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# ALMOST SURE ASYMPTOTIC EXPANSIONS FOR PROFILES OF SIMPLY GENERATED RANDOM TREES

This paper is a continuation of the analysis of Edgeworth expansions for one-split branching random walk and its application to random trees. We provide new results for profile, mode and width for several simply generated random trees, in particular for random recursive trees, $p$-oriented recursive trees and $D$-ary random trees. Our results are corollaries of a general Edgeworth expansion for a one-split branching random walk proved by Kabluchko, Marynych and Sulzbach [*The Annals of Applied Probability* 27(6): 3478–3524, 2017]. We derive an additional characterization of the random variables appearing in the coefficients of the asymptotic expansions by calculating explicitly corresponding fixed-point equations of a branching type. We further provide numerical simulations justifying our theoretical findings.

## 1. Introduction

The main purpose of this paper is three-fold. First, we obtain asymptotic expansions for the profile, mode and width of simply generated random trees, such as $D$-ary recursive trees, random recursive trees and $p$-oriented recursive trees, when the number of vertices of those trees tends to infinity. This part of the paper consists mostly of corollaries of the theorems proved by Kabluchko, Marynych and Sulzbach, see [14], for the profile of a one-split branching random walk using mod-$\phi$ convergence technique. The reader can find rigorous definitions of this type of convergence and some general results in [9] and we also refer to [13] and [16] for examples of its usage. The second purpose of the paper is to provide a further characterization of the random coefficients of the aforementioned asymptotic expansions by providing explicit stochastic fixed-point equations characterizing their distributions. Last but not least, we provide numerical simulations supporting our theoretical results. Simply generated random trees and in particular the aforementioned random trees are of course well-studied. The obtained results will be compared in details to previously known limit theorems for profiles and related quantities. For better understanding of random trees and their applications we refer the reader to the book [6].

The paper has the following structure. In Section 2 we collect all necessary definitions. Section 3 contains our main results, almost sure asymptotic expansions for the profile, mode and width of simply generated random trees. Furthermore, in this section we provide a characterization of random variables appearing in the main asymptotic expansions via stochastic fixed-point equations. A comparison of our expansions with already known results is also contained in this section. Section 4 is devoted to numerical simulations and their comparison with our theoretical results.

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## 2. Random trees overview

2.1. **One-split branching random walk.** Consider a system of particles living on the lattice $\mathbb{Z}$. At time 0, the system contains a single particle usually located at 0. The system evolves in the following way: in each discrete moment of time one of the particles is chosen uniformly at random (if it exists). This particle (the mother) is replaced by a random cluster of particles (children) whose relative displacements are described by a point process $\zeta = \sum_{i=1}^{N} \delta_{Z_i}$. That is, if the mother was located at $x$ then its children are located at $x + Z_1, \dots, x + Z_N$. We assume that the number of particles $N$ is a.s. finite. All random mechanisms in the described process are independent. A sequence of point processes on $\mathbb{Z}$ describing the positions of particles throughout generations is called one-split branching random walk (one-split BRW).

Let us denote by $S_n$ the number of particles at time $n$, and by $x_{1,n}, \dots, x_{S_n,n}$ their positions. The profile of a one-split BRW at moment $n$ is the following function:

$$\mathbb{L}_n(k) = \#\{1 \le j \le S_n : x_{j,n} = k\}, \quad n \in \mathbb{N}, \quad k \in \mathbb{Z}, \tag{1}$$

that is, $\mathbb{L}_n(k)$ is the number of particles at site $k \in \mathbb{Z}$ at time $n$.

Denote by $\nu_k$ the expected number of particles at site $k \in \mathbb{Z}$ in the cluster process $\zeta$:

$$\nu_k = \mathbb{E}\zeta(\{k\}) = \mathbb{E}\left[\sum_{i=1}^{N} \mathbb{1}_{\{Z_i = k\}}\right], \quad k \in \mathbb{Z}. \tag{2}$$

Further, denote by $m(\beta)$ the moment generating function of the intensity of the cluster process $\zeta$ minus 1:

$$m(\beta) = \sum_{k \in \mathbb{Z}} e^{\beta k} \nu_k - 1 = \mathbb{E}\left[\sum_{i=1}^{N} e^{\beta Z_i}\right] - 1. \tag{3}$$

We assume that there exists an open interval $\mathscr{I}$ containing 0 where the function $m$ is finite. Then $m$ is well-defined for $\beta \in \{z \in \mathbb{C} : \operatorname{Re} z \in \mathscr{I}\}$ and is strictly convex and infinitely differentiable on $\mathscr{I}$. We define the function

$$\varphi(\beta) = \frac{m(\beta)}{m(0)}, \quad \operatorname{Re}\beta \in \mathscr{I}.$$

Denote by $(\beta_-, \beta_+) \subset \mathscr{I}$ the open interval where $\varphi'(\beta)\beta < \varphi(\beta)$:

$$\beta_- = \inf\{\beta \in \mathscr{I} : \varphi'(\beta)\beta < \varphi(\beta)\}, \tag{4}$$

$$\beta_+ = \sup\{\beta \in \mathscr{I} : \varphi'(\beta)\beta < \varphi(\beta)\}. \tag{5}$$

The interval $(\beta_-, \beta_+)$ contains 0 and, thus, is non-empty.

Let $\nu$ be the intensity measure of the point process $\zeta$:

$$\nu = \sum_{k \in \mathbb{Z}} \nu_k \delta_k. \tag{6}$$

The (normalized) moment-generating function of the one-split BRW profile is defined, for $\operatorname{Re}\beta \in \mathscr{I}$, by

$$W_n(\beta) = \frac{1}{n^{\varphi(\beta)}} \sum_{i=1}^{S_n} e^{\beta x_{i,n}}. \tag{7}$$

Under certain conditions (see Section 3) $W_n$ converges as $n \to \infty$ to a random analytic function $W_\infty$ with probability 1. Note that $W_\infty(0) = m(0)$ since $W_n(0) = S_n/n$.

We define the following random variables:

$$\chi_1(0) = (\log W_\infty)'(0) = \frac{W'_\infty(0)}{m(0)}, \tag{8}$$

$$\chi_2(0) = (\log W_\infty)''(0) = \frac{W''_\infty(0)m(0) - (W'_\infty(0))^2}{m^2(0)} \tag{9}$$

and, more generally, $\chi_j(\beta) = (\log W_\infty)^{(j)}(\beta)$ $(j \geq 1)$.

2.2. **Random trees models.** A simply generated random tree is a tree constructed by the following iterative process. There are nodes of two types: external and internal. At time 0 the tree usually contains only one external node, which is the root of the tree. At each discrete time moment an external node is chosen uniformly at random, is usually declared internal and a new finite (possibly random) subtree is attached to the chosen node. There is a natural procedure of encoding the evolution of a simply generated random tree using a one-split BRW. In this encoding the external nodes correspond to alive particles, internal nodes which have been chosen at some step, to dead particles, and the depth, that is, the distance to the root, of each node is the position of the corresponding particle.

2.2.1. *$D$-ary trees ($D \geq 2$).* At time 0 a $D$-ary tree contains only its root. At each discrete time moment, one random external node is picked and replaced by an internal node with $D$ children (external nodes) attached. In other words, all external nodes of $D$-ary tree are its leaves, and on each step $D$ sons are added to a random leaf. Functional limit theorems for profiles of those trees can be found in the paper [19].

FIGURE 1. A construction rule of a $D$-ary tree with $D = 3$ and an example of its realization at time $n = 3$.

In case of $D = 2$ the tree becomes a well-known binary search tree (BST), which can also be constructed using a uniformly distributed data input. It is one of the most studied type of random trees, see for example [1, 2, 5, 7, 8, 11] to name but a few.

Let us also mention that for $D > 2$ the notion of $D$-ary tree is **not** the same as the notion of $D$-ary search tree. For the comprehensive analysis of search trees we refer the reader to [6].

The evolution of a $D$-ary tree can be represented by a one-split BRW with the deterministic displacement point process $\zeta = D\delta_1$. We have

$$\varphi(\beta) = \frac{De^\beta - 1}{D-1}, \quad m(0) = D - 1, \quad \varphi^{(j)}(0) = \frac{D}{D-1}, \quad j \in \mathbb{N}.$$

The constants $\beta_-$ and $\beta_+$ are the solution to $De^\beta(1-\beta) = 1$.

2.2.2. *Random recursive trees.* A random recursive tree (RRT) is constructed using the following procedure. At time 0 RRT consists of one node, the root. At each discrete time moment a random existing node is chosen and a new node is attached to the chosen one. In this representation each node is considered as external. Some recenet results for RRT can be found in [5, 7, 11].

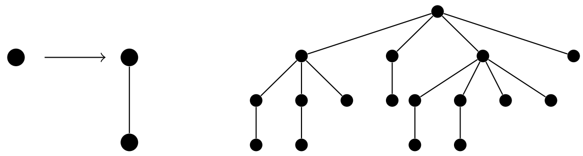

FIGURE 2. A rule for creating a RRT and an example of its realization at time $n = 16$.

A one-split BRW which corresponds to a RRT stems from the deterministic displacement point process $\zeta = \delta_0 + \delta_1$ for which

$$\varphi(\beta) = e^{\beta}, \quad m(0) = 1, \quad \varphi^{(j)}(0) = 1, j \in \mathbb{N}, \quad \beta_- = -\infty, \quad \beta_+ = 1.$$

2.2.3. *p-oriented recursive trees* ($p \geq 2$). A $p$-oriented recursive tree starts with an internal node (the root) and one external node attached to it. The construction rule is depicted on Figure 3 (left) for $p = 2$. For arbitrary $p = 2, 3, \ldots$ the rule is similar but with $p$ instead of two external nodes in the middle.

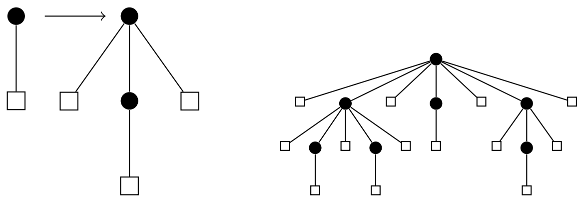

FIGURE 3. A rule for creating a $p$-oriented recursive tree with $p = 2$ and an example of its realization in a moment of time $n = 6$.

For $p = 2$ the tree becomes a well-known plane-oriented recursive tree (PORT). We refer to [12, 20, 15] for some recent results about profiles of PORTs.

One-split BRW which corresponds to a $p$-oriented recursive tree starts with an original particle located at $x_{1,0} = 1$ and has the deterministic displacement point process $\zeta = p\delta_0 + \delta_1$ for which

$$\varphi(\beta) = \frac{1}{p}(e^{\beta} + p - 1), \quad m(0) = p, \quad \varphi^{(j)}(0) = \frac{1}{p}, j \in \mathbb{N}, \quad \beta_- = -\infty$$

and $\beta_+$ is a solution of $e^{\beta}(\beta - 1) = p - 1$.

2.3. **Functionals on random trees.** There are many various functionals which can be defined on trees. In this paper we focus only on some of them. The depth (or level) of a node is by definition its distance to the root. Denote by $\mathbb{L}_n(k)$ the number of external nodes of the tree at time $n$ at depth $k$. This definition is equal (except for $p$-oriented trees) to (1) given on a one-split BRW corresponding to the random tree. This functional is called an external profile of the random tree (or occupation number) at level $k$.

The mode $u_n$ of a random tree is defined by

$$u_n = \arg\max_{k \in \mathbb{N}_0} \mathbb{L}_n(k), \quad n \in \mathbb{N}, \tag{10}$$

that is, $u_n$ is the level containing the largest number of external nodes. The width $M_n$ of a random tree is defined by the following formula

$$M_n = \max_{k \in \mathbb{N}_0} \mathbb{L}_n(k), \tag{11}$$

and is equal to the largest number of external nodes on the same level.

## 3. Results for random trees

We start with a lemma listing some properties of the one-split branching random walks corresponding to the aforementioned three types of random trees.

**Lemma 3.1.** *For the one-split BRWs corresponding to a $D$-ary tree, a RRT or a $p$-oriented recursive tree the following statements are true:*

(a) $\nu_k > 0$ *for at least one* $k \in \mathbb{Z}\backslash\{0\}$*;*

(b) *the cluster point process $\zeta$ is a.s. non-empty, and has at least 2 particles with positive probability;*

(c) *the function $m(\beta)$ is finite on some non-empty open interval $\mathscr{I}$ containing* 0*;*

(d) *$\nu$ is not concentrated on any proper additive subgroup of $\mathbb{Z}$;*

(e) *for any $\beta \in (\beta_-, \beta_+)$ there is $\gamma = \gamma(\beta) > 1$ such that*

$$\mathbb{E}\left[\left(\sum_{i=1}^{N} e^{\beta Z_i}\right)^{\gamma}\right] < \infty.$$

*Proof.* *(a)* For each process there is at least one particle $\delta_1$.

*(b)* For each process there are at least 2 particles (exactly 2 for a RRT, $D$ for a $D$-ary tree and $p+1$ for a $p$-oriented recursive tree).

*(c)* The function $m(\beta)$ is a finite sum of exponents, it's finite on any finite interval $\mathscr{I}$.

*(d)* For each process there is at least one particle $\delta_1$.

*(e)* For each process the sum under expectation is a.s. bounded by a non-random finite constant; $\gamma$ can be taken, for example, as $\gamma = 2$. ☐

We state this lemma for one-split BRWs introduced above but it holds also for a much larger class of one-split BRWs. One of the results which follows from Lemma 3.1 is existence of $W_\infty(\beta)$ such that $W_n(\beta) \xrightarrow[n\to\infty]{a.s.} W_\infty(\beta)$ on an open neighborhood in $\mathbb{C}$ of interval $(\beta_-, \beta_+)$, see [2, 14]. From now on we assume that the argument $\beta$ lies in this open set, where $W_\infty(\beta)$ is well-defined.

The following result provides a characterization of the random variables $W_\infty(\beta)$ and $\chi_j(\beta)$ via stochastic fixed-point equations.

**Theorem 3.2.** *For a one-split BRW such that assumptions of Lemma 3.1 hold and the displacement point process $\zeta$ is deterministic, a random variable $W_\infty(\beta)$ satisfies the stochastic fixed-point equation*

$$W_\infty(\beta) \stackrel{\mathrm{d}}{=} \sum_{i=1}^{N} e^{\beta Z_i} U_i^{\varphi(\beta)} W_\infty^{(i)}(\beta) \tag{12}$$

*where $W_\infty^{(i)}(\beta)$ are independent distributional copies of $W_\infty(\beta)$ and an independent random vector $(U_i)_{i=1}^N$ has a Dirichlet distribution with parameter $\frac{1}{N-1}$.*

The following proof relies on a notion of the Polya-Eggenberger Urn model. In this urn model there is a single urn containing balls of $N$ different colors. The urn evolves in discrete time. At each step we pick a ball from the urn uniformly at random, note the color of the ball $i \in \overline{1,N}$ and then return it back to the urn. Depending on the color $i$ of the drawn ball we also add $a_{i,j}$ balls of color $j \in \overline{1,N}$ into the urn. The replacement scheme is usually represented as matrix $A = (a_{i,j})_{i,j=1}^N$. The values $a_{i,j}$ can be zeros or negative. For example, if $a_{2,4} = -3$, then we pull 3 balls of color 4 from the urn when the ball having color 2 was drawn. The Polya-Eggenberger Urn is called *tenable* if it's possible to continue the urn evolving process indefinitely on every possible stochastic path. If all $a_{i,j}$ are not negative, then the corresponding urn model is obviously tenable. However, if some $a_{i,j}$ is negative there might be a situation when a replacement rule requires to pick $|a_{i,j}|$ balls of color $j$ which are not contained in the urn. Tenable urns and properties of tenability among most studied for the Polya-Eggenberger Urn model. For more information about urn models and some classic results we refer the reader to the book [17].

*Proof of Theorem 3.2.* The fact that the point process $\zeta$ is deterministic means that the number of particles $N$ as well as their positions $Z_i$ are constants. From Lemma 3.1 it readily follows that $N > 1$.

At moment $n = 0$ there is only one particle located at $x_{1,0}$. At moment $n = 1$ it splits into $N$ particles located at $x_{1,0} + Z_1, \dots, x_{1,0} + Z_N$. Each particle now evolves independently and its evolution is the same as evolution of the original particle.

Consider the one-split BRW at moment of time $n+1$ ($n > 0$). We can rewrite (7) by using values $W_k^{(i)}(\beta)$ ($k \in \overline{1,n}$) of subprocesses on particles created after the first split event as follows

$$W_{n+1}(\beta) \stackrel{\text{d}}{=} \frac{1}{(n+1)^{\varphi(\beta)}} \sum_{i=1}^{N} (e^{\beta Z_i} (V_n^{(i)})^{\varphi(\beta)} W^{(i)}_{V_n^{(i)}}(\beta) \mathbb{1}\{V_n^{(i)} > 0\} + e^{\beta(x_{1,0}+Z_i)} \mathbb{1}\{V_n^{(i)} = 0\}), \tag{13}$$

where $V_n^{(i)}$ is the number of split events occurred in the subprocess created on particle $x_{1,0} + Z_i$ (which was created after first split) and since $n$ split events happened in total:

$$\sum_{i=1}^{N} V_n^{(i)} = n. \tag{14}$$

By symmetry all $V_n^{(i)}$ have the same distribution for $i = 1, \dots, N$, since the subprocesses are independent and identically distributed. Also $V_n^{(i)} \xrightarrow[n\to\infty]{a.s.} \infty$ and thus $W^{(i)}_{V_n^{(i)}}(\beta) \xrightarrow[n\to\infty]{a.s.} W_\infty^{(i)}(\beta)$ for $i = 1, \dots, N$.

It only remains to find the limit distribution of $V_n^{(i)}/(n+1)$ as $n$ goes to infinity. It can be rewritten in the following way:

$$\lim_{n\to\infty} \frac{V_n^{(i)}}{n+1} = \lim_{n\to\infty} \frac{V_n^{(i)}}{n} = \lim_{n\to\infty} \frac{V_n^{(i)}(N-1)}{n(N-1)} = \lim_{n\to\infty} \frac{1 + V_n^{(i)}(N-1)}{N + n(N-1)} = = \lim_{n\to\infty} \frac{S_n^{(i)}}{S_n^*} \tag{15}$$

where $S_n^{(i)}$ is a sum of particles in the $i$-th subprocess and $S_n^* = S_{n+1}$ – the sum of particles in all subprocesses.

Note that each split event occurs by picking a random particle among all present particles, then $N-1$ particles are added to a subprocess corresponding to the picked particle. It means that the probability that the split event occurs in a $i$-th subprocess is proportional to the current number of particles in in this subprocess. The above process can be described in terms of a Polya-Eggenberger Urn model with a single ball of each color present at the beginning and with the replacement scheme $A = (a_{i,j})_{i,j=1}^N$ with $a_{i,i} = N - 1$ and $a_{i,j} = 0$ for $i \neq j$. Here the balls of color $i$ correspond to particles in the $i$-th subprocess. It is a tenable urn model for which Gouet provided the following result, see [10],

$$\lim_{n\to\infty} \frac{S_n^{(i)}}{S_n^*} = U_i \tag{16}$$

where vector $(U_i)_{i=1}^N$ has the Dirichlet distribution with parameter $\frac{1}{N-1}$.

The equation (12) follows by combining the above formulas. □

*Remark* 3.3. A random vector $(x_1, x_2, \ldots, x_N)$ has the Dirichlet distribution with parameter $\alpha > 0$ if $x_i \geq 0$, $i = 1, \ldots, N$, $\sum_{i=1}^N x_i = 1$ and the probability density function is given by

$$f(x_1, \ldots, x_N, \alpha) = \frac{\Gamma(\alpha N)}{\Gamma(\alpha)^N} \prod_{i=1}^N x_i^{\alpha-1}, \tag{17}$$

where $\Gamma(\alpha)$ is the gamma function.

**Corollary 3.4.** *Under the assumptions of Theorem 3.2 the random variable $\chi_1(0)$ satisfies the stochastic fixed-point equation*

$$\chi_1(0) \stackrel{\mathrm{d}}{=} \sum_{i=1}^N (Z_i U_i + \varphi'(0) U_i \log U_i + U_i \chi_1^{(i)}(0)) \tag{18}$$

*where $\chi_1^{(i)}(0)$ are independent distributional copies of $\chi_1(0)$ and an independent random vector $(U_i)_{i=1}^N$ has the Dirichlet distribution with parameter $\frac{1}{N-1}$.*

*Proof.* Recall that $W_\infty(0) = m(0)$ and $\varphi(0) = 1$. Taking the logarithmic derivative in (12) and plugging $\beta = 0$ we obtain

$$\frac{W'_\infty(0)}{m(0)} = \frac{\sum_{i=1}^N (Z_i U_i m(0) + \varphi'(0) U_i \log U_i m(0) + U_i W'_\infty(0))}{\sum_{i=1}^N U_i m(0)}. \tag{19}$$

From the definition (8) of $\chi_1(0)$ it follows $W'_\infty(0) = m(0)\chi_1(0)$. By the definition of a Dirichlet distribution we have $\sum_{i=1}^N U_i = 1$. Combining these equations we derive (18). □

*Remark* 3.5. By the same reasoning as in the proof of Corollary 3.4 one can derive similar equations also for $\chi_j(\beta)$.

**Example 3.6.** Let us calculate the above quantities for a BST, that is for a $D$-ary tree with $D = 2$, see Section 2.2. From Theorem 3.2 we obtain the stochastic fixed-point equation for BST:

$$W_\infty(\beta) \stackrel{\mathrm{d}}{=} e^\beta U_1^{2e^\beta - 1} W_\infty^{(1)}(\beta) + e^\beta U_2^{2e^\beta - 1} W_\infty^{(2)}(\beta). \tag{20}$$

The vector $(U_1, U_2)$ has the Dirichlet distribution with parameter 1. It can be written as $(\frac{Y_1}{Y_1+Y_2}, \frac{Y_2}{Y_1+Y_2})$, where $Y_1$, $Y_2$ are independent random variables with the standard exponential distribution, see [4]. We have for $x \in [0, 1]$:

$$\begin{aligned} \mathbb{P}\left\{\frac{Y_1}{Y_1+Y_2} \leq x\right\} &= \mathbb{P}\left\{Y_1 \leq \frac{xY_2}{1-x}\right\} = \int_0^\infty e^{-y_2} \int_0^{\frac{xy_1}{1-x}} e^{-y_1} \mathrm{d}y_1 \mathrm{d}y_2 = \\ &= \int_0^\infty e^{-y_2}\left(1 - e^{\frac{xy_2}{x-1}}\right) \mathrm{d}y_2 = x. \end{aligned} \tag{21}$$

It means that the random variable $\frac{Y_1}{Y_1+Y_2}$ has the standard uniform distribution on $[0, 1]$, thus the vector $(\frac{Y_1}{Y_1+Y_2}, \frac{Y_2}{Y_1+Y_2})$ has the same distribution as the vector $(U, 1-U)$ with $U$ being uniformly distributed on $[0, 1]$.

Thus, we can rewrite (20) as follows

$$W_\infty(\beta) \overset{\mathrm{d}}{=} e^\beta U^{2e^\beta-1} W_\infty^{(1)}(\beta) + e^\beta (1-U)^{2e^\beta-1} W_\infty^{(2)}(\beta). \tag{22}$$

By Corollary 3.4 we also have the stochastic fixed-point equation

$$\chi_1(0) \overset{\mathrm{d}}{=} 1 + 2U\log U + U\chi_1^{(1)}(0) + 2(1-U)\log(1-U) + (1-U)\chi_2^{(1)}(0). \tag{23}$$

It is a well-known fixed-point equation for the Quicksort law, see [18], which means that $\chi_1(0)$ has the Quicksort distribution (up to an additive constant).

### 3.1. **Results for external profiles.**

**Theorem 3.7.** *Let $\mathbb{L}_n(k)$ $(k \in \mathbb{Z})$ be the external profile of a $D$-ary tree with $(D-1)n+1$ external nodes. For every $r \in \mathbb{N}_0$ we have*

$$(\log n)^{\frac{r+1}{2}} \sup_{k\in\mathbb{Z}} \left| \frac{1}{n}\mathbb{L}_n(k) - H_n(k) \sum_{j=0}^{r} G_j\left( \frac{k - \frac{D}{D-1}\log n}{\sqrt{\frac{D}{D-1}\log n}}; 0 \right) \frac{1}{(\log n)^{j/2}} \right| \xrightarrow[n\to\infty]{a.s.} 0, \tag{24}$$

*where $G_j$ is given in Remark 3.10 below, and*

$$H_n(k) = \frac{(D-1)^{3/2}}{\sqrt{2D\pi\log n}} e^{-\frac{(k-\frac{D}{D-1}\log n)^2}{\frac{2D}{D-1}\log n}}.$$

This theorem can be easily compared to the following central limit theorem which was proved by Chauvin, Drmota and Jabbour-Hattab for BST, see [1],

$$\sup_{k\in\mathbb{Z}} \left| \frac{1}{n}\mathbb{L}_n(k) - \frac{1}{\sqrt{4\pi\log n}} e^{-\frac{(k-2\log n)^2}{4\log n}} \right| = O\left(\frac{1}{\log n}\right) \quad \text{a.s.} \tag{25}$$

This result follows from Theorem 3.7 with $D = 2$ and $r = 0$, however the aforementioned theorem provides a much more accurate result.

**Theorem 3.8.** *Let $\mathbb{L}_n(k)$ $(k \in \mathbb{Z})$ be the external profile of a random recursive tree with $n+1$ nodes. For every $r \in \mathbb{N}_0$ we have*

$$(\log n)^{\frac{r+1}{2}} \sup_{k\in\mathbb{Z}} \left| \frac{1}{n}\mathbb{L}_n(k) - \frac{e^{-\frac{(k-\log n)^2}{2\log n}}}{\sqrt{2\pi\log n}} \sum_{j=0}^{r} G_j\left( \frac{k-\log n}{\sqrt{\log n}}; 0 \right) \frac{1}{(\log n)^{j/2}} \right| \xrightarrow[n\to\infty]{a.s.} 0, \tag{26}$$

*where $G_j$ is given in Remark 3.10 below.*

**Theorem 3.9.** *Let $\mathbb{L}_n(k)$ $(k \in \mathbb{Z})$ be the external profile of a $p$-oriented recursive tree with $pn+1$ external nodes. For every $r \in \mathbb{N}_0$ we have*

$$(\log n)^{\frac{r+1}{2}} \sup_{k\in\mathbb{Z}} \left| \frac{1}{n}\mathbb{L}_n(k) - H_n(k) \sum_{j=0}^{r} G_j\left( \frac{k - 1 - \frac{1}{p}\log n}{\sqrt{\log n/p}}; 0 \right) \frac{1}{(\log n)^{j/2}} \right| \xrightarrow[n\to\infty]{a.s.} 0, \tag{27}$$

*where $G_j$ is given in Remark 3.10 below and*

$$H_n(k) = \frac{p^{3/2}}{\sqrt{2\pi\log n}} e^{-\frac{p(k-1-\frac{1}{p}\log n)^2}{2\log n}}.$$

*Proof of theorems 3.7, 3.8, 3.9.* These results are corollaries from Theorem 3.11 in [14] for one-split BRWs corresponding to random trees as described in Section 2.2 and Lemma 3.1. □

*Remark* 3.10. $G_j(x;\beta)$ is a polynomial of degree at most $3j$ given by

$$G_j(x;\beta) = \frac{(-1)^j}{j!} e^{\frac{1}{2}x^2} B_j(D_1, \dots, D_j) e^{\frac{1}{2}x^2}, \tag{28}$$

where $B_j$ is the $j$-th (exponential) complete Bell polynomial, see [3, Chapter 3.3], and $D_1, D_2, \dots$ are differential operators with random coefficients given by

$$D_j = \frac{\varphi^{(j+2)}(\beta)}{(j+1)(j+2)} \left( \frac{1}{\sqrt{\varphi''(\beta)}} \frac{\mathrm{d}}{\mathrm{dx}} \right)^{j+2} + \chi_j(\beta) \left( \frac{1}{\sqrt{\varphi''(\beta)}} \frac{\mathrm{d}}{\mathrm{dx}} \right)^{j}. \tag{29}$$

In particular,

$$G_0(x;0) = 1. \tag{30}$$

**Theorem 3.11.** *Let $\mathbb{L}_n(k)$ $(k \in \mathbb{Z})$ be the external profile of a $D$-ary tree with $(D-1)n+1$ external nodes. For every $r \in \mathbb{N}_0$ and every compact set $L \subset (\frac{D}{D-1}e^{\beta_-}, \frac{D}{D-1}e^{\beta_+})$ we have*

$$(\log n)^{r+1} \sup_{k \in \mathbb{Z} \cap (\log n)L} \left| n^{\frac{1}{D-1}} \left( \frac{(D-1)k}{De \log n} \right)^k \mathbb{L}_n(k) - \frac{1}{\sqrt{2\pi k}} \sum_{j=0}^{r} \frac{F_{2j}(0; \log \frac{(D-1)k}{D \log n})}{(\log n)^j} \right| \xrightarrow[n\to\infty]{a.s.} 0,$$

*where $F_{2j}$ is given in Remark 3.14*

**Theorem 3.12.** *Let $\mathbb{L}_n(k)$ $(k \in \mathbb{Z})$ be the external profile of a random recursive tree with $n+1$ nodes. For every $r \in \mathbb{N}_0$ and every compact set $L \subset (0, e)$ we have*

$$(\log n)^{r+1} \sup_{k \in \mathbb{Z} \cap (\log n)L} \left| \left( \frac{k}{e \log n} \right)^k \mathbb{L}_n(k) - \frac{1}{\sqrt{2\pi k}} \sum_{j=0}^{r} \frac{F_{2j}(0; \log \frac{k}{\log n})}{(\log n)^j} \right| \xrightarrow[n\to\infty]{a.s.} 0,$$

*where $F_{2j}$ is given in Remark 3.14.*

**Theorem 3.13.** *Let $\mathbb{L}_n(k)$ $(k \in \mathbb{Z})$ be the external profile of a $p$-oriented recursive tree with $pn+1$ external nodes. For every $r \in \mathbb{N}_0$ and every compact set $L \subset (0, \frac{1}{p}e^{\beta_+})$ we have*

$$(\log n)^{r+1} \sup_{k-1 \in \mathbb{Z} \cap (\log n)L} \left| n^{\frac{1-p}{p}} \left( \frac{p(k-1)}{e \log n} \right)^{k-1} \mathbb{L}_n(k) - \sum_{j=0}^{r} \frac{F_{2j}(0; \log \frac{p(k-1)}{\log n})}{\sqrt{2\pi(k-1)}(\log n)^j} \right| \xrightarrow[n\to\infty]{a.s.} 0,$$

*where $F_{2j}$ is given in Remark 3.14.*

*Proof of Theorems 3.11, 3.12, 3.13.* These results follow from Theorem 3.15 in [14] for one-split BRWs corresponding to random trees as described in Section 2.2 and Lemma 3.1. □

*Remark* 3.14. We have $F_{2j}(0;\beta) = W_\infty(\beta) G_{2j}(0;\beta)$ and $F_{2j}(0;\beta)$ is a linear combination of $1, W_\infty(\beta), \dots, W_\infty^{(2j)}(\beta)$. In particular,

$$F_0(0;\beta) = W_\infty(\beta). \tag{31}$$

**Theorem 3.15.** *Let $\mathbb{L}_n(k)$ $(k \in \mathbb{Z})$ be the external profile of a $D$-ary tree with $(D-1)n+1$ external nodes. Put $\mathbb{L}_n^\circ(k) := \mathbb{L}_n(k) - \mathbb{E}[\mathbb{L}_n(k)]$ and let $(k_n)_{n \in \mathbb{N}}$ be a deterministic integer sequence.*

(a) *If $k_n = \frac{D}{D-1}\log n + \alpha\sqrt{\frac{D}{D-1}\log n} + o(\sqrt{\log n})$ for some $\alpha \in \mathbb{R}$, then*

$$\frac{\log n}{n} \mathbb{L}_n^\circ(k_n) \xrightarrow[n\to\infty]{a.s.} \frac{(D-1)^2 \alpha e^{-\frac{1}{2}\alpha^2}}{D\sqrt{2\pi}} (\chi_1(0) - \mathbb{E}[\chi_1(0)]).$$

(b) *If $k_n = \frac{D}{D-1}\log n + c_n$ where $c_n = o(\log n)$ and $\lim_{n\to\infty} |c_n| = \infty$, then*

$$\frac{(\log n)^{3/2}}{n c_n e^{c_n}} \left( \frac{(D-1)k_n}{D \log n} \right)^{k_n} \mathbb{L}_n^\circ(k_n) \xrightarrow[n\to\infty]{a.s.} \frac{(D-1)(\chi_1(0) - \mathbb{E}[\chi_1(0)])}{(\frac{D}{D-1})^{3/2}\sqrt{2\pi}}.$$

*In particular, if* $c_n = o(\sqrt{\log n})$ *and* $\lim_{n\to\infty} |c_n| = \infty$*, then*

$$\frac{(\log n)^{3/2}}{nc_n}\mathbb{L}_n^\circ(k_n) \xrightarrow[n\to\infty]{a.s.} \frac{(D-1)(\chi_1(0) - \mathbb{E}[\chi_1(0)])}{(\frac{D}{D-1})^{3/2}\sqrt{2\pi}}.$$

(c) *If* $k_n = \frac{D}{D-1}\log n + c_n$ *where* $c_n = O(1)$*, then*

$$\frac{(\log n)^{3/2}}{n}\mathbb{L}_n^\circ(k_n) - \frac{D-1}{(\frac{D}{D-1})^{3/2}\sqrt{2\pi}}(R(c_n) - \mathbb{E}[R(c_n)]) \xrightarrow[n\to\infty]{a.s.} 0,$$

*where* $R(c_n) = (c_n + \frac{1}{2})\chi_1(0) - \frac{\chi_1^2(0) - \chi_2(0)}{2}$.

**Theorem 3.16.** *Let* $\mathbb{L}_n(k)$ $(k \in \mathbb{Z})$ *be the external profile of a random recursive tree with* $n+1$ *nodes. Put* $\mathbb{L}_n^\circ(k) := \mathbb{L}_n(k) - \mathbb{E}[\mathbb{L}_n(k)]$ *and let* $(k_n)_{n\in\mathbb{N}}$ *be a deterministic integer sequence.*

(a) *If* $k_n = \log n + \alpha\sqrt{\log n} + o(\sqrt{\log n})$ *for some* $\alpha \in \mathbb{R}$*, then*

$$\frac{\log n}{n}\mathbb{L}_n^\circ(k_n) \xrightarrow[n\to\infty]{a.s.} \frac{\alpha e^{-\frac{1}{2}\alpha^2}}{\sqrt{2\pi}}(\chi_1(0) - \mathbb{E}[\chi_1(0)]).$$

(b) *If* $k_n = \log n + c_n$ *where* $c_n = o(\log n)$ *and* $\lim_{n\to\infty} |c_n| = \infty$*, then*

$$\frac{(\log n)^{3/2}}{nc_n e^{c_n}}\left(\frac{k_n}{\log n}\right)^{k_n}\mathbb{L}_n^\circ(k_n) \xrightarrow[n\to\infty]{a.s.} \frac{\chi_1(0) - \mathbb{E}[\chi_1(0)]}{\sqrt{2\pi}}.$$

*In particular, if* $c_n = o(\sqrt{\log n})$ *and* $\lim_{n\to\infty} |c_n| = \infty$*, then*

$$\frac{(\log n)^{3/2}}{nc_n}\mathbb{L}_n^\circ(k_n) \xrightarrow[n\to\infty]{a.s.} \frac{\chi_1(0) - \mathbb{E}[\chi_1(0)]}{\sqrt{2\pi}}.$$

(c) *If* $k_n = \log n + c_n$ *where* $c_n = O(1)$*, then*

$$\frac{(\log n)^{3/2}}{n}\mathbb{L}_n^\circ(k_n) - \frac{1}{\sqrt{2\pi}}(R(c_n) - \mathbb{E}[R(c_n)]) \xrightarrow[n\to\infty]{a.s.} 0,$$

*where* $R(c_n) = (c_n + \frac{1}{2})\chi_1(0) - \frac{\chi_1^2(0) - \chi_2(0)}{2}$.

**Theorem 3.17.** *Let* $\mathbb{L}_n(k)$ $(k \in \mathbb{Z})$ *be the external profile of a* $p$*-oriented recursive tree with* $pn+1$ *external nodes. Put* $\mathbb{L}_n^\circ(k) := \mathbb{L}_n(k) - \mathbb{E}[\mathbb{L}_n(k)]$ *and let* $(k_n)_{n\in\mathbb{N}}$ *be a deterministic integer sequence.*

(a) *If* $k_n = \frac{1}{p}\log n + 1 + \alpha\sqrt{\frac{\log n}{p}} + o(\sqrt{\log n})$ *for some* $\alpha \in \mathbb{R}$*, then*

$$\frac{\log n}{n}\mathbb{L}_n^\circ(k_n) \xrightarrow[n\to\infty]{a.s.} \frac{p^2\alpha e^{-\frac{1}{2}\alpha^2}}{\sqrt{2\pi}}(\chi_1(0) - \mathbb{E}[\chi_1(0)]).$$

(b) *If* $k_n = \frac{1}{p}\log n + 1 + c_n$ *where* $c_n = o(\log n)$ *and* $\lim_{n\to\infty} |c_n| = \infty$*, then*

$$\frac{(\log n)^{3/2}}{nc_n e^{c_n}}\left(\frac{pk_n}{\log n}\right)^{k_n}\mathbb{L}_n^\circ(k_n) \xrightarrow[n\to\infty]{a.s.} \frac{p^{5/2}(\chi_1(0) - \mathbb{E}[\chi_1(0)])}{\sqrt{2\pi}}.$$

*In particular, if* $c_n = o(\sqrt{\log n})$ *and* $\lim_{n\to\infty} |c_n| = \infty$*, then*

$$\frac{(\log n)^{3/2}}{nc_n}\mathbb{L}_n^\circ(k_n) \xrightarrow[n\to\infty]{a.s.} \frac{p^{5/2}(\chi_1(0) - \mathbb{E}[\chi_1(0)])}{\sqrt{2\pi}}.$$

(c) *If* $k_n = \frac{1}{p}\log n + 1 + c_n$ *where* $c_n = O(1)$, *then*

$$\frac{(\log n)^{3/2}}{n}\mathbb{L}_n^{\circ}(k_n) - \frac{p^{5/2}}{\sqrt{2\pi}}(R(c_n) - \mathbb{E}[R(c_n)]) \xrightarrow[n\to\infty]{a.s.} 0,$$

*where* $R(c_n) = (c_n + \frac{1}{2})\chi_1(0) - \frac{\chi_1^2(0)-\chi_2(0)}{2}$.

*Proof of Theorems 3.15, 3.16, 3.17.* These results are corollaries of Theorem 3.25 in [14] for one-split BRWs corresponding to random trees as described in Section 2.2 and Lemma 3.1. □

*Remark* 3.18. Note that all results for $p$-oriented recursive trees are shifted by $x_{1,0} = 1$ which is the position of the original particle of the corresponding one-split BRW.

### 3.2. Results for mode.

**Theorem 3.19.** *There is an a.s. finite random variable* $K$ *such that for* $n > K$, *the mode* $u_n$ *of a* $D$*-ary tree with* $(D-1)n+1$ *external nodes is equal to one of the numbers*

$$\lfloor \frac{D}{D-1}\log n + \chi_1(0) - \frac{1}{2} \rfloor \text{ or } \lceil \frac{D}{D-1}\log n + \chi_1(0) - \frac{1}{2} \rceil.$$

**Theorem 3.20.** *There is an a.s. finite random variable* $K$ *such that for* $n > K$, *the mode* $u_n$ *of a random recursive tree with* $n+1$ *nodes is equal to one of the numbers*

$$\lfloor \log n + \chi_1(0) - \frac{1}{2} \rfloor \text{ or } \lceil \log n + \chi_1(0) - \frac{1}{2} \rceil.$$

**Theorem 3.21.** *There is an a.s. finite random variable* $K$ *such that for* $n > K$, *the mode* $u_n$ *of a* $p$*-oriented recursive tree with* $pn+1$ *external nodes is equal to one of the numbers*

$$\lfloor \frac{1}{p}\log n + \chi_1(0) + \frac{1}{2} \rfloor \text{ or } \lceil \frac{1}{p}\log n + \chi_1(0) + \frac{1}{2} \rceil.$$

*Proof of Theorems 3.19, 3.20, 3.21.* These results are corollaries of Theorem 3.17 in [14] for one-split BRWs corresponding to random trees as described in Section 2.2 and Lemma 3.1. □

### 3.3. Results for width.

**Theorem 3.22.** *Let* $M_n$ *be the width of a* $D$*-ary tree with* $(D-1)n+1$ *external nodes, then it satisfies*

$$\frac{\sqrt{2D\pi\log n}}{(D-1)^{3/2}n}M_n \xrightarrow[n\to\infty]{a.s.} 1.$$

**Theorem 3.23.** *Let* $M_n$ *be the width of a random recursive tree with* $n+1$ *nodes, then it satisfies*

$$\frac{\sqrt{2\pi\log n}}{n}M_n \xrightarrow[n\to\infty]{a.s.} 1.$$

The above result is known in the literature, see [7]. However, it was proved by a completely different technique.

**Theorem 3.24.** *Let* $M_n$ *be the width of a* $p$*-oriented recursive tree with* $pn+1$ *external nodes, then it satisfies*

$$\frac{\sqrt{2\pi\log n}}{p^{3/2}n}M_n \xrightarrow[n\to\infty]{a.s.} 1.$$

*Proof of theorems 3.22, 3.23, 3.24.* These results are corollaries of Theorem 3.20 in [14] for one-split BRWs corresponding to random trees as described in Section 2.2, and Lemma 3.1. □

**Theorem 3.25.** *Let $M_n$ be the width of a $D$-ary tree with $(D-1)n+1$ external nodes. With probability 1, the set of subsequential limits of the sequence*

$$\tilde{M}_n := 2\frac{D}{D-1}\log n\left(1 - \frac{\sqrt{2\pi\log n}\sqrt{\frac{D}{D-1}}M_n}{(D-1)n}\right), \quad n \in \mathbb{N},$$

*is the interval $[\chi_2(0) - 1/12, \chi_2(0) + 1/6]$. Furthermore, with $\theta_n = \min_{k\in\mathbb{Z}}|\frac{D}{D-1}\log n + \chi_1(0) - 1/2 - k|$ we have*

$$\tilde{M}_n - \theta_n^2 \xrightarrow[n\to\infty]{a.s.} \chi_2(0) - \frac{1}{12}.$$

**Theorem 3.26.** *Let $M_n$ be the width of a random recursive tree with $n+1$ external nodes. With probability 1, the set of subsequential limits of the sequence*

$$\tilde{M}_n := 2\log n\left(1 - \frac{\sqrt{2\pi\log n}M_n}{n}\right), \quad n \in \mathbb{N},$$

*is the interval $[\chi_2(0)-1/12, \chi_2(0)+1/6]$. Furthermore, with $\theta_n = \min_{k\in\mathbb{Z}}|\log n+\chi_1(0)-1/2-k|$ satisfied:*

$$\tilde{M}_n - \theta_n^2 \xrightarrow[n\to\infty]{a.s.} \chi_2(0) - \frac{1}{12}.$$

**Theorem 3.27.** *Let $M_n$ be the width of a $p$-oriented recursive tree with $pn+1$ external nodes. With probability 1, the set of subsequential limits of the sequence*

$$\tilde{M}_n := \frac{2}{p}\log n\left(1 - \frac{\sqrt{2\pi\log n}M_n}{p^{3/2}n}\right), \quad n \in \mathbb{N},$$

*is the interval $[\chi_2(0) - 1/12, \chi_2(0) + 1/6]$. Furthermore, with $\theta_n = \min_{k\in\mathbb{Z}}|\frac{1}{p}\log n + \chi_1(0) - 1/2 - k|$ we have*

$$\tilde{M}_n - \theta_n^2 \xrightarrow[n\to\infty]{a.s.} \chi_2(0) - \frac{1}{12}.$$

*Proof of theorems 3.25, 3.26, 3.27.* These results are corollaries of Theorem 3.21 in [14] for one-split BRWs corresponding to random trees as described in Section 2.2 and Lemma 3.1. □

## 4. Simulations

The results presented in Section 3 provide an almost sure convergence which means that the corresponding asymptotic relation must hold for a.s. every realization of a random tree. Being results of pure theoretical nature, nevertheless it would be interesting to compare them with real-world data. Simulations presented in this chapter are performed for a single growing tree when $n$ varies. Furthermore, the simulations were performed according to the procedure described in Section 2.2.

Let us consider first a 3-ary tree. From Theorem 3.7 with $r = 0$ we get for every $k \in \mathbb{Z}$:

$$\mathbb{L}_n(k) \xrightarrow[n\to\infty]{a.s.} \mathbb{L}_n^*(k) = \frac{2n}{\sqrt{3\pi\log n}}e^{-\frac{(k-\frac{3}{2}\log n)^2}{3\log n}}. \tag{32}$$

From Theorem 3.19 we get that mode $u_n$ for large $n$ should be equal to $\lfloor u_n^*\rfloor$ or $\lceil u_n^*\rceil$, where

$$u_n^* = \frac{3}{2}\log n - \frac{1}{2} \tag{33}$$

and the random variable $\chi_1(0)$ is taken as 0 without computing.

From Theorem 3.22 we obtain that $M_n/M_n^* \xrightarrow[n\to\infty]{a.s.} 1$ where

$$M_n^* = \frac{2n}{\sqrt{3\pi \log n}}. \tag{34}$$

Figure 4 represents evolution of the profile $\mathbb{L}_n(k)$ and its expected value $\mathbb{L}_n^*(k)$ of a simulated 3-ary tree over different $n$. In Table 1 we compare simulated and theoretical values of the mode and the width of this tree when $n$ varies.

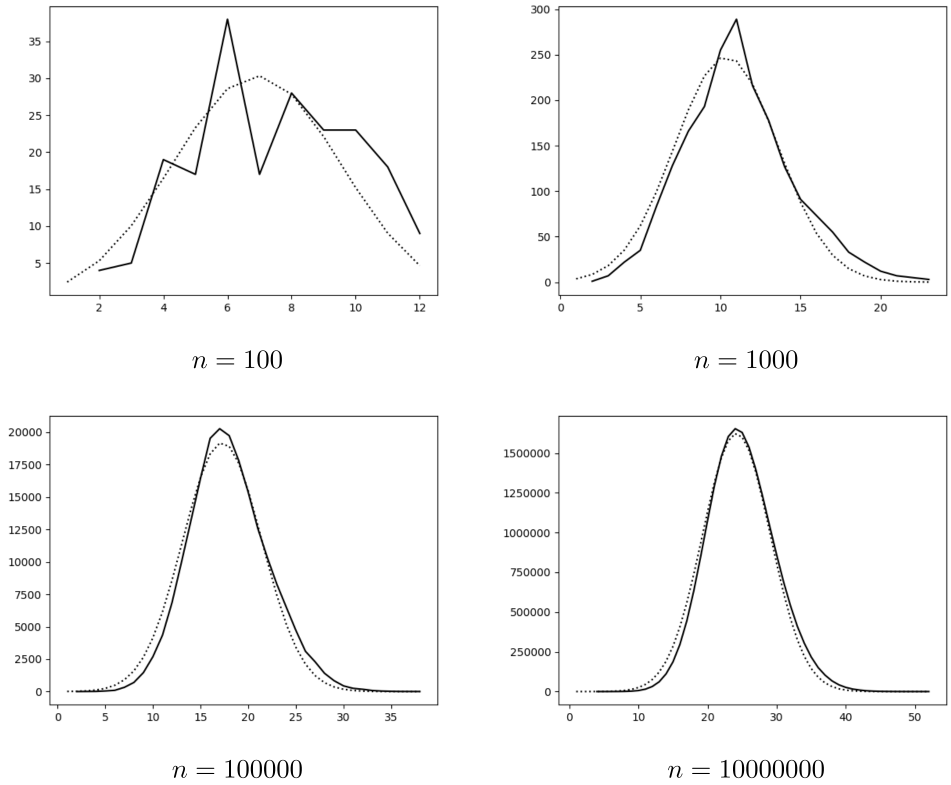


FIGURE 4. Evolution of the profile $\mathbb{L}_n(k)$ of a 3-ary tree (solid line) and its theoretical value $\mathbb{L}_n^*(k)$ (dotted line)

| n | $M_n$ | $M_n^*$ | $M_n/M_n^*$ | $u_n$ | $u_n^*$ |
|---|---|---|---|---|---|
| 100 | 38 | 30.4 | 1.252 | 6 | 6.41 |
| 1000 | 289 | 247.9 | 1.166 | 11 | 9.86 |
| 10000 | 2511 | 2146.6 | 1.170 | 14 | 13.32 |
| 100000 | 20269 | 19200.0 | 1.056 | 17 | 16.77 |
| 1000000 | 179494 | 175271.4 | 1.024 | 21 | 20.22 |
| 10000000 | 1653079 | 1622697.5 | 1.019 | 24 | 23.68 |

TABLE 1. Evolution of the width and the mode of a 3-ary tree

We proceed with simulations of PORTs. From the results of Section 3 we obtain

$$\mathbb{L}_n^*(k) = \frac{2n}{\sqrt{\pi \log n}} e^{-\frac{(k-1-\frac{1}{2}\log n)^2}{\log n}}, \tag{35}$$

$$u_n^* = \frac{\log n}{2} + \frac{1}{2}, \tag{36}$$

$$M_n^* = \frac{2n}{\sqrt{\pi \log n}}. \tag{37}$$

Simulations are presented on Figure 5 and in Table 2.

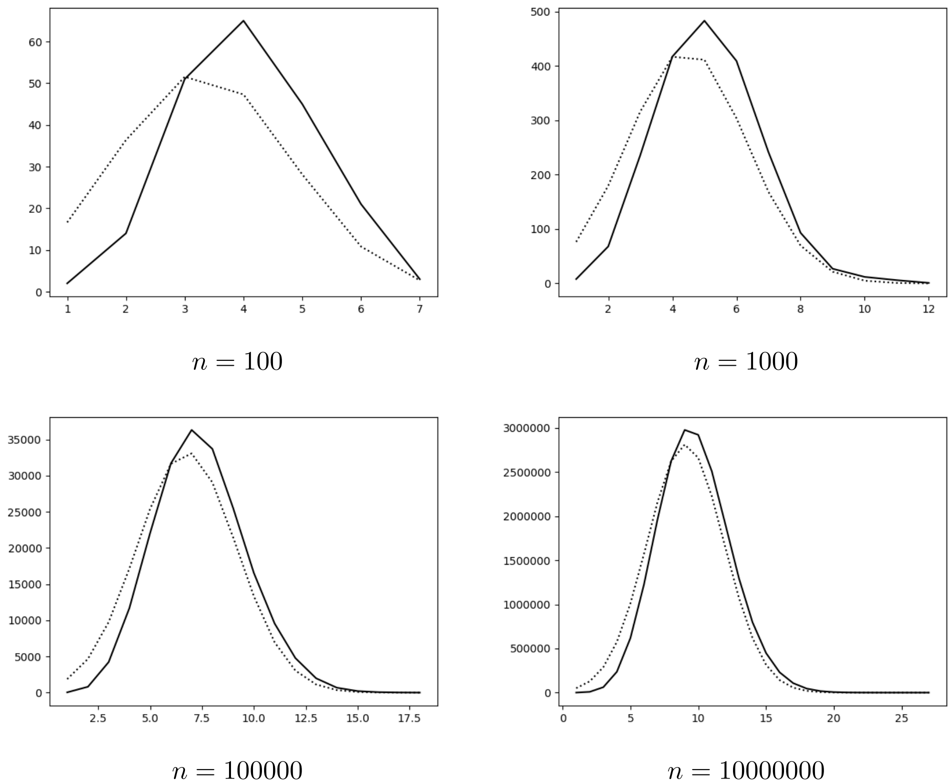


FIGURE 5. Evolution of the profile $\mathbb{L}_n(k)$ of a PORT (solid line) and its theoretical value $\mathbb{L}_n^*(k)$ (dotted line)

| n | $M_n$ | $M_n^*$ | $M_n/M_n^*$ | $u_n$ | $u_n^*$ |
|---|---|---|---|---|---|
| 100 | 65 | 52.6 | 1.236 | 4 | 2.80 |
| 1000 | 483 | 429.3 | 1.125 | 5 | 3.95 |
| 10000 | 4114 | 3718.1 | 1.106 | 6 | 4.11 |
| 100000 | 36333 | 33255.4 | 1.093 | 7 | 6.26 |
| 1000000 | 327194 | 303578.9 | 1.078 | 8 | 7.41 |
| 10000000 | 2976165 | 2810594.5 | 1.059 | 9 | 8.56 |

TABLE 2. Evolution of the width and the mode of a PORT

From the above simulations it is plausible that real values indeed converge to their theoretical counterparts. However, the speed of convergence is quite slow.

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