\phi_M(y),$$

where β is an arbitrary fixed positive number. If $k \leqslant 2$ or if $n = 2$ and $k \leqslant 3$, then there is a limit of $\gamma_\varepsilon(\psi,F,A)$ in L_2 for any Borel $A \subset [0,1]^n$. If $\phi_M(w) > 0$ at some $w \in H \cap M$ and A has non-zero Lebesgue measure then the limit is not zero.

Proof. It is well-known that there is a unique strong solution to given SDE, and it is a time-homogeneous Markov process, which transition density p is continuous and satisfies Gaussian upper bounds (these bounds can be found, for example, in [6]): there are positive constants C and γ, such that for all $x \in \mathbb{R}^d$, $y \in \mathbb{R}^d$, $t \in [0,1]$

$$p(t,x,y) \leqslant Ct^{-d/2}e^{-\gamma|x-y|^2/t}$$

If $d \geqslant 3$ we obtain: for all $x \in M_1$, $y \in M_1$

$$\int_0^1 p(t,x,y)dt \leqslant C|x-y|^{2-d}$$

If $d = 2$ this inequality contains logarithm, which does not fit in our framework, but we can bound it with small negative power: there are $C > 0$ and $\delta > 0$ such that for all $x \in M_1$, $y \in M_1$

$$\int_0^1 p(t,x,y)dt \leqslant C|x-y|^{-\delta}$$

So if $d = 3$ we have (3) with $Q = d$ and $p_i = 1$, and if $d = 2$ we have (3) with $Q = 3$ and $p_i = \frac{1}{\delta}$. Therefore we can apply Theorem 3 and obtain the desired conclusion. Note that if all p_i are equal, as we have in this case, then $m_f(x) = kp_1$ everywhere, and then the sufficient condition in Theorem 3 has form $k < \frac{2n}{n-1}$ if $d = 3$. If $d = 2$ we have $k < \frac{2n}{n-1}(1-\delta/2)$ which is essentially the same, since we can choose $\delta > 0$ arbitrarily (it only should be smaller then β). This condition can be resolved as $k \leqslant 2$ or $k = 3$, $n = 2$ to complete the proof (here we only consider $n \geqslant 2$). □

Remark 7. Here we obtained a way to define self-intersection local times for 2- and 3- dimensional Brownian motion (including multiple self-intersections in 2-dimensional case) as partial cases of Theorem 5. It is known that these local times generally speaking do not exist in a standard definition, but in 2-dimensional case renormalization can be applied (see [11]). Now we see that we do not need renormalization if we add a multiplier inside time integral. Such multiplier contains a random component, partly independent from the process, for which local times are constructed. It seems that the existence of local time becomes possible, because, for example in case of double self-intersection, multiplier is close to zero, when both time variables t,s in double time integral are close to each other. We may conjecture that the multiplier of the form $|t-s|^a$ with large enough a also leads to existence of the corresponding limit (such case, when multiplier is a function from time, needs separate treatment).

Theorem 6. *Let $d \geqslant 2$ and $X(t)$ be a solution of SDE:*

$$dX(t) = a(X(t))dt + \sigma(X(t))dW(t)$$

where $W(t)$ is a d-dimensional standard Brownian motion, a, σ are bounded Lipshitz functions and $\sigma\sigma^T(x) \geqslant cI$ for all $x \in \mathbb{R}^d$ and some $c > 0$. Let $n = 2$ and

$$\psi(y_1,y_2) = (\int_0^1 p(t,y_1,y_2))^{-1}\phi_M(y), y_1 \neq y_2; \psi(y_1,y_1) = 0$$

where ϕ_M is any continious non-negative bounded function with support inside M. If $k \leqslant 3$, then there is a limit of $\gamma_\varepsilon(\psi,F,[0,1]^2)$ in L_2. If $\phi_M(w) > 0$ at some $w \in H \cap M$ then the limit is not zero.

Proof. Again we use Gaussian bounds, but this time we also need lower bounds as well:

$$C_1 t^{-d/2} e^{-\gamma_1|x-y|^2/t} \leqslant p(t,x,y) \leqslant Ct^{-d/2}e^{-\gamma|x-y|^2/t}$$

These bounds guarantee that

$$g(t,a,b) = \frac{\int_0^t p(s,a,b)ds}{\int_0^1 p(s,a,b)ds} = 1 - \frac{\int_t^1 p(s,a,b)ds}{\int_0^1 p(s,a,b)ds}, a \neq b$$

has continuous extension to $a = b$ as needed. Using Theorem 4 we obtain the result. □

In the following Theorem we show what happens with multiple self-intersection local times for 2-dimensional Brownian motion with simplest weights only depending on the process itself. This case needs a special approach (since the continuity of h_σ from Theorem 1 can not be shown here), so what we obtain is not a direct consequence of Theorems proved earlier, even though we follow the same ideas as before.

Theorem 7. *Let $X(t) = (W_1(t), W_2(t))$ be a 2-dimensional Brownian motion started at $X(0) = x$. Suppose that φ is a bounded continuous function on $\mathbb{R}^{2n}$. Assume that $k = d$ and $f(x) = x$ and set*

$$\psi(y_1, \dots, y_n) = \varphi(y_1, \dots, y_n) \prod_{i=1}^{n-1} |\ln|y_{i+1} - y_i||^{-1}$$

Then there is a limit $\gamma(\varphi)$ of $\gamma_\varepsilon(\psi, F, [0,1]^n)$ in L_p for integer $p \geqslant 2$ as $\varepsilon \to 0+$. Moreover (9)

$$E\gamma(\varphi)^p = (2\pi)^{-pn} 2^{p(n-1)} (n!)^p \sum_{\sigma \in S_p} \int_{\mathbb{R}^{2p}} q(x, z_{\sigma(1)}, \dots, z_{\sigma(k)}, [0,1]^p) \prod_{i=1}^{p} \varphi(z_i, \dots, z_i) dz$$

Proof. We have as in the proof of Theorem 1:

$$E\gamma_{\varepsilon_1}(\psi, F, A)\gamma_{\varepsilon_2}(\psi, F, A) = \\ = \sum_{\sigma \in S_{2n}} \int_{\mathbb{R}^{2nd}} q_{2n}(x, y_{\sigma(1)}, \dots, y_{\sigma(2n)}, [0,1]^n) \psi(y_1, \dots, y_n) \psi(y_{n+1}, \dots, y_{2n}) \\ f_{m,\varepsilon_1}(F(y_1, \dots, y_n)) f_{m,\varepsilon_2}(F(y_{n+1}, \dots, y_{2n})) dy$$

We take $w_1 = y_{\sigma(1)} - x$, $w_2 = y_{\sigma(k+1)} - y_{\sigma(k)}$ for some $k = 1, \dots, n$ (to be chosen later differently for each σ) in such way that exactly one of $\sigma(k+1)$ and $\sigma(k)$ is less or equal n. The we change variables $y \mapsto u, v, w_1, w_2$ in the integral with $u_i = y_{i+1} - y_i$ and $v_i = y_{n+i+1} - y_{n+i}$, $i = 1, \dots, n$, and due to the specific form of F and since $f(x) = x$ we obtain:

$$E\gamma_{\varepsilon_1}(\psi, F, A)\gamma_{\varepsilon_2}(\psi, F, A) = \\ = \sum_{\sigma \in S_{2n}} \int_{\mathbb{R}^{2n-2}} \int_{\mathbb{R}^{2n-2}} \int_{\mathbb{R}^4} (q_{2n}(x, y_{\sigma(1)}, \dots, y_{\sigma(2n)}, [0,1]^n) \varphi(y_1, \dots, y_n) \varphi(y_{n+1}, \dots, y_{2n})) \\ \prod_{i=1}^{n-1} |\ln|u_i| \ln|v_i||^{-1} f_{m,\varepsilon_1}(u) f_{m,\varepsilon_2}(v) dw_1 dw_2 du_1 \dots du_{n-1} dv_1 \dots dv_{n-1}$$

where $y = y(u, v, w_1, w_2)$ used inside integral is a function describing old variables as functions from new variables:

$$y_i = S_i^u + w_1 + x, i = 1, \dots, n, \\ y_{n+i} = S_i^v + w_2 + S^u + w_1 + x, i = 1, \dots, n,$$

where S_i^u, S^u are linear combinations of u_j with coefficients 1, -1 or 0 (depending on σ), and S_i^v are linear combinations of v_j with coefficients 1, -1 or 0. Using that (we have $m = 2(n-1)$ for given F)

$$f_{m,\varepsilon}(u) = \varepsilon^{-d} \phi_m(\frac{u}{\varepsilon})$$

we can make the change of variables by multipliying u by ε_1 and v by ε_2 (we keep the same letters for new variables):

$$\sum_{\sigma\in S_{2n}}\int_{\mathbb{R}^{2n-2}}\int_{\mathbb{R}^{2n-2}}\int_{\mathbb{R}^4}(q_{2n}(x,y_{\sigma(1)},\ldots,y_{\sigma(2n)},[0,1]^n)\varphi(y_1,\ldots,y_n)\varphi(y_{n+1},\ldots,y_{2n}))$$
$$\prod_{i=1}^{n-1}|\ln|\varepsilon_1u_i|\ln|\varepsilon_2v_i||^{-1}\phi_m(u)\phi_m(v)dw_1dw_2du_1\ldots du_{n-1}dv_1\ldots dv_{n-1}$$

and new $y=y(\varepsilon_1,\varepsilon_2,u,v,w_1,w_2)$ have form

$$y_i=\varepsilon_1S_i^u+w_1+x, i=1,\ldots,n,$$
$$y_{n+i}=\varepsilon_2S_i^v+w_2+\varepsilon_1S^u+w_1+x, i=1,\ldots,n,$$

To prove convergence of each integral for all σ as $(\varepsilon_1,\varepsilon_2)\to 0+$ it is enough to show convergence of the function under integral for almost all w_1,w_2,u,v w.r.t Lebesgue measure and find such $\beta>0$ and $\gamma>0$ that the following integral

$$(10)\quad \int_{\mathbb{R}^{2n-2}}\int_{\mathbb{R}^{2n-2}}\int_{\mathbb{R}^4}(q_{2n}(x,y_{\sigma(1)},\ldots,y_{\sigma(2n)},[0,1]^n)|\varphi(y_1,\ldots,y_n)\varphi(y_{n+1},\ldots,y_{2n})|)^{1+\gamma}$$
$$\prod_{i=1}^{n-1}|\ln|\varepsilon_1u_i|\ln|\varepsilon_2v_i||^{-1-\gamma}(1+|w_1|)^{(2+\beta)\gamma}(1+|w_2|)^{(2+\beta)\gamma}$$
$$\phi_m(u)\phi_m(v)dw_1dw_2du_1\ldots du_{n-1}dv_1\ldots dv_{n-1}$$

is uniformly bounded over $(\varepsilon_1,\varepsilon_2)$ in some neighbourhood of zero (this is similar to what we did in Theorem 1 but with slightly different integral). The last condition gives us uniform integrability of the function under integral multiplied by $(1+|w_1|)^{2+\beta}(1+|w_2|)^{2+\beta}$ w.r.t. the finite measure

$$(1+|w_1|)^{-2-\beta}(1+|w_2|)^{-2-\beta}\phi_m(u)\phi_m(v)dw_1dw_2du_1\ldots du_{n-1}dv_1\ldots dv_{n-1},$$

so the convergence follows from this by well-known arguments.

Let us consider q_{2n}, knowing that $p(t,x,y)=(2\pi t)^{-1}e^{-\frac{|x-y|^2}{2t}}$:

$$q_{2n}(x,y_{\sigma(1)},\ldots,y_{\sigma(2n)},[0,1]^n)=$$
$$(2\pi)^{-2n}\int_{0<t_1<\ldots<t_{2n}<1}t_1^{-1}e^{-\frac{|x-y_{\sigma(1)}|^2}{2t_1}}\ldots(t_{2n}-t_{2n-1})^{-1}e^{-\frac{|y_{\sigma(2n)}-y_{\sigma(2n-1)}|^2}{2(t_{2n}-t_{2n-1})}}dt$$

To estimate this integral the following lemma will be useful. Denote

$$\Pi(a)=\int_0^1 t^{-1}e^{-\frac{a}{2t}}dt$$

Lemma 3. *We have*

$$\Pi(a)\sim-\ln a, a\to 0+$$

and

$$\Pi(a)\leqslant\frac{2}{a}e^{-\frac{a}{2}}, a>0$$

Proof. With change of variables $s=\frac{t}{a}$ we obtain

$$\Pi(a)=\int_0^{\frac{1}{a}}s^{-1}e^{-\frac{1}{2t}}dt$$

which gives us the asymptotics as $a \to 0+$. Another change of variables $s = \frac{a}{2t}$ produces

$$\Pi(a) = \int_{\frac{a}{2}}^{+\infty} s^{-1} e^{-s} ds$$

and gives us the bound. □

We can find an upper bound for q_{2n} by simply extending the domain of the integral to $0 < t_1 < 1, 0 < t_2 - t_1 < 1, \dots, 0 < t_{2n} - t_{2n-1} < 1$:

$$(11)\quad q_{2n}(x, y_{\sigma(1)}, \dots, y_{\sigma(2n)}, [0,1]^n) \leqslant (2\pi)^{-2n} \Pi(|y_{\sigma(1)} - x|^2) \dots \Pi(|y_{\sigma(2n)} - y_{\sigma(2n-1)}|^2)$$

We recall the following definitions (from the proof of Theorem 3): $N_1 = \{1\}$,

$$N_2 = \{i = 2, \dots, 2n : \sigma(i) \in \{1, 2, \dots, n\}, \sigma(i-1) \in \{n+1, n+2, \dots, 2n\}\},$$
$$N_3 = \{i = 2, \dots, 2n : \sigma(i) \in \{1, 2, \dots, n\}, \sigma(i-1) \in \{1, 2, \dots, n\}\},$$
$$N_4 = \{i = 2, \dots, 2n : \sigma(i) \in \{n+1, n+2, \dots, 2n\}, \sigma(i-1) \in \{1, 2, \dots, n\}\},$$
$$N_5 = \{i = 2, \dots, 2n : \sigma(i) \in \{n+1, n+2, \dots, 2n\}, \sigma(i-1) \in \{n+1, n+2, \dots, 2n\}\}.$$

The expressions for y_i give us for $i \in N_3$:

$$y_{\sigma(i)} - y_{\sigma(i-1)} = \varepsilon_1 \tilde{S}_i^u,$$

where $\tilde{S}_i^u$ are linear combinations of u_j with coefficients 1, -1 or 0 (depending on σ), for $i \in N_5$:

$$y_{\sigma(i)} - y_{\sigma(i-1)} = \varepsilon_2 \tilde{S}_i^v,$$

where $\tilde{S}_i^v$ are linear combinations of v_j with coefficients 1, -1 or 0, and for $i \in N_2 \cup N_4$:

$$y_{\sigma(i)} - y_{\sigma(i-1)} = w_2 + \varepsilon_1 \tilde{S}_i^u + \varepsilon_2 \tilde{S}_i^v$$

where $\tilde{S}_i^u$, $\tilde{S}_i^v$ have the same meaning as above. Also we have similar formula for $i = 1$, for example supposing that $\sigma(1) \in \{1, 2, \dots, n\}$ (the other case is analogous):

$$y_{\sigma(1)} - x = w_1 + \varepsilon_1 \tilde{S}_1^u$$

From this we can conclude that for $i \in N_3$ we have

$$\Pi(|y_{\sigma(i+1)} - y_{\sigma(i)}|^2) \sim -2 \ln \varepsilon_1$$

and for $i \in N_5$:

$$\Pi(|y_{\sigma(i+1)} - y_{\sigma(i)}|^2) \sim -2 \ln \varepsilon_2$$

as $(\varepsilon_1, \varepsilon_2) \to 0+$. For $i \in N_1 \cup N_2 \cup N_4$ and under condition $w_1 \neq 0$ and $w_2 \neq 0$ we have that $\Pi(|y_{\sigma(i+1)} - y_{\sigma(i)}|^2)$ is bounded for small $(\varepsilon_1, \varepsilon_2)$. But it is clear that the size of each N_3 and N_5 is less or equal $n-1$ so if either $|N_3| < n-1$ or $|N_5| < n-1$, then:

$$\lim_{(\varepsilon_1, \varepsilon_2) \to 0+} q_{2n}(x, y_{\sigma(1)}, \dots, y_{\sigma(2n)}, [0,1]^n) (\ln \varepsilon_1)^{-(n-1)} (\ln \varepsilon_2)^{-(n-1)} = 0$$

for almost all u, v, w_1, w_2 (we have to assume that $w_1 \neq 0$ and $w_2 \neq 0$). It means that we have the convergence of the function under integral to 0 unless $|N_3| = n-1$ and $|N_5| = n-1$. For such cases we have that $N_3 = \{2, \dots, n\}$, $N_5 = \{n+2, \dots, 2n\}$ or $N_5 = \{2, \dots, n\}$, $N_3 = \{n+2, \dots, 2n\}$. Assuming the first situation (the second situation is similar due to symmetry) we choose $k = n$, so that $w_2 = y_{\sigma(n+1)} - y_{\sigma(n)}$ and find more precise upper and lower bounds for q_{2n}.

We make a change of variables $s_1 = t_1, s_2 = t_2 - t_1, \dots, s_{2n} = t_{2n} - t_{2n-1}$ in the integral for q_{2n} and increase the domain of integral to $s_1 + s_{n+1} < 1, s_i < 1, i \neq 1, n+1$, $s_i > 0$, so it gives us the following upper bound:

$$q_{2n}(x, y_{\sigma(1)}, \dots, y_{\sigma(2n)}, [0,1]^n) \leqslant \prod_{i=1}^{n-1} \Pi(|y_{\sigma(i+1)} - y_{\sigma(i)}|^2) \prod_{i=1}^{n-1} \Pi(|y_{\sigma(n+i+1)} - y_{\sigma(n+i)}|^2)$$

$$(2\pi)^{-2n} \int\limits_{s_1+s_{n+1}<1, s_1>0, s_{n+1}>0} s_1^{-1} e^{-\frac{|w_1|^2}{2s_1}} s_{n+1}^{-1} e^{-\frac{|w_2|^2}{2s_{n+1}}} ds_1 ds_{n+1}$$

Taking into account that for $i = 1, \dots, n$

$$\Pi(|y_{\sigma(i+1)} - y_{\sigma(i)}|^2) \sim -2\ln\varepsilon_1$$

$$\Pi(|y_{\sigma(n+i+1)} - y_{\sigma(n+i)}|^2) \sim -2\ln\varepsilon_2$$

we see that the right hand side, multiplied by $(\ln\varepsilon_1)^{-(n-1)}(\ln\varepsilon_2)^{-(n-1)}$, converges to

$$(2\pi)^{-2n} 2^{2n-2} \int\limits_{s_1+s_{n+1}<1, s_1>0, s_{n+1}>0} s_1^{-1} e^{-\frac{|w_1|^2}{2s_1}} s_{n+1}^{-1} e^{-\frac{|w_2|^2}{2s_{n+1}}} ds_1 ds_{n+1}$$

as $(\varepsilon_1, \varepsilon_2) \to 0+$.

But if we decrease the domain of integral to $(2n-2)s_i + s_1 + s_{n+1} < 1$, $i \neq 1, n+1$, $s_i > 0$ we obtain for each $\delta > 0$ the lower bound:

$$q_{2n}(x, y_{\sigma(1)}, \dots, y_{\sigma(2n)}, [0,1]^n) \geqslant$$
$$(2\pi)^{-2n} \int\limits_{s_1+s_{n+1}<1, s_1>0, s_{n+1}>0} \prod_{i=1}^{n-1} \Pi((2n-2)\frac{|y_{\sigma(i+1)} - y_{\sigma(i)}|^2}{1 - s_1 - s_{n+1}}) \cdot$$
$$\cdot \prod_{i=1}^{n-1} \Pi((2n-2)\frac{|y_{\sigma(n+i+1)} - y_{\sigma(n+i)}|^2}{1 - s_1 - s_{n+1}}) s_1^{-1} e^{-\frac{|w_1|^2}{2s_1}} s_{n+1}^{-1} e^{-\frac{|w_2|^2}{2s_{n+1}}} ds_1 ds_{n+1}$$

Similarly as before we have for $i = 1, \dots, n$

$$\Pi((2n-2)\frac{|y_{\sigma(i+1)} - y_{\sigma(i)}|^2}{1 - s_1 - s_{n+1}}) \sim -2\ln\varepsilon_1$$

$$\Pi((2n-2)\frac{|y_{\sigma(n+i+1)} - y_{\sigma(n+i)}|^2}{1 - s_1 - s_{n+1}}) \sim -2\ln\varepsilon_2$$

and since $\Pi(a) \leqslant C\max(-\ln a, 1)$ for some C (from the Lemma) the right hand side converges multiplied by $(\ln\varepsilon_1)^{-(n-1)}(\ln\varepsilon_2)^{-(n-1)}$ (by Lebesgue dominated covergence Theorem), to

$$(2\pi)^{-2n} 2^{2n-2} \int\limits_{s_1+s_{n+1}<1, s_1>0, s_{n+1}>0} s_1^{-1} e^{-\frac{|w_1|^2}{2s_1}} s_{n+1}^{-1} e^{-\frac{|w_2|^2}{2s_{n+1}}} ds_1 ds_{n+1}$$

as $(\varepsilon_1, \varepsilon_2) \to 0+$. But then we deduce that in fact:

$$\lim_{(\varepsilon_1,\varepsilon_2)\to 0+} q_{2n}(x, y_{\sigma(1)}, \dots, y_{\sigma(2n)}, [0,1]^n)(\ln\varepsilon_1)^{-(n-1)}(\ln\varepsilon_2)^{-(n-1)} =$$
$$(2\pi)^{-2n} 2^{2n-2} \int\limits_{s_1+s_{n+1}<1, s_1>0, s_{n+1}>0} s_1^{-1} e^{-\frac{|w_1|^2}{2s_1}} s_{n+1}^{-1} e^{-\frac{|w_2|^2}{2s_{n+1}}} ds_1 ds_{n+1}$$

In order to prove the bound for (10) we use (11) again with additional assumption that $k = \min(N_2 \cap N_4)$ (it is the first moment i, when exactly one of the numbers $\sigma(i+1)$ and $\sigma(i)$ is less or equal n). We gather multipliers in (11) as follows

$$q_{2n}(x, y_{\sigma(1)}, \dots, y_{\sigma(2n)}, [0,1]^n) \leqslant (2\pi)^{-2n}\Pi(|w_1|^2)\Pi(|w_2|^2)$$
$$\prod_{i\in N_2\cap N_4\setminus k} \Pi(|w_2 + \varepsilon_1\tilde{S}_i^u + \varepsilon_2\tilde{S}_i^v|^2) \prod_{i\in N_3\cap N_5} \Pi(|y_{\sigma(i+1)} - y_{\sigma(i)}|^2)$$

We may assume that u and v are bounded (since ϕ_m has bounded support) and therefore for $i \in N_3$:

$$\Pi(|y_{\sigma(i+1)} - y_{\sigma(i)}|^2) \leqslant -C(\ln \varepsilon_1 + \ln |S_i^u|)$$

and for $i \in N_5$

$$\Pi(|y_{\sigma(i+1)} - y_{\sigma(i)}|^2) \leqslant -C(\ln \varepsilon_1 + \ln |S_i^v|)$$

For small enough ε_1 we have $\ln |\varepsilon_1 u_j| < -L < 0$ and consequently for $i \in N_3$:

$$\frac{\Pi(|y_{\sigma(i+1)} - y_{\sigma(i)}|^2)}{-\ln |\varepsilon_1 u_j|} \leqslant -C\frac{\ln \varepsilon_1 + \ln |S_i^u|}{-\ln |\varepsilon_1 u_j|} \leqslant C + C\frac{|\ln |S_i^u| - \ln |u_j||}{L}$$

and similarly for $i \in N_5$

$$\frac{\Pi(|y_{\sigma(i+1)} - y_{\sigma(i)}|^2)}{-\ln |\varepsilon_1 v_j|} \leqslant C + C\frac{|\ln |S_i^v| - \ln |v_j||}{L}$$

To find the bound on (10) we integrate the bound obtained above w.r.t. $dw_1 dw_2$ and use Cauchy inequality to separate integrals of powers of $(1 + |w_1|)^{(2+\beta)\gamma}\Pi(|w_1|^2)$, $(1 + |w_2|)^{(2+\beta)\gamma}\Pi(|w_2|^2)$ and each $\Pi(|w_2+\varepsilon_1\tilde{S}_i^u+\varepsilon_2\tilde{S}_i^v|^2)$. Then all these integrals are bounded uniformly over u, v and small ε_1, ε_2. But the rest is already a function that does not depend on ε_1, ε_2 and integrable w.r.t. u, v in any power and so the Theorem is proved. □

Remark 8. If we look carefully at the form of 9 we can see that in fact:

$$\gamma(\varphi) = (2\pi)^{-n}2^{n-1}n! \int_0^1 \varphi(W(t), \ldots, W(t))dt$$

It means that the weights we used are in fact "killing" self-intersections and what is left in $\gamma(\varphi)$ is just an additive functional of $W(t)$. It seems that the same may happen in Theorem 6 or more generally in Theorem 4, but unfortunately we can not provide more details here. Such "killing" is clearly not happening in Theorems 3 and 5, since the limit is still not zero if A does not contain the neighbourhood of the diagonal $t_1 = t_2 = \ldots = t_n$.

Clearly it is possible to consider many different Levy processes as X, since there are a lot of known upper bounds for the density of Levy processes. Here we only study the simplest case of symmetric stable processes.

Theorem 8. *Let $d \geqslant 2$ and $X(t)$ be a symmetric stable process in $\mathbb{R}^d$ of index $\alpha \in (0, 2)$. Suppose that $n = 2$ or $k < d$. Let*

$$\psi(y_1, \ldots, y_n) = \prod_{i \neq j} |y_i - y_j|^{d-\alpha}\phi_M(y),$$

where ϕ_M is any continious non-negative bounded function with support inside M. If $k < \frac{\alpha n}{n-1}$, then there is a limit of $\gamma_\varepsilon(\psi, F, A)$ in L_2 for any Borel $A \subset [0, 1]^n$. If $\phi_M(w) > 0$ at some $w \in H \cap M$ and A has non-zero Lebesgue measure then the limit is not zero.

Proof. From the well-known results for the density of symmetric stable processes we have (see for example [1])

$$C_1|x - y|^{\alpha-d} \leqslant \int_0^1 p(t, x, y)dt \leqslant C|x - y|^{\alpha-d}$$

with some constants $C_1 > 0$, $C > 0$. Therefore we can apply Theorem 3 with $Q = 1$ and $p_i = \frac{1}{d-\alpha}$ similarly as in Theorem 5 to prove this Theorem. For the case $n = 2$ we just have to use Theorem 4 instead and note, as before, that on a compact the multiplier $|y_1 - y_2|^{d-\alpha}$ is equal to $(\int_0^1 p(t, y_1, y_2)dt)^{-1}$ multiplied by a bounded continuous function. □

This result also leads to a new definition of self-intersection local time for symmetric stable processes, as long as $d < \frac{\alpha n}{n-1}$. Note that it is well-known that under condition $d < \frac{\alpha n}{n-1}$ the symmetric stable process has n-fold intersections with probability 1 (see [8]), which corresponds to our result perfectly, meaning that we are able to construct the corresponding local times for all cases, where self-intersections are known to exist.

Let $X(t)$ be a solution of the following SDE:

$$dX(t) = \sum_{i=1}^{l} L_i(X(t)) \circ dW_i(t), X(0) = x \in \mathbb{R}^d \tag{12}$$

We fix the choice of $L_1, \ldots, L_k$ such that they are a basis of the first level of stratification of Lie algebra of left-invariant vectors fields of a Carnot group $G = (\mathbb{R}^d, \bullet)$ (stratified Lie group, obtained by introducing a specific group action $\bullet$ on $\mathbb{R}^d$, see [2] for details). We call this process a Brownian motion on Carnot group (by analogy with Brownian motions on Lie group, introduced by Ito [7]). The framework of this paper was introduced with the aim to deal with local times for such processes, and now we are ready to show that our general theory is indeed applicable to this case. We will need the following notation: $\tilde{\rho}$ is a natural distance on the given Carnot group G (Carnot-Caratheodory distance, see [2]), $\tilde{p}_i$ are such that in a fixed coordinate system $(x_1, \ldots, x_d) \mapsto (\lambda^{\tilde{p}_1} x_d, \ldots, \lambda^{\tilde{p}_n} x_d)$ is a group automorphism for all $\lambda > 0$ (dilations in G, see [2]), and $\tilde{Q} = \sum_{i=1}^{d} \tilde{p}_i$ is a homogeneous dimension of G. We will assume that $\tilde{Q} \geqslant 3$, since for $\tilde{Q} \leqslant 2$ Carnot group is just a Euclidean space with usual addition and we have standard Brownian motion as X.

Theorem 9. *Let $X(t)$ be a Brownian motion on Carnot group defined as above. Suppose that $k < d$. Let*

$$\psi(y_1, \ldots, y_n) = \prod_{i \neq j} \tilde{\rho}(y_i, y_j)^{\tilde{Q}-2} \phi_M(y)$$

where ϕ_M is any continious non-negative bounded function with support inside M. If $m_f(y_i) \leqslant 2$ or $m_f(y_i) = 3$, $n = 2$ for all y_i, $i = 1, \ldots, n$ such that $(y_1, \ldots, y_n) \in H \cap M$, then there is a limit of $\gamma_\varepsilon(\psi, F, A)$ in L_2 for any Borel $A \subset [0, 1]^n$. If $\phi_M(w) > 0$ at some $w \in H \cap M$ and A has non-zero Lebesgue measure then the limit is not zero.

Proof. The following well-known facts can be found in [2] (see also [13] for density estimates and [10] for the comparison of pseudo-distances). There are positive constants C and γ, such that for all $x \in \mathbb{R}^d$, $y \in \mathbb{R}^d$, $t \in [0, 1]$

$$p(t, x, y) \leqslant C t^{-\tilde{Q}/2} e^{-\gamma \tilde{\rho}(x,y)^2/t}$$

and with obvious calculations we get: for all $x \in M_1$, $y \in M_1$

$$\int_0^1 p(t, x, y) dt \leqslant C \tilde{\rho}(x, y)^{2-\tilde{Q}}$$

Moreover the distance $\tilde{\rho}$ is locally equivalent to ρ in our definition with $S(x, y) = x^{-1} \bullet y$ (x^{-1} is the inverse of x in G), $Q = \tilde{Q}$ and $p_i = \tilde{p}_i$ and using the same coordinate system in the definition of ρ, as the one in the definition of dilations. But since both $\tilde{\rho}$ and ρ are bounded on compacts, they are equivalent on any compact. It means our assumptions hold and Theorem 3 can be applied. Note, that since p_i are positive integers, the condition $m_f(y_i) < \frac{2n}{n-1}$ ($\sum_{l=1}^{d} p_l - Q = 0$ in this case) can be resolved as $m_f(y_i) \leqslant 2$ or $m_f(y_i) = 3$, $n = 2$, which completes the proof of the Theorem. □

This theorem leads to a new definition of self-intersection local times on Carnot group (see [12] for a result related to this).

3. Local times for independent processes

3.1. **n-fold local time on the surface for independent processes.** We want to be able to cover the classical case of intersection local time for two or more independent and possibly different processes, but unfortunately Theorem 3 can not provide that. Moreover to take the independence into account we are forced to go all the way back to the definition of local time, change it to reflect the presence of independent processes and prove the corresponding versions of Theorems 1, 2. We could also try to use Theorems 1, 2 directly (since technically the case of the intersection of independent processes is still included in our general formulation), but that seems to lead to some complications related to the nature of our assumptions (for example in (5) the independence appears as some complicated structural property of S and F). Be aware that the notation in this section will be slightly different from the previous section, meaning that we use the same letters for objects that can be different (so we have to repeat all basic definitions).

Suppose that $Y_1(t), \dots, Y_n(t)$ are n independent Markov processes, each taking values in $\mathbb{R}^d$ and there are non-negative functions $p(k, t, x, y)$, continuous for $t > 0$, which are densities of the distribution of $Y_k(t)$ w.r.t. y, given that $Y_k(0) = x$. Let m be a positive integer, $F : \mathbb{R}^{nd} \to \mathbb{R}^m$ be an infinitely differentiable function, $\psi : \mathbb{R}^{nd} \to \mathbb{R}$ be a non-negative bounded continuous function. We define approximations of local time as follows

$$\gamma_\varepsilon(\psi, F, A) = \int_A \psi(Y_1(t_1), \dots, Y_n(t_n)) f_{m,\varepsilon}(F(Y_1(t_1), \dots, Y_n(t_n))) dt_1 \dots dt_n$$

where $A \subset [0,1]^n$ is a Borel set and $f_{m,\varepsilon}$ is as defined earlier.

We denote $H = \{z = (z_1, \dots, z_n) \in \mathbb{R}^{nd} | F(z_1, \dots, z_n) = 0\}$ and assume that the matrix of derivatives F' of F at z has maximal rank for all $z \in H$. Since F is defined as before, we also have similar definitions of θ_z and ν.

Define

$$q_{2n}(x_1, \dots, x_n, z_1, \dots, z_{2n}, B) = \\ = \int_B \prod_{k=1}^{n} 1_{t_k < t_{k+n}} p(k, t_k, x_k, z_k) p(k, t_{k+n} - t_k, z_k, z_{k+n}) dt_1 \dots dt_{2n}$$

for any Borel set $B \subset [0,1]^{2n}$. Note that there is an injection τ_σ of $\sigma = (\sigma_1, \dots, \sigma_n) \in S_2^n$ into S_{2n} permuting j and $j+n$ according to σ_j:

$$\tau_\sigma(j) = j + n(\sigma_j(1) - 1), \tau_\sigma(j+n) = j + n(\sigma_j(2) - 1); j = 1, \dots, n$$

which we will use in the following as a covenient abbreviation.

Theorem 10. *Fix $Y_k(0) = x_k \in \mathbb{R}^d$, $k = 1, \dots, n$ and non-negative bounded continuous function ψ on $\mathbb{R}^{nd}$ with $\operatorname{supp} \psi \subset M$. If for all $\sigma = (\sigma_1, \dots, \sigma_n) \in S_2^n$ we have*

$$q_{2n}(x_1, \dots, x_n, y_{\tau_\sigma(1)}, y_{\tau_\sigma(2)}, \dots, y_{\tau_\sigma(2n)}, A_\sigma)) < +\infty$$

for almost all $y \in M^2$ w.r.t. Lebesgue measure, where

$$A_\sigma = \{t : (t_{\tau_\sigma(1)}, \dots, t_{\tau_\sigma(n)}) \in A; (t_{\tau_\sigma(n+1)}, \dots, t_{\tau_\sigma(2n)}) \in A\},$$

and there is a function $h_\sigma(x, y)$, which satisfies the equality

$$h_\sigma(x, y) = \psi(y_1, \dots, y_n) \psi(y_{n+1}, \dots, y_{2n}) q_{2n}(x_1, \dots, x_n, y_{\tau_\sigma(1)}, y_{\tau_\sigma(2)}, \dots, y_{\tau_\sigma(2n)}, A_\sigma),$$

for almost all $y \in M^2$ w.r.t. Lebesgue measure, such that $h_\sigma(x,y)$ is continuous for almost all $y \in M^2$ w.r.t. $\nu(0, dy_1 \dots dy_n)\nu(0, dy_{n+1} \dots dy_{2n})$ and there are such positive numbers δ and β that

$$\sup_{|u|<\delta, |v|<\delta} \sum_{\sigma \in S_2^n} \int_{M^2} (h_\sigma(x,y))^{1+\beta} \nu(u, dy_1 \dots dy_n)\nu(v, dy_{n+1} \dots dy_{2n}) < +\infty \tag{13}$$

then there is a limit of $\gamma_\varepsilon(\psi, F, A)$ in L_2. If additionally $h_\sigma(x,y) > 0$ for y in a neighbourhood of some $z \in (H \cap M)^2$, then the limit is not zero.

Proof. The proof is similar to the proof of Theorem 1, except for the formula for h_σ, which can be found as follows.

$$E\gamma_{\varepsilon_1}(\psi, F, A)\gamma_{\varepsilon_2}(\psi, F, A) =$$
$$= E \int_A \psi(Y_1(t_1), \dots, Y_n(t_n)) f_{m,\varepsilon_1}(F(Y_1(t_1), \dots, Y_n(t_n)))dt_1 \dots dt_n$$
$$\int_A \psi(Y_1(t_{n+1}), \dots, Y_n(t_{2n})) f_{m,\varepsilon_1}(F(Y_1(t_{n+1}), \dots, Y_n(t_{2n})))dt_{n+1} \dots dt_{2n} =$$
$$\sum_{\sigma \in S_2^n} \int_{A \times A} \prod_{k=1}^n 1_{t_{\tau_\sigma(k)} < t_{\tau_\sigma(k+n)}} E\psi(Y_1(t_1), \dots, Y_n(t_n)) f_{m,\varepsilon_1}(F(Y_1(t_1), \dots, Y_n(t_n)))$$
$$\psi(Y_1(t_{n+1}), \dots, Y_n(t_{2n})) f_{m,\varepsilon_1}(F(Y_1(t_{n+1}), \dots, Y_n(t_{2n})))dt =$$
$$= \sum_{\sigma \in S_2^n} \int_{A_\sigma} \int_{\mathbb{R}^{2nd}} \prod_{k=1}^n 1_{t_k < t_{k+n}} p(k, t_k, x_k, y_{\tau_\sigma(k)}) p(k, t_{k+n} - t_k, y_{\tau_\sigma(k)}, y_{\tau_\sigma(k+n)})$$
$$\psi(y_1, \dots, y_n)\psi(y_{n+1}, \dots, y_{2n})$$
$$f_{m,\varepsilon_1}(F(y_1, \dots, y_n)) f_{m,\varepsilon_2}(F(y_{n+1}, \dots, y_{2n}))dydt =$$
$$= \sum_{\sigma \in S_2^n} \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} \int_{\mathbb{R}^{2nd}} h_\sigma(x,y) f_{m,\varepsilon_1}(u) f_{m,\varepsilon_2}(v) \nu(u, dy_1 \dots dy_n)\nu(v, dy_{n+1} \dots dy_{2n})dudv$$

□

Let $M_1(j)$, $j = 1, \dots, n$ be a family of bounded open sets such that $M \subset M_1(1) \times \dots \times M_1(n)$. We make the same assumptions as before on p.

(1) Suppose that $p(j, t, x, y)$ is continuous at $(0, x, y)$ for all $x \neq y$ and $j = 1, \dots, n$.
(2) Let a family of differentiable functions $S(j, \cdot) : \mathbb{R}^{2d} \to \mathbb{R}^d$, $j = 1, \dots, n$ be such that $S(j, x, x) = 0$ for all x, $S(j, x, y) \neq 0$ for $x \neq y$ and the derivatives of $S(j, x, y)$ w.r.t. x and w.r.t. y are non-degenerate (as two separate $d \times d$ matrices) for all x, y and $j = 1, \dots, n$. We suppose that for all $j = 1, \dots, n$ there are positive numbers $p_i(j) > 0$, $i = 1, \dots, d$, $Q(j) > 2$ and $C_1 > 0$, such that for all $x \in M_1(j)$, $y \in M_1(j)$ with $x \neq y$:

$$\int_0^1 p(j, t, x, y)dt \leqslant C_1(\rho(j, x, y))^{2-Q(j)}. \tag{14}$$

where $\rho(j, x, y) = \max_{i=1,\dots,d} |S_i(j, x, y)|^{1/p_i(j)}$.

Note that we do not need pseudo-triangle inequality for ρ, which was needed for self-intersection case.

Denote

$$R_{\sigma,x}(y_1,\dots,y_{2n}) = (S(1,x_1,y_{\tau_\sigma(1)}),S(1,y_{\tau_\sigma(1)},y_{\tau_\sigma(n+1)}),\dots$$
$$\dots,S(n,x_n,y_{\tau_\sigma(n)}),S(n,y_{\tau_\sigma(n)},y_{\tau_\sigma(2n)})).$$

Let $T(z)$ be any $2nd \times (2nd-2m)$ matrix composed of the vectors, forming a basis of the tangent space at z of $H \times H$, written in columns.

Theorem 11. *Fix $\sigma \in S_2^n$, $z = (z_1, z_2, \dots, z_{2n}) \in (H \cap M)^2$ and real numbers $k_1,\dots,k_{2n}$. If for all $\lambda_1 \geqslant 0,\dots,\lambda_{2n} \geqslant 0$ not all zero, but with $\lambda_{2j} = 0$ if $z_{\tau_\sigma(j)} \neq z_{\tau_\sigma(j+n)}$ for $j = 1,\dots,n$, $\lambda_{2j-1} = 0$ if $x_j \neq z_{\tau_\sigma(j)}$ for $j = 1,\dots,n$ and $\lambda_j = 0$ if $k_j \geqslant 0$ for $j = 1,\dots,2n$ (if all λ are forced to be zero the condition is trivially fulfilled), there exists $I = (i_1,\dots,i_{2(nd-m)}) \in N(R'_{\sigma,x}(z)T(z))$, such that*

$$\sum_{j=1}^{2n} \lambda_j\Big(\sum_{s:[i_s/d]=j} p_{i_s \, mod\, d}([j/2]) + k_j\Big) > 0$$

then there is an open neighbourhood U_z of z, $\delta > 0$ and $\beta > 0$ such that

$$\sup_{|u|<\delta,|v|<\delta} \int_{U_z} \prod_{j=1}^{n} (\rho(j,x_j,y_{\tau_\sigma(j)})^{k_{2j-1}} \rho(j,y_{\tau_\sigma(j)},y_{\tau_\sigma(j+n)})^{k_{2j}})^{1+\beta} \nu(u,dy_1\dots dy_n)\nu(v,dy_{n+1}\dots dy_{2n}) < +\infty \tag{15}$$

Proof. The proof is the same as the proof of Theorem 2 with natural adjustments for the different form of $R_{\sigma,x}$. □

3.2. **Intersection local time for independent processes.** We consider the following case

$$F(y_1,\dots,y_n) = (f_1(y_1) - f_2(y_2),\dots,f_{n-1}(y_{n-1}) - f_n(y_n))$$

where $f_i : \mathbb{R}^d \to \mathbb{R}^k$, $i = 1,\dots,n$ ($k \leqslant d$ is fixed) are continuously differentiable functions with derivative of maximal rank at all points.

Denote as $D_S(z_1,\dots,z_n)$ the matrix of the derivatives w.r.t. $(y_1,y_2,\dots,y_n)$ of the function $(S(z_1,y_1),\dots,S(z_n,y_n))$ at $(y_1,\dots,y_n) = (z_1,\dots,z_n)$, which have $d \times d$ blocks equal to $S_2'(z_1,z_1),\dots,S_2'(z_n,z_n)$ on diagonal, and zeros elsewhere. Let $T_H(z_1,\dots,z_n)$ be any $nd \times (nd-k(n-1))$ matrix composed of the vectors, forming a basis of the tangent space at $z_1,\dots,z_n$ of H, written in columns.

Theorem 12. *Let $\psi(y_1,\dots,y_n) = \phi_M(y)$, where ϕ_M is any continious non-negative bounded function with support inside M. Suppose that for all $(z_1,\dots,z_n) \in H \cap M$ and for all non-negative $\lambda_1,\dots,\lambda_n$, not all equal to zero, there is a multiindex*

$$I \in N(D_S(z_1,\dots,z_n)T_H(z_1,\dots,z_n)),$$

such that

$$\sum_{j=1}^{n} \lambda_j\Big(\sum_{s:[i_s/d]=j} p_{i_s \, mod\, d}(j) + 2 - Q(j)\Big) > 0 \tag{16}$$

Then there is a limit of $\gamma_\varepsilon(\psi,F,A)$ in L_2 for any Borel $A \subset [0,1]^n$. If $\phi_M(w) > 0$ at some $w \in H \cap M$ and for some $\sigma \in S_{2n}$ we can find $z \in (H \cap M)^2$ in any neighbourhood of (w,w), such that $q_{2n}(x_1,\dots,x_n,z_{\tau_\sigma(1)},\dots,z_{\tau_\sigma(2n)},A_\sigma) > 0$ and $x_j \neq z_{\tau_\sigma(j)}$, $z_{\tau_\sigma(j)} \neq z_{\tau_\sigma(j+n)}$ for all $j = 1,\dots,n$, then the limit is not zero ($x_i = Y_i(0)$).

Proof. We can define $h_\sigma(x,y)$ exactly as in the equality in Theorem 10 if

$$q_{2n}(x_1,\dots,x_n,y_{\tau_\sigma(1)},\dots,y_{\tau_\sigma(2n)},A_\sigma) < +\infty$$

and zero otherwise. By Proposition 2 such $h_\sigma(x,\cdot)$ is continuous at y if $x_j \neq y_{\tau_\sigma(j)}$ and $y_{\tau_\sigma(j)} \neq y_{\tau_\sigma(j+n)}$ for all $j = 1,\dots,n$. Since under our assumptions $\nu(0, dy_1 \dots dy_n)$ is zero on any set $\{y|y_j = a\}$ we obtain that the set

$$\{y|\exists j \in \{1,\dots,n\} : x_j = y_{\tau_\sigma(j)}\} \cup \{y|\exists j \in \{1,\dots,n\} : y_{\tau_\sigma(j)} = y_{\tau_\sigma(j+n)}\}$$

have zero measure w.r.t. $\nu(0, dy_1 \dots dy_n)\nu(0, dy_{n+1} \dots dy_{2n})$ and therefore such h_σ satisfies the continuity condition of Theorem 10. To check the integrability condition (13) of Theorem 10 it is enough to check the condition (15) in Theorem 11 for a fixed $z \in (H \cap M)^2$. Note that if the limit exists it is non-zero, since by our construction there is a point $z \in (H \cap M)^2$ (in a neigbourhood of (w,w)), such that $h_\sigma(x,z) > 0$ and $h_\sigma(x,\cdot)$ is continuous in a neighbourhood of z. Therefore in the following we can fix z and $\sigma \in S_2^n$ and focus on proving (15) using Theorem 11. We assume that the starting points of the processes $Y_j(0) = x_j$ are also fixed.

Let us describe the structure of $R'_{\sigma,x}(z)T(z)$ in our special case. The matrix $R'_{\sigma,x}(z)$ can be seen to have the following structure in the column basis where $y_1,\dots,y_{2n}$ has coordinates $y_{\tau_\sigma(1)}, y_{\tau_\sigma(2)},\dots,y_{\tau_\sigma(2n)}$: only non-zero elements are in $2d \times 2d$ diagonal blocks constructed as shown:

$$\begin{pmatrix} S'_2(j, x_j, y_{\tau_\sigma(j)}) & 0 \\ S'_1(j, y_{\tau_\sigma(j)}, y_{\tau_\sigma(j+n)}) & S'_2(j, y_{\tau_\sigma(j)}, y_{\tau_\sigma(j+n)}) \end{pmatrix}$$

where j is an index of the $2d \times 2d$ block.

To describe $T(z)$ we split rows in $2n$ blocks of size d, and we split columns into two blocks of size k and $2n$ blocks of size $d-k$. Denote $G_j(z) = f'_j(z)^T (f'_j(z) f'_j(z)^T)^{-1}$. In the first block of columns row blocks $1, 2 \dots, n$ are equal to $G_1(z_1), \dots, G_n(z_n)$ correspondingly. In the second block of columns row blocks $n+1, n+2, \dots, 2n$ are equal to $G_1(z_{n+1}), \dots, G_n(z_{2n})$ correspondingly. In the column block $i+2$, $i = 1,\dots,2n$ the row block i is equal to $L_i(z_i)$, where $L_i(a)$ be any $d \times (d-k)$ matrix consisting from vectors, giving basis of tangent space of $\{u \in \mathbb{R}^d : f_i(u) = f_i(a)\}$ at a, written in columns (those vectors are orthogonal to column vectors of $G_i(a)$). The rest of the blocks contain only zeros. It is easy to see that all column vectors are such that directional derivatives of $(F(z_1,\dots,z_n), F(z_{n+1},\dots,z_{2n}))$ along them are zero and they are linearly independent, so we have a basis in the tangent space of $H \times H$ at z.

We split the set of indices $\{1,\dots,n\}$ into two disjoint sets N_1, N_2 according to σ:

$$N_1 = \{j = 1,\dots,n : \sigma_j(1) = 1\}, N_2 = \{j = 1,\dots,n : \sigma_j(1) = 2\}$$

Denote for $j \in N_1$

$$A_j(x,y) = \begin{pmatrix} S'_2(j, x_j, y_{\tau_\sigma(j)}) G_j(y_{\tau_\sigma(j)}) & 0 \\ S'_1(j, y_{\tau_\sigma(j)}, y_{\tau_\sigma(j+n)}) G_j(y_{\tau_\sigma(j)}) & S'_2(j, y_{\tau_\sigma(j)}, y_{\tau_\sigma(j+n)}) G_j(y_{\tau_\sigma(j+n)}) \end{pmatrix}$$

and for $j \in N_2$

$$A_j(x,y) = \begin{pmatrix} 0 & S'_2(j, x_j, y_{\tau_\sigma(j)}) G_j(y_{\tau_\sigma(j)}) \\ S'_2(j, y_{\tau_\sigma(j)}, y_{\tau_\sigma(j+n)}) G_j(y_{\tau_\sigma(j+n)}) & S'_1(j, y_{\tau_\sigma(j)}, y_{\tau_\sigma(j+n)}) G_j(y_{\tau_\sigma(j)}) \end{pmatrix}$$

Also for all $j = 1,\dots,n$

$$C_j(x,y) = \begin{pmatrix} S'_2(j, x_j, y_{\tau_\sigma(j)}) L_j(y_{\tau_\sigma(j)}) & 0 \\ S'_1(j, y_{\tau_\sigma(j)}, y_{\tau_\sigma(j+n)}) L_j(y_{\tau_\sigma(j)}) & S'_2(j, y_{\tau_\sigma(j)}, y_{\tau_\sigma(j+n)}) L_j(y_{\tau_\sigma(j+n)}) \end{pmatrix}$$

Then it easy to see that $R'_{\sigma,x}(z)T(z)$ is equal to (we changed the order of $d \times d$ column blocks starting from the third according to τ_σ for convenience)

$$\begin{pmatrix} A_1(x,z) & C_1(x,z) & 0 & \dots & 0 \\ A_2(x,z) & 0 & C_2(x,z) & \dots & 0 \\ \dots & & & & \\ A_n(x,z) & 0 & 0 & \dots & C_n(x,z) \end{pmatrix}$$

We recall that we need to prove that for all $\lambda_j \geqslant 0$, not all zero, we can select linearly independent rows from this matrix according to multiindex I, such that

$$\sum_{j=1}^{2n} \lambda_j (\sum_{s:[i_s/d]=j} p_{i_s \, mod \, d}([j/2]) + 2 - Q(j)) > 0,$$

where $\lambda_{2j-1} = 0$ if $x_j \neq y_{\tau_\sigma(j)}$ and $\lambda_{2j} = 0$ if $y_{\tau_\sigma(j)} \neq y_{\tau_\sigma(j+n)}$. Note that if $y_{\tau_\sigma(j)} = y_{\tau_\sigma(j+n)}$ we have that $S_1' = -S_2'$ in the same row block of A_j and C_j. So for such j we can transform matrix to eliminate S_1' inside C_j (by adding second column block to the first inside C_j, which does not change the rest of the matrix). Moreover we can add the first column block to the second, and obtain that if $y_{\tau_\sigma(j)} = y_{\tau_\sigma(j+n)}$, then we have zeros in the second column block intersecting second row block inside A_j. This allows us to separate all second row blocks inside A_j for such j from the rest of the matrix: we may choose to select $l(d-k)+k$ linearly independent rows from these column blocks separately from the rest, where l is the numbers of such blocks. Note that those blocks, after we drop zero columns, will join into matrix $D_S(z_1,\dots,z_n)T_H(z_1,\dots,z_n)$, if T_H is bulit similarly to T using matrices G and L, with some row blocks skipped and zero columns removed (some rows in A_j may have different sign, but it can be changed easily, without impacting the linear independence of the selection). It means that according to our assumption we may select rows from it to get that the part of sum that corresponds to λ_{2j}, with j taken from the selection, is positive (we can set all other λ_j in the assumption (16) to zero). In the remainder of the matrix only one column block remain in A_j, and it contains $S_2'(j, x_j, y_{\tau_\sigma(j)})G_j(y_{\tau_\sigma(j)})$ in its first row block for all j. So now we can consider all first rows blocks from all A_j such that $x_j = y_{\tau_\sigma(j)}$, and determine that the selection from such blocks can also be done separately if we choose to select $l(d-k)+k$ rows from these l blocks. Moreover the submatrix for such selection is again has the form of $D_S(z_1,\dots,z_n)T_H(z_1,\dots,z_n)$, with some row blocks skipped and zero columns removed. It means that the part of the sum that corresponds to λ_{2j-1} for such j is also positive. But the remainder of the sum is zero and the remainder of the selection of linearly independent rows is always possible, so the Theorem is proved.

□

Remark 9. Note that the sufficient condition in Theorem 12 is much more complicated, then it was in Theorem 3. This is because we wanted to take advantage of possible interactions between f_j in the applications, so such interaction had to be present in our condition. It is possible to give a more simple sufficient condition with interaction removed:

$$\sum_{j\neq i} m_{f_j}(y_j) < \sum_{j=1}^{n} (\sum_{l=1}^{d} p_l(j) + 2 - Q(j)), i = 1,\dots,n \tag{17}$$

for all $(y_1,\dots,y_n) \in H \cap M$, which provides the condition (16) in Theorem 12 and is similar to the condition (6) in Theorem 3.

Remark 10. In Theorem 12 the starting point of the process and the choice of ψ do not play any role, in the sense that our conditions provide the existence for any starting points and any suitable ψ. We do not know if it is possible to find weaker assumptions than (16) such that they provide existence, but only for some starting points and/or for a special choice of ψ.

3.3. **Applications.** Some applications we presented for self-intersection local times, namely Theorems 5, 8, 9 has their counterparts for intersection local times of independent process, if we assume that all Y_i have the same transition density. We will not provide the details, only note that the condition (17) gives exactly the same conditions in all cases (except the conditions related to the multiplier ψ, which are not needed) for the existence of intersection local times of independent process, as we had for self-intersection local times.

Here we provide an interesting application for interaction between f_i in the condition (16) of Theorem 12 in the case of Brownian motion on Carnot group.

Theorem 13. *Let $n = 2$, $d = 3$ and processes Y_1, Y_2 are Brownian motions on Carnot group as defined by (12) with $l = 2$ and $L_1(x) = (1, 0, x_2)$, $L_2(x) = (0, 1, -x_1)$ for both processes. If $k = d = 3$ and at some point $z = (z_1, z_2) \in \mathbb{R}^3$ such that $f_1(z_1) = f_2(z_2)$ the vectors $L_1f_1(z_1)$, $L_2f_1(z_1)$, $L_1f_2(z_2)$, $L_1f_2(z_2)$ span the whole $\mathbb{R}^3$, then there is $\delta > 0$, such that for any ψ with $\operatorname{supp}\psi \subset \{(y_1, y_2) : |y_1 - z_1| < \delta, |y_2 - z_2| < \delta\}$ and $\psi(z_1, z_2) > 0$ there is a nonzero limit of $\gamma_\varepsilon(\psi, F, [0, 1]^2)$ in L_2.*

Proof. Using Theorem 12 we can see that we only need to show that our conditions provide the condition (16). We take $S(x, y) = x^{-1} \bullet y$, where $\bullet$ is a group action on $\mathbb{R}^3$ and $x^{-1} = -x$ is the inverse element to x in this group:

$$x \bullet y = (x_1+y_1, x_2+y_2, x_3+y_3+x_2y_1-x_1y_2), x^{-1} \bullet y = (y_1-x_1, y_2-x_2, y_3-x_3+x_1y_2-x_2y_1),$$

Then the results of [13] show that the inequality (14) holds with such S and $p_1 = p_2 = 1$, $p_3 = 2$, $Q = 4$. But for this S we obtain

$$D_S(z_1, z_2) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -(z_1)_2 & (z_1)_1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -(z_2)_2 & (z_2)_1 & 1 \end{pmatrix}$$

and since we can also define (in terms of 3×3 blocks)

$$T_H(z_1, z_2) = \begin{pmatrix} f_1'(z_1)^{-1} \\ f_2'(z_2)^{-1} \end{pmatrix}$$

then (again in terms of 3×3 blocks)

$$D_S(z_1, z_2)T_H(z_1, z_2) = \begin{pmatrix} \Lambda^{-1}(z_1)f_1'(z_1)^{-1} \\ \Lambda^{-1}(z_2)f_2'(z_2)^{-1} \end{pmatrix}$$

where we denoted

$$\Lambda(z_i) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ (z_i)_2 & -(z_i)_1 & 1 \end{pmatrix}, \Lambda^{-1}(z_i) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -(z_i)_2 & (z_i)_1 & 1 \end{pmatrix}$$

Note that $L_1f_1(z_1)$, $L_2f_1(z_1)$ are two first columns of $f_1'(z_1)\Lambda(z_1)$ and $L_1f_2(z_2)$, $L_1f_2(z_2)$ are two first columns of $f_2'(z_2)\Lambda(z_2)$. Consequently the row 3 of $D_S(z_1, z_2)T_H(z_1, z_2)$ is orthogonal to both $L_1f_1(z_1)$, $L_2f_1(z_1)$ and the row 6 of $D_S(z_1, z_2)T_H(z_1, z_2)$ is orthogonal to both $L_1f_2(z_2)$, $L_2f_2(z_2)$, meaning that these two rows are linearly independent under the conditions of the Theorem.

It follows that we can choose rows 3, 6 and one of the rows 1, 2 of $D_S(z_1, z_2)T_H(z_1, z_2)$, that they are linearly independent (if both rows 1 and 2 can be written as a linear combination of rows 3, 6, then the rows 1, 2, 3 are linearly dependent, which is a contradiction). This gives us coefficients 1, 0 near λ_1, λ_2 correspondingly in the condition (16) (the sum of p_i equal to 3 and 2 correspondingly). Similarly choosing rows 3, 6 and one of the rows 4, 5 we obtain coefficients 0, 1 near λ_1, λ_2 correspondingly, which proves the condition (16). $\square$

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