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SOME SELECTED TOPICS FOR THE BOOTSTRAP OF THE EMPIRICAL AND QUANTILE PROCESSES

Dedicated to the memory of Djalil Kateb

In the present work, we consider the asymptotic distributions of L_p functionals of bootstrapped weighted uniform quantile and empirical processes. The asymptotic laws obtained are represented in terms of Gaussian integrals. We investigate the strong approximations for the bootstrapped Vervaat process and the weighted bootstrap for Bahadur-Kiefer process. We obtain new results on the precise asymptotics in the law of the logarithm related to complete convergence and a.s. convergence, under some mild conditions, for the weighted bootstrap of empirical and the quantile processes. In addition we consider the strong approximation of the hybrids of empirical and partial sums processes when the sample size is random.

1. Introduction

Bootstrap samples were introduced and first investigated in [34]. Since this seminal paper, bootstrap methods have been proposed, discussed, investigated and applied in a huge number of papers in the literature. Being one of the most important ideas in the practice of statistics, the bootstrap also introduced a wealth of innovative probability problems, which in turn formed the basis for the creation of new mathematical theories. The asymptotic theory of the bootstrap with statistical applications has been reviewed in the books among others [17], [63], [40], [16], [31], [73], [47] and [59]. A major application for an estimator is in the calculation of confidence intervals. By far the most favored confidence interval is the standard confidence interval based on a normal or a Student t-distribution. Such standard intervals are useful tools, but they are based on an approximation that can be quite inaccurate in practice. Bootstrap procedures are an attractive alternative. One way to look at them is as procedures for handling data when one is not willing to make assumptions about the parameters of the populations from which one sampled. The most that one is willing to assume is that the data are a reasonable representation of the population from which they come. One then resamples from the data and draws inferences about the corresponding population and its parameters. The resulting confidence intervals have received the most theoretical study of any topic in the bootstrap analysis. Roughly speaking, it is known that the bootstrap works in the i.i.d. case if and only if the central limit theorem holds for the random variable under consideration. For further discussion we refer the reader to the landmark paper by [39]. The following notation is needed for the statement of our results. Let $X_1, X_2, \ldots$ be a sequence of i.i.d. random variables [rv's] with common df

$$F(t) = \mathbb{P}(X_1 \leq t).$$

For each $n \geq 1$, the empirical distribution function of $X_1, \ldots, X_n$, is given by

$$F_n(t) = n^{-1}\#\{X_i \leq t : 1 \leq i \leq n\}, \quad \text{for} \quad -\infty < t < \infty,$$

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where # stands for cardinality. The quantile function [qf] pertaining to $F(\cdot)$, is defined, for $u \in (0,1)$, by

$$F^{-1}(u) = \inf\{x : F(x) \geq u\}.$$

The empirical quantile function is given, for each $n \geq 1$ and $u \in (0,1)$, by

$$F_n^{-1}(u) = \inf\{x : F_n(x) \geq u\}.$$

Given the sample $X_1, \ldots, X_n$, let $X_1^*, \ldots, X_m^*$ be conditionally independent rv's with common distribution function $F_n(\cdot)$. Let

$$F_{m,n}(t) = m^{-1}\#\{X_i^* \leq t : 1 \leq i \leq m\}, \quad \text{for} \quad -\infty < t < \infty,$$

denote the *classical* Efron (or multinomial) bootstrap (see, e.g. [34] and [35] for more details). Consider also the bootstrapped empirical quantile function, belonging to $F_{m,n}(\cdot)$,

$$F_{m,n}^{-1}(u) = \inf\{x : F_{m,n}(x) \geq u\}, \quad \text{for} \quad 0 < u < 1.$$

Define the *bootstrapped empirical* and *quantile processes*, respectively, by

$$\xi_{m,n}(t) := m^{1/2}(F_{m,n}(t) - F_n(t)), \quad \text{for} \quad -\infty < t < \infty, \tag{1.1}$$

and

$$\zeta_{m,n}(t) := m^{1/2}(F_{m,n}^{-1}(t) - F_n^{-1}(t)), \quad \text{for} \quad 0 < t < 1. \tag{1.2}$$

[10] investigated the weak convergence of the processes in (1.1) and (1.2), which make possible to obtain the asymptotic validity of the bootstrap method in forming confidence bounds for $F(\cdot)$. [70] provided an elegant proof of weak convergence of the process in (1.1) [see also [71], Section 23.1]. The generalization of the work of Bickel and Freedman was given in the multivariate setting as well as in very general sample spaces and for various set and function-indexed random objects [see, for example [9], [37]]. The most advanced results for the bootstrap are due to [39] and [30]. For a survey of further results on weighted bootstrap the reader is referred to [6], for recent reference see [13]. One of important question (both in probability and in statistics) is about the rates of convergence and formed the basis of works for great number of authors (see [20], [57], [51], [11] and the references therein).

In this paper, we consider several selected topics for the bootstrap for empirical and quantile processes. We are first concerned with the characterization of the asymptotic distributions of L_p functionals of bootstrapped weighted uniform quantile and empirical processes. We consider also the bootstrap of the Vervaat process, which is an important tools in several applications, see for example [28]. We investigate the behavior of the weighted bootstrap for the well know Bahadur-Kiefer processes, see [5], [55, 56]). [77] investigated the uniform empirical process and obtained the precise asymptotics in the Baum-Katz and Davis law of large numbers given by [42] and [43] for a sequence of i.i.d. random. For further details we refer to [60], [45], [44], [67], [75], [76], [41, 46], [2, 3, 4] and the references therein. The legendary paper by [52] introducing the concept of "complete convergence" is to be cited here. The last mentioned reference generated a series of papers, in particular [7]'s seminal work which provided necessary and sufficient conditions for the convergence of the series

$$\sum_{n=1}^{\infty} n^{r/p-2}\mathbb{P}\left(\left|\sum_{i=1}^{n} X_i\right| \geq \epsilon\right)$$

for suitable values of r and p. We consider the Baum-Katz and Davis law of large numbers for weighted bootstrap for the empirical and quantile processes. We consider the strong approximation of the hybrid processes when the sample size is random. The motivation for introducing random simple sizes is that in some applied situations the number of elements in the sample is not fixed a priori because of constraints in time, costs or space,

see [8] where the Kac process appear as a particular case of randomly indexed empirical processes. [68] study the limit behavior of the empirical process with random index for a broad class of indices $\widetilde{\nu}_n$ say. [61] showed that the approximation on the tails with Poisson processes is better than that with Brownian bridge if $d = 1$. [48] obtained the same rates for the tail approximations of the multivariate empirical process using Kac's representation. Later [49] stated some weighted approximations of the multivariate empirical process with Poisson bridges and proved that the Poisson approximation is better on the tail in the case of heavy weight functions when the approximating process is derived from Kac's representation. [12] obtained some approximations of the multivariate empirical copula process with Poisson bridges. The construction of the Poisson bridges is based on Kac's representation of empirical processes of [49] combined with Bahadur-Kiefer representation of the empirical copula process. Notice that the present work extends largely, in many directions, the scope of our previous work [1].
To best of our knowledge, some of the results presented here, respond to a problems those have not been studied systematically until present, and it gives the main motivation to the present investigation.

The present work is organized as follows. In Section 2, we recall some elementary definitions for the empirical processes and the gaussian processes. In Section 3, we provide our results concerning the distributions of L_p norms of bootstrapped weighted uniform empirical and quantile processes. These results are largely inspired by the results in [24] combined with those in [30]. In Section 4, we investigate the multinomial bootstrap for the Lorenz curves, in the same spirit of [28, 29]. In Section 5, we study the weighted bootstrap for the Bahadur-Kiefer processes. In Section 6, we investigate the moment convergence for the weighted bootstrap for the empirical and quantile processes, in the same spirit as in [15] in connection with main tools of the strong approximations obtained in [1]. In Section 7, we investigate the weighted bootstrap for the hybrid processes when the sample size is random. To avoid interrupting the flow of the presentation, all proofs are relegated to the Section 8.

2. Some preliminary results

Let $U := U_1, U_2, \ldots$ be i.i.d. rv's on a probability space $(\Omega, \mathscr{A}, \mathbb{P})$. Let U follow a uniform on $(0, 1)$ law, which is denoted by $U \stackrel{\mathscr{D}}{=} \mathcal{U}(0, 1)$. For each $n \geq 1$, let $U_{n,1} \leq \cdots \leq U_{n,n}$ be the order statistics of $U_1, \ldots, U_n$, and set $U_{0,n} = 0$ and $U_{n+1,n} = 1$ for $n \geq 0$. Since the strict inequalities

$$0 = U_{0,n} < U_{1,n} < \cdots < U_{n,n} < U_{n+1,n} = 1, \tag{2.1}$$

hold almost surely for all $n \geq 0$, we will work, without loss of generality, on the event (2.1), of probability 1. Let

$$\mathbb{U}_n(u) := n^{-1}\#\{U_i \leq u : 1 \leq i \leq n\}, \text{ for } 0 \leq u \leq 1,$$

be the empirical df based upon $U_1, \ldots, U_n$. Define the empirical qf, pertaining to $\mathbb{U}_n(\cdot)$, by

$$\mathbb{U}_n^{-1}(v) := \inf\{u \geq 0 : \mathbb{U}_n(u) \geq v\}, \text{ for } 0 \leq v \leq 1;$$
$$\mathbb{U}_n^{-1}(v) := 0, v < 0, \text{ and } \mathbb{U}_n^{-1}(v) := 1, v \geq 1.$$

Denote the uniform empirical (resp. quantile) process by

$$\alpha_n(u) := n^{1/2}\left(\mathbb{U}_n(u) - u\right) \text{ and } \beta_n(u) := n^{1/2}\left(\mathbb{U}_n^{-1}(u) - u\right), \text{ for } u \in [0, 1].$$

Let us now introduce some definitions and notations. Let $W = \{W(s) : s \geq 0\}$ and $B = \{B(u) : u \in [0, 1]\}$ be the standard Wiener process and Brownian bridge, that is, the centered Gaussian processes with continuous sample paths, and covariance functions

$$\mathbb{E}(W(s)W(t)) = s \wedge t \quad \text{for} \quad s, t \geq 0$$

and

$$\mathbb{E}(B(u)B(v)) = u \wedge v - uv \quad \text{for} \quad u, v \in [0,1].$$

A Kiefer process $K = \{K(s,u) : s \geq 0, u \in [0,1]\}$ is a two-parameters centered Gaussian process, with continuous sample paths, and covariance function

$$\mathbb{E}(K(s,u)K(t,v)) = (s \wedge t)\,(u \wedge v - uv) \quad \text{for} \quad s,t \geq 0 \quad \text{and} \quad u,v \in [0,1].$$

It satisfies the following distributional identities:

$$\{K(s,u) : u \in [0,1]\} \overset{\mathscr{D}}{=} \left\{\sqrt{s}\, B(u) : u \in [0,1]\right\} \quad \text{for} \quad s \geq 0$$

and

$$\{K(s,u) : s \geq 0\} \overset{\mathscr{D}}{=} \left\{\sqrt{u(1-u)}\, W(s) : s \geq 0\right\} \quad \text{for} \quad u \in [0,1].$$

The interested reader may refer to [20] for details on the Gaussian processes mentioned above.

3. The distributions of L_p norms of bootstrapped weighted uniform empirical and quantile processes

We assume, without loss of generality, that the underlying probability space $(\Omega, \mathscr{A}, \mathbb{P})$, constructed in [22], is so rich that it accommodates all the r.v.'s and processes introduced so far and also later on. This space carries a sequence of Brownian bridges $\{B_i\}_{i\geq 1}$ such that (3.1) holds true. [22] obtained the following deep results

$$\begin{array}{rcl} \displaystyle\sup_{0<s<1} \frac{|\alpha_n(s) - B_n^*(s)|}{(s(1-s))^{1/2-\nu_1}} & = & O_{\mathbb{P}}(n^{-\nu_1}), \\ \displaystyle\sup_{\lambda/n<s<1-\lambda/n} \frac{|\beta_n(s) - B_n(s)|}{(s(1-s))^{1/2-\nu_2}} & = & O_{\mathbb{P}}(n^{-\nu_2}), \end{array} \tag{3.1}$$

for all $0 < \lambda < \infty$ and $0 \leq \nu_1 < \frac{1}{4}$, $0 \leq \nu_2 < \frac{1}{2}$ where $B_n^*(s) = B_n(s)$ if $1/n \leq s \leq 1 - 1/n$ and zero otherwise. Throughout the paper, we use the notation $\log_{(2)} x := \log\log x$ for $x > 3$ and we will denote by $\|\cdot\|$ the sup-norm i.e., $\|\cdot\| = \sup_{0\leq t\leq 1} |\cdot(t)|$. For the same construction we also have

$$\begin{array}{rcl} \displaystyle\sup_{0<s<1} |\alpha_n(s) - B_n(s)| & = & O(n^{-1/4}(\log n)^{1/2}(\log_{(2)} n)^{1/4}) \ a.s., \\ \displaystyle\sup_{0<s<1} |\beta_n(s) - B_n(s)| & = & O(n^{-1/2}\log n) \ a.s. \end{array} \tag{3.2}$$

[As in [30], alternatively, we could choose $(\Omega, \mathscr{A}, \mathbb{P})$ to be the space constructed by [66] with ν_1 and ν_2 transposed in (3.1) and (3.3) below and the rate sequences transposed in (3.2)]. As in [30], now extend $(\Omega, \mathscr{A}, \mathbb{P})$ to obtain a probability space $(\widetilde{\Omega}, \widetilde{\mathscr{A}}, \widetilde{\mathbb{P}})$, which besides $\{U_i\}$ and $\{B_i\}$, carries another sequence of independent uniform (0,1) r.v.'s $\xi_1, \xi_2, \ldots$ and another sequence of Brownian bridges $\widetilde{B}_1, \widetilde{B}_2, \ldots$ such that the sets of random elements

$$\{U_i\}_{i=1}^{\infty} \cup \{B_i(s) : 0 \leq s \leq 1\}$$

and

$$\{\xi_i\}_{i=1}^{\infty} \cup \{\widetilde{B}_i(s) : 0 \leq s \leq 1\}$$

are independent. We define the uniform empirical quantile function by $\xi_m(0) = 0$,

$$\xi_m(s) = \xi_{k,m}, \ (k-1)/m < s \leq k/m, \quad \text{for} \quad k = 1, \ldots, m,$$

and the uniform quantile process

$$k_m(s) = m^{1/2}(s - \xi_m(s)), \quad \text{for} \quad 0 < s \leq 1.$$

Define also the uniform empirical distribution function

$$E_m(s) = \begin{cases} 0, & \xi_{1,m} > 0 \\ k/m, & \xi_{k,m} \le s < \xi_{k+1,m} \qquad \text{for} \quad k = 1, \ldots, m-1, \\ 1, & \xi_{m,m} \le s, \end{cases}$$

and

$$e_m(s) = m^{1/2}(E_m(s) - s), \quad \text{for} \quad 0 \le s \le 1.$$

We have

$$\begin{aligned} \sup_{0<s<1} \frac{|e_m(s) - \widetilde{B}_m^*(s)|}{(s(1-s))^{1/2-\nu_1}} &= O_{\widetilde{\mathbb{P}}}(m^{-\nu_1}), \\ \sup_{\lambda/m<s<1-\lambda/m} \frac{|k_m(s) - \widetilde{B}_m(s)|}{(s(1-s))^{1/2-\nu_2}} &= O_{\widetilde{\mathbb{P}}}(m^{-\nu_2}), \end{aligned} \tag{3.3}$$

for all $0 < \lambda < \infty$ and $0 \le \nu_1 < \frac{1}{4}$, $0 \le \nu_2 < \frac{1}{2}$ where $\widetilde{B}_m^*(s) = \widetilde{B}_m(s)$ if $1/m \le s \le 1 - 1/m$ and zero otherwise. As pointed out in [30], $(\widetilde{\Omega}, \widetilde{\mathscr{A}}, \widetilde{\mathbb{P}})$ can be obtained by taking the product $(\Omega, \mathscr{A}, \mathbb{P})$with itself. Now P can be replaced by $\widetilde{\mathbb{P}}$ in (3.1). Introduce

$$\mathbb{U}_{m,n}(s) = E_m(\mathbb{U}_n(s)) \text{ and } \mathbb{U}_{m,n}^{-1}(s) = \mathbb{U}_n^{-1}(\xi_m(s)), \quad \text{for} \quad 0 \le s \le 1. \tag{3.4}$$

The bootstrapped uniform empirical and quantile processes are defined, respectively, by

$$\begin{aligned} \alpha_{m,n}(s) &= m^{1/2}(\mathbb{U}_{m,n}(s) - \mathbb{U}_n(s)) \text{ and} \\ \beta_{m,n}(s) &= m^{1/2}(\mathbb{U}_n^{-1}(s) - \mathbb{U}_{m,n}^{-1}(s)), \text{ for } 0 \le s \le 1. \end{aligned} \tag{3.5}$$

Set $\ell(n) = n^{-1/4}(\log n)^{1/2}(\log_{(2)} n)^{1/4}$ for the rate sequence figuring in (3.2). [30] showed the following result. We make no claim of originality for the results that we highlight in this section. In fact, the results presented heavily rely on those of [24] and [30]. It is worth noticing that these results are not stated elsewhere.

Theorem 3.1. *For any sequence $m = m(n) \to \infty$ of positive integers and each $0 \le \nu < 1/4$,*

$$\sup_{U_{1,n} \le s < U_{n,n}} \frac{|\alpha_{m,n}(s) - \widetilde{B}_m^*(s)|}{(s(1-s))^{1/2-\nu}} = O_{\widetilde{\mathbb{P}}}((m \wedge n)^{-\nu}), \tag{3.6}$$

and

$$\sup_{0 \le s \le 1} |\alpha_{m,n}(s) - \widetilde{B}_m^*(s)| = (\ell(m) \vee \ell(n)) \ a.s., \tag{3.7}$$

and whenever $m = m(n)$ satisfies the condition that for two constants $0 < C_1 < C_2$,

$$C_1 m \le n \le C_2 m, \ n = 1, 2, \ldots, \tag{3.8}$$

and for any $0 \le \lambda < \infty$ and $0 \le \nu < 1/4$,

$$\sup_{\lambda/m \le s < 1-\lambda/m} \frac{|\beta_{m,n}(s) - \widetilde{B}_m(s)|}{(s(1-s))^{1/2-\nu}} = O_{\widetilde{\mathbb{P}}}(m^{-\nu}), \tag{3.9}$$

and

$$\sup_{0 \le s \le 1} |\beta_{m,n}(s) - \widetilde{B}_m(s)| = O(\ell(m)) \ a.s. \tag{3.10}$$

Let $\{\zeta_{1,n}\}$ and $\{\zeta_{2,n}\}$ be sequences of positive numbers, such that, as $n \to \infty$

$$1 \le \zeta_{1,n} \le n, \ \zeta_{1,n} \to \infty \tag{3.11}$$

$$\zeta_{1,n}/n \to 0, \tag{3.12}$$

and

$$1 \leq \zeta_{2,n} \leq n, \ \zeta_{2,n} \to \infty \tag{3.13}$$
$$\zeta_{2,n}/n \to 0. \tag{3.14}$$

Let $\{\zeta_{1,m}\}$ and $\{\zeta_{2,m}\}$ be sequences of positive numbers, such that, as $n \to \infty$

$$\begin{aligned} 1 \leq \zeta_{1,m} \leq n, \ \zeta_{1,m} \to \infty \\ \zeta_{1,m}/m \to 0, \end{aligned} \tag{3.15}$$

and

$$\begin{aligned} 1 \leq \zeta_{2,m} \leq n, \ \zeta_{2,m} \to \infty \\ \zeta_{2,m}/m \to 0. \end{aligned} \tag{3.16}$$

The following theorem is the bootstrapped version of Theorem 2.1 of [24].

Theorem 3.2. *Let $q(\cdot)$ be a positive function on $(0, 1/2]$, $1 \leq p < \infty$ and assume that*

$$\int_0^{1/2} s^{p/2}/q(s)ds < \infty. \tag{3.17}$$

Then with $\{\zeta_{1,m}\}$ as in (3.15) we have, as $n \to \infty$,

$$\int_{\zeta_{1,m}/m}^{1/2} |\beta_{m,n}(s)|^p/q(s)ds \ \to_{\mathscr{D}} \ \int_0^{1/2} |B(s)|^p/q(s)ds, \tag{3.18}$$

$$\int_{1/(m+1)}^{\zeta_{1,m}/m} |\beta_{m,n}(s)|^p/q(s)ds \ \to_{\mathscr{D}} \ 0, \tag{3.19}$$

and

$$\int_{U_{m,n}(\zeta_{1,m}/m)}^{U_{m,n}(1/2)} |\alpha_{m,n}(s)|^p/q(s)ds \ \to_{\mathscr{D}} \ \int_0^{1/2} |B(s)|^p/q(s)ds, \tag{3.20}$$

$$\int_{U_{m,n}(1/m)}^{U_{m,n}(\zeta_{1,m}/m)} |\alpha_{m,n}(s)|^p/q(s)ds \ \to_{\mathscr{D}} \ 0. \tag{3.21}$$

Theorem 3.3. *As $n \to \infty$, we have*

$$\int_{1/(m+1)}^{m/(m+1)} \frac{||\beta_{m,n}(s)|^p - |\widetilde{B}_m(s)|^p|}{(s(1-s))^{p/2+1}} ds = O_{\widetilde{\mathbb{P}}}(1), \tag{3.22}$$

and

$$\int_{\lambda/(m+1)}^{1-\lambda/(m+1)} \frac{||\alpha_{m,n}(s)|^p - |\widetilde{B}^*_m(s)|^p|}{(s(1-s))^{p/2+1}} ds \ = \ O_{\widetilde{\mathbb{P}}}(1), \ \textit{for all } \lambda > 0, \tag{3.23}$$

$$\int_{U_{1,m}}^{U_{m,m}} \frac{||\alpha_{m,n}(s)|^p - |\widetilde{B}^*_m(s)|^p|}{(s(1-s))^{p/2+1}} ds \ = \ O_{\widetilde{\mathbb{P}}}(1). \tag{3.24}$$

From now on we assume that the weight function $q(\cdot)$ is regularly varying at zero. This means that $q(s) = s^{\nu}L(s)$, $-\infty < \nu < \infty$, where $L(\cdot)$ is a slowly varying function, i.e., $L(s)$ is positive on $(0, 1/2]$, Lebesgue measurable and

$$\lim_{s \to 0} L(\lambda s)/L(s) = 1, \ \text{for all } \lambda > 0. \tag{3.25}$$

We mention as in [24], that the condition (3.17) holds true for all $q(\cdot)$ regularly varying at zero with exponent $\nu \leq 1 + p/2$.

Theorem 3.4. *Let $L(\cdot)$ be slowly varying at zero and $\{\zeta_{1,m}\}$ as in (3.15). If $-\infty < \nu < 1 + p/2$, then as $n \to \infty$,*

$$\left(\frac{\zeta_{1,m}}{m}\right)^{\nu-(p/2+1)} L\left(\frac{\zeta_{1,m}}{m}\right) \int_{1/(m+1)}^{\zeta_{1,m}/m} \frac{|\beta_{m,n}(s)|^p}{s^\nu L(s)} ds \to_{\mathscr{D}} \int_0^1 s^{-\nu} |W(s)|^p ds, \tag{3.26}$$

and

$$\left(\frac{\zeta_{1,m}}{m}\right)^{\nu-(p/2+1)} L\left(\frac{\zeta_{1,m}}{m}\right) \int_{U_{m,n}(1/(m+1))}^{U_{m,n}(\zeta_{1,m}/m)} \frac{|\alpha_{m,n}(s)|^p}{s^\nu L(s)} ds \to_{\mathscr{D}} \int_0^1 s^{-\nu} |W(s)|^p ds. \tag{3.27}$$

Corollary 3.1. *Let $m/(m+1) \le \zeta_{1,m} < \zeta_{2,m} \le m^2/(m+1)$ and assume that, as $n \to \infty$,*

$$\frac{\zeta_{2,m}}{\zeta_{1,m}} \frac{m - \zeta_{1,m}}{m - \zeta_{2,m}} \to \infty. \tag{3.28}$$

Then we have

$$\left(2D \log\left(\frac{\zeta_{2,m}}{\zeta_{1,m}} \frac{m - \zeta_{1,m}}{m - \zeta_{2,m}}\right)\right)^{-1/2} \cdot \left\{\int_{\zeta_{1,m}/m}^{\zeta_{2,m}/m} \frac{|\beta_{m,n}(s)|^p}{(s(1-s))^{p/2+1}} - \mu \log\left(\frac{\zeta_{2,m}}{\zeta_{1,m}} \frac{m - \zeta_{1,m}}{m - \zeta_{2,m}}\right)\right\} \to_{\mathscr{D}} N(0,1),$$

where $D = D(\mathbb{P})$is positive constant, $\mu = \mu(p) = \mathbb{E}|N(0,1)|^p$, and $N(0,1)$ stands for the standard normal r.v.

Corollary 3.2. *Let $0 \le \zeta_{1,m} < \zeta_{2,m} \le m$ and assume (3.28). Then, with $\zeta^*_{1,m} = (\zeta_{1,m} \vee 1)$ and $\zeta^*_{2,m} = (\zeta_{2,m} \wedge 1)$, we have*

$$\left(2D \log\left(\frac{\zeta^*_{2,m}}{\zeta^*_{1,m}} \frac{m - \zeta^*_{1,m}}{m - \zeta^*_{2,m}}\right)\right)^{-1/2} \cdot \left\{\int_{U_{m,n}(\zeta_{1,m}/m)}^{U_{m,n}(\zeta_{2,m}/m)} \frac{|\beta_{m,n}(s)|^p}{(s(1-s))^{p/2+1}} - \mu \log\left(\frac{\zeta^*_{2,m}}{\zeta^*_{1,m}} \frac{m - \zeta^*_{1,m}}{m - \zeta^*_{2,m}}\right)\right\} \to_{\mathscr{D}} N(0,1), \tag{3.29}$$

where $D = D(\mathbb{P})$, $\mu = \mu(p) = \mathbb{E}|N(0,1)|^p$, and $N(0,1)$ are as in Corollary (3.1).

4. Bootstrap approach for the Lorenz Curves

Let us define the Lorenz curve by

$$\widetilde{L}_F(t) = \frac{1}{\vartheta} \int_0^t F^{-1}(s) ds, \text{ for } 0 \le t \le 1,$$

where

$$\vartheta = \int_{\mathbb{R}} x dF(x) = \int_0^1 F^{-1}(s) ds < \infty$$

when X admits a df $F(\cdot)$ and $X = F^{-1}(U)$, $\vartheta \neq 0$. In econometrics it is customary to interpret $\widetilde{L}_F(\cdot)$ as the proportion of total amount of "wealth" that is owned by the least fortunate $t \times 100$ percent of a "population." For some details on the variety of situations where estimating the curve $\widetilde{L}_F(\cdot)$ is of importance, we may refer, for example to [28]. The empirical Lorenz curve is defined to be

$$\widetilde{L}_n(t) = \frac{1}{\vartheta_n} \int_0^t F_n^{-1}(s) ds, \quad \text{for} \quad 0 \le t \le 1,$$

where ϑ_n denotes the usual empirical mean. Let $\mathbb{U}_n(\cdot)$ the uniform empirical distribution function corresponding to $U_1 = F(X_1), \ldots, U_n = F(X_n)$ and $\widetilde{\mathbb{V}}_n(\cdot)$ denotes the Vervaat process defined by

$$\begin{aligned}\widetilde{\mathbb{V}}_n(t) &= \int_0^t \left(F_n^{-1}(s) - F^{-1}(s)\right) ds \\ &\quad + \int_{-\infty}^{F^{-1}(t)} \left(F_n(s) - F(s)\right) ds, \quad \text{for} \quad 0 \leq t \leq 1. \end{aligned} \tag{4.1}$$

It is worth noticing that the Lorenz curve and the Vervaat process are related by the following relation (refer to the equation (2.1) in [29])

$$\widetilde{L}_n - \widetilde{L}_F = \frac{1}{\vartheta_n}\widetilde{\mathbb{V}}_n + \left(1 + \frac{\vartheta - \vartheta_n}{\vartheta_n}\right) l\left(\mathbb{U}_n - I\right),$$

where

$$\begin{aligned} l\left(\mathbb{U}_n(t) - t\right) &= -\frac{1}{\vartheta}\int_{-\infty}^{F^{-1}(t)} \left(\mathbb{U}_n(F(x)) - F(x)\right) dx \\ &\quad + \frac{1}{\vartheta} L_F(t) \int_{-\infty}^{+\infty} \left(\mathbb{U}_n(F(x)) - F(x)\right) dx. \end{aligned}$$

We will denote by $\mathbb{V}_n(t)$ the version of $\widetilde{\mathbb{V}}_n(t)$ where we replace $F_n(t)$ resp. $F_n^{-1}(t)$, resp $F(t)$ by $\mathbb{U}_n(t)$ resp. $\mathbb{U}_n^{-1}(t)$, resp. t. Recall that we have to deal with a sequence $X_1^*, \ldots, X_m^*$ conditionally independent r.v.'s with common distribution function $F_n(x)$ and $X_{1:m}^* \leq \ldots \leq X_{m:m}^*$ their order statistics and let

$$\bar{X}_{m,n} = \frac{1}{m}\sum_{i=1}^{m} X_m^* = \frac{1}{m}\sum_{i=1}^{m} X_{i:m}^*.$$

Let us define

$$\widetilde{L}_{m,n}(u) = \begin{cases} \dfrac{1}{\bar{X}_{m,n}} \dfrac{1}{m} \displaystyle\sum_{i=1}^{\lfloor mu \rfloor + 1} X_{i:m}^*, & \text{if} \quad 0 \leq u < 1, \\ 1, & \text{if} \quad\quad u = 1, \end{cases}$$

and

$$\widetilde{\mathbb{L}}_{m,n}(u) = \sqrt{m}\left(\widetilde{L}_{m,n}(u) - \widetilde{L}_n(u)\right).$$

[21] provide the following result (see also [23]).

Theorem 4.1. *If $F^{-1}(\cdot)$ is continuous on $[0,1)$, and*

$$0 < \liminf_{n\to\infty}(m/n) \leq \limsup_{n\to\infty}(m/n) < \infty,$$

then there is a sequence of Gaussian processes

$$\{\Lambda_m(u) : 0 \leq u \leq 1\} \stackrel{d}{=} \{\Lambda_m^*(u) : 0 \leq u \leq 1\}$$

and as, $m \wedge n \to \infty$

$$\sup_{0\leq u\leq 1} |\widetilde{\mathbb{L}}_{m,n}(u) - \Lambda_m^*(u)| \stackrel{\mathbb{P}}{\to} 0,$$

where

$$\Lambda_m^*(u) = \frac{1}{\vartheta}\left\{-\int_0^u \widetilde{B}_m(s) dF^{-1}(s) + \widetilde{L}_F(u)\int_0^1 \widetilde{B}_m(s) dF^{-1}(s)\right\},$$

where $\{\widetilde{B}_m(\cdot) : m \geq 1\}$ denotes a sequence of Brownian bridges.

Let us introduce

$$L_n(t) = \frac{1}{\vartheta_n}\int_0^t \mathbb{U}_n^{-1}(s)ds, \quad \text{for} \quad 0 \le t \le 1.$$

Let $\mathcal{H}$ denote the so called Finkelstein set, consisting of all absolutely continuous functions $h : [0,1] \longrightarrow \mathbb{R}$ such that

$$h(0) = 0 = h(1) \quad \text{and} \quad \int_0^1 \{h'(s)\}^2 ds \le 1.$$

Let $D[0,1]$ denote the set of all left-continuous functions on $[0,1]$ that have right-hand limits at each point. Let $\mathcal{L}$ denote the set $\{\widetilde{L}_h : \ h \in \mathcal{H}\}$, where

$$\widetilde{L}_h := -\frac{1}{\vartheta}\int_0^{F^{-1}(t)} h(F(x))dx + \frac{1}{\vartheta}\widetilde{L}_F(t)\int_0^{\infty} h(F(x))dx.$$

Recall also the following arguments given in [78]. Let

$$\gamma_n(t) = \mathbb{U}_n(t) - t, \quad \text{for} \quad 0 \le t \le 1. \tag{4.2}$$

It is well known that

$$\left\{\sqrt{n/2\log_{(2)} n}\right\}\gamma_n \text{ is relatively compact in } \mathcal{H}, \ a.s.$$

see [36] and $\sqrt{n}\gamma_n(t) \to_{\mathscr{D}} B(t)$ ([32] in the space $D(0,1)$ endowed, respectively, with the uniform and Skorohod J_1 topology, where $B(\cdot)$ denotes a Brownian bridge on $[0,1]$. The combination of these two results is summarized in Theorem 1.1 of [78], that we state

$$\left\{\frac{n}{\log_{(2)} n}\right\}\mathbb{V}_n \text{ is relatively compact in } \mathcal{H}^2 := \left\{h^2 : \ h \in \mathcal{H}\right\} \ a.s.$$

Moreover

$$2n\mathbb{V}_n(t) \to_{\mathscr{D}} B^2(t),$$

holds true in the space $C[0,1]$ endowed with the topology of uniform convergence. Let $\mathbb{V}_{m,n}(\cdot)$ a uniform on $(0,1)$ version of $\widetilde{\mathbb{V}}_n(\cdot)$ where we consider also that $\mathbb{U}_{m,n}(\cdot)$ replaces $F_n(\cdot)$. Recall the definition of $\mathbb{U}_{m,n}(\cdot)$ in (3.4). Let us define

$$L_{m,n}(t) = \frac{1}{\vartheta_{m,n}}\int_0^t \mathbb{U}_{m,n}^{-1}(s)ds, \quad \text{for} \quad 0 \le t \le 1.$$

and

$$\vartheta_m = \vartheta_{m,n} = \frac{1}{m}\sum_{i=1}^{m}\xi_i.$$

Let us introduce

$$d_2(F,G) = \left(\int_0^1 \left(F^{-1}(t) - G^{-1}(t)\right)^2 dt\right)^{1/2},$$

where $F(\cdot)$ and $G(\cdot)$ are two dfs in $\mathcal{F}_2$, where

$$\mathcal{F}_2 = \left\{F : F \text{ is a df such that } \int_{-\infty}^{\infty} x^2 dF(x) < \infty\right\},$$

refer to [71], see pp.62-63. Recall the definition of $\alpha_{m,n}(\cdot)$ in (3.5). Let us introduce

$$\widetilde{\mathbb{L}}_{m,n} = \sqrt{m}\{L_{m,n} - L_n\}.$$

We summarize our first result in the following theorem.

Theorem 4.2. *Assume that the following conditions are satisfied :*

- **(H1):** *The functions* $F(\cdot)$ *and* $F^{-1}(\cdot)$ *are continuous;*
- **(H2):** $\mathbb{E}[X]^{2+\epsilon} < \infty$ *for some* $\epsilon > 0$.

We have

$$\frac{\vartheta_M\widetilde{\mathbb{L}}_{m,n}}{\sqrt{m\log_{(2)} m\mu_{m,n}}} = \frac{1}{\sqrt{\log_{(2)} m}}\left\{-\frac{1-L_n}{\vartheta_{m,n}}\int_0^t \alpha_{m,n}(s)ds + \frac{L_n}{\vartheta_{m,n}}\int_t^1 \alpha_{m,n}(s)ds\right\}, \ a.s.,$$

in the space $D[0,1]$ *with respect to the norm sup* $\|\cdot\|$.

Remark 4.1. In the proof of Lemma 2.1 of [29] the study of the behavior of $\mathbb{V}_{m,n}(t)$ is based on the study of the behavior of

$$\|\mathbb{V}_{m,n}\| = \|\mathbb{V}_{m,n}\|_{[0,\delta_m]} \vee \|\mathbb{V}_{m,n}\|_{[\delta_m,1-\delta_m]} \vee \|\mathbb{V}_{m,n}\|_{[1-\delta_m,1]},$$

where $\|\cdot\|_{\mathbf{\Theta}}$ denotes the supremum norm on $\mathbf{\Theta} \subseteq [0,1]$.

Inspired by Theorem 4.1 of [78], and keeping in mind the definitions of $\mathbb{V}_{m,n}(t)$ and $\gamma_n(t)$ given by (4.2), we obtain the following result.

Theorem 4.3. *Under the assumptions of Theorem 4.2, we have*

$$\limsup_{m\longrightarrow\infty}\left(\frac{m^{1/2}}{l(n)\vee l(m)}\right)^{1/2}\left\|\mathbb{V}_{m,n} - \frac{1}{2}\{\gamma_n\}^2\right\| = 0, \ a.s.$$

5. Weighted bootstrap for Bahadur-Kiefer processes

We recall the result due to [57] [refer also to [58]], which is one of the deepest results in probability theory.

Theorem 1. *For each* n*, there exists a sequence of Brownian bridges* $\{B_n^{(1)}(t) : 0 \le t \le 1\}$ *such that*

$$\mathbb{P}\left\{\sup_{0\le t\le 1}|\alpha_n(t) - B_n^{(1)}(t)| \ge n^{-1/2}(c_1\log n + x)\right\} \le c_2\exp(-c_3 x), \tag{5.1}$$

for all $x \ge 0$*, where* c_1*,* c_2 *and* c_3 *are positive constants.*

In his manuscript, [62] details the original proof of (5.1). The following result is due to [27] improved in Theorem 3.2.1 of [25].

Theorem 2. *For each* n*, there exists a sequence of Brownian bridges* $\{B_n^{(2)}(t) : 0 \le t \le 1\}$ *such that*

$$\mathbb{P}\left\{\sup_{0\le t\le 1}|\beta_n(t) - B_n^{(2)}(t)| \ge n^{-1/2}(c_4\log n + x)\right\} \le c_5\exp(-c_6 x),$$

for all $x \ge 0$*, where* c_4*,* c_5 *and* c_6 *are positive constants.*

Let $Z_1, Z_2, \dots$ be a sequence of positive independent, identically distributed random variables with a df $H(\cdot)$ and

$$\mathbb{E}Z_1 = 1 \ \text{ and } \ \text{var}Z_1 = 1. \tag{5.2}$$

We assume that

$$\mathbb{E}\exp(tZ_1) < \infty, \ |t| \le t_0 \ \text{ with some } \ t_0 > 0, \tag{5.3}$$

and finally

$$\{U_i, 1 \le i < \infty\} \ \text{ and } \ \{Z_i, 1 \le i < \infty\} \text{ are independent.} \tag{5.4}$$

For all $n \ge 1$, let $T_n = Z_1 + \cdots + Z_n$ and define the random weights,

$$W_{i;n} := \frac{Z_i}{T_n}, \ \text{ for } \ i = 1, \dots, n. \tag{5.5}$$

The generalized bootstrapped empirical distribution becomes

$$\mathbb{U}_{n,\mathscr{W}}(t) := \sum_{i=1}^{n} W_{i;n} \mathbb{1}_{\{U_i \leq t\}}, \quad \text{for} \quad 0 \leq t \leq 1. \tag{5.6}$$

The bootstrapped empirical quantile function $\mathbb{U}_{n,\mathscr{W}}^{-1}(\cdot)$ is the left-continuous inverse of $\mathbb{U}_{n,\mathscr{W}}(\cdot)$

$$\mathbb{U}_{n,\mathscr{W}}^{-1}(s) := \inf\left\{t : \mathbb{U}_{n,\mathscr{W}}(t) \geq s\right\}, \quad \text{for} \quad 0 \leq s \leq 1. \tag{5.7}$$

We now define the generalized bootstrapped empirical process to be

$$\alpha_{n,\mathscr{W}}(t) := n^{1/2}\left\{\mathbb{U}_{n,\mathscr{W}}(t) - \mathbb{U}_n(t)\right\}, \quad \text{for} \quad 0 \leq t \leq 1, \tag{5.8}$$

and the generalized bootstrapped quantile process to be

$$\beta_{n,\mathscr{W}}(s) := n^{1/2}\left\{\mathbb{U}_n^{-1}(s) - \mathbb{U}_{n,\mathscr{W}}^{-1}(s)\right\}, \text{ for } 0 \leq s \leq 1. \tag{5.9}$$

[1] proved the following theorems.

Theorem 3. *Let assumptions (5.2)-(5.4) hold. Then, it is possible to define a sequence of Brownian bridges $\{B_{n,\mathscr{W}}^{(1)}(t); 0 \leq t \leq 1\}$ such that, for n large enough,*

$$\mathbb{P}\left(\sup_{0<t<1}\left|\alpha_{n,\mathscr{W}}(t) - B_{n,\mathscr{W}}^{(1)}(t)\right| > n^{-1/2}(c_7 \log n + x)\right) \leq c_8 \exp(-c_9 x), \tag{5.10}$$

for all $x \geq 0$, where c_7, c_8 and c_9 are positive universal constants.

The following theorem establishes the strong approximation of the generalized bootstrapped uniform quantile process $\{\beta_{n,\mathscr{W}}(s) : 0 \leq s \leq 1\}$.

Theorem 4. *Let assumptions (5.2)-(5.4) hold. Then, it is possible to define a sequence of Brownian bridges $\{B_{n,\mathscr{W}}^{(2)}(t); 0 \leq t \leq 1\}$ such that, for n large enough,*

$$\mathbb{P}\left(\sup_{0<t<1}\left|\beta_{n,\mathscr{W}}(t) - B_{n,\mathscr{W}}^{(2)}(t)\right| > n^{-1/2}(c_{10} \log n + x)\right) \leq c_{11} \exp(-c_{12} x), \tag{5.11}$$

for all $x \geq 0$, where c_{10}, c_{11} and c_{12} are positive universal constants.

Remark 5.1. The system of weights defined in (5.5) appears in [65], p.1617, where it is shown that it satisfies assumptions $(\mathscr{W}_I)$, $(\mathscr{W}_{II})$ and $(\mathscr{W}_{III})$ on p.1612 of the same reference, so that all the results therein hold for the objects to be treated in this paper. In particular, weak convergences for the bootstrapped empirical and quantile processes to a Brownian bridges are proved.

Let us introduce the object that we are interested in. The sum

$$R_n(t) = \alpha_n(t) + \beta_n(t), \text{ for } t \in [0,1]$$

of the empirical and quantile processes is known in the literature as the Bahadur-Kiefer process (cf. [5], [55, 56]). Define the bootstrapped Bahadur-Kiefer process, by

$$R_{n,\mathscr{W}}(t) = \alpha_{n,\mathscr{W}}(t) + \beta_{n,\mathscr{W}}(t), \text{ for } t \in [0,1].$$

Theorem 5.1. *We have, as $n \to \infty$,*

$$\sup_{0 \leq t \leq 1}\left|R_{n,\mathscr{W}}(t) - R_n(t)\right| = O\left(n^{-1/2}\log n\right), \ a.s.$$

Theorem 5.2. *We have, as $n \to \infty$,*

$$\limsup_{n\to\infty} n^{1/4}(\log n)^{-1/2}(\log_{(2)} n)^{-1/4}\|R_{n,\mathscr{W}}\| \leq 2^{-1/4}, \ a.s. \tag{5.12}$$

and

$$\liminf_{n\to\infty} n^{1/4}(\log n)^{-1/2}(\log_{(2)} n)^{1/4}\|R_{n,\mathscr{W}}\| \leq 8^{-1/4}\pi^{1/2}, \ a.s. \tag{5.13}$$

Let

$$I_{n,\mathscr{W}}(t) = \int_0^t R_{n,\mathscr{W}}(s)ds$$

and

$$V_{n,\mathscr{W}}(t) = 2n^{1/2} I_{n,\mathscr{W}}(t).$$

In the following result, we obtain an upper-bound for $\|V_{n,\mathscr{W}} - n^{-1/2}\alpha_n^2\|$ as a consequence of Theorem 5.1 and from Theorem 4.1 of [78].

Corollary 5.1. *We have, as $n \to \infty$,*

$$\|V_{n,\mathscr{W}} - n^{-1/2}\alpha_n^2\| = O(n^{-1/2}\log n), \ a.s.$$

6. Moment convergence rates

Motivated by the moment convergence rates for the uniform empirical process (also for the uniform quantile process) stated by [15], see Theorem 5 below, we state analogue results for the bootstrapped processes $\alpha_{n,\mathscr{W}}(\cdot)$ and $\beta_{n,\mathscr{W}}(\cdot)$ defined respectively in (5.8) and (5.9). [15] result for the uniform empirical process is given in the following theorem, where $\{x\}_+ = \max\{x, 0\}$.

Theorem 5. *Let $a > -1$, then*

$$\lim_{\epsilon \searrow \frac{\sqrt{a+1}}{2}} \left(1 - \frac{a+1}{4\epsilon^2}\right)^{1/2} \sum_{n=1}^{+\infty} n^a \mathbb{E}\left\{\|\alpha_n\| - \epsilon\sqrt{2\log n}\right\}_+ = \frac{\sqrt{\pi/2}}{a+1},$$

and

$$\lim_{\epsilon \searrow \frac{\sqrt{a+1}}{2}} \left(1 - \frac{a+1}{4\epsilon^2}\right)^{1/2} \sum_{n=1}^{+\infty} \frac{(\log n)^a}{n} \mathbb{E}\left\{\|\alpha_n\| - \epsilon\sqrt{2\log n}\right\}_+ = \frac{\sqrt{\pi/2}}{a+1}.$$

Roughly speaking, our aim now, is to study the same kind of results when we replace $\alpha_n(\cdot)$ by the generalized bootstrapped empirical process to $\alpha_{n,\mathscr{W}}(\cdot)$ and the generalized bootstrapped quantile process $\beta_{n,\mathscr{W}}(\cdot)$. Let us recall Proposition 2.1 of [15], stating the following result for the Brownian bridge $\{B(t), 0 \leq t \leq 1\}$.

Theorem 6. *Let $a > -1$, $a_n = o(1/\log n)$, then*

$$\lim_{\epsilon \searrow \frac{\sqrt{a+1}}{2}} \left(1 - \frac{a+1}{4\epsilon^2}\right)^{1/2} \sum_{n=1}^{+\infty} n^a \mathbb{E}\left\{\|B\| - (\epsilon + a_n)\sqrt{2\log n}\right\}_+ = \frac{\sqrt{\pi/2}}{a+1}.$$

Theorem 6.1. *Let $a > -1$, then*

$$\lim_{\epsilon \searrow \frac{\sqrt{a+1}}{2}} \left(1 - \frac{a+1}{4\epsilon^2}\right)^{1/2} \sum_{n=1}^{+\infty} n^a \mathbb{E}\left\{\|\alpha_{n,\mathscr{W}}\| - \epsilon\sqrt{2\log n}\right\}_+ = \frac{\sqrt{\pi/2}}{a+1},$$

and

$$\lim_{\epsilon \searrow \frac{\sqrt{a+1}}{2}} \left(1 - \frac{a+1}{4\epsilon^2}\right)^{1/2} \sum_{n=1}^{+\infty} \frac{(\log n)^a}{n} \mathbb{E}\left\{\|\alpha_{n,\mathscr{W}}\| - \epsilon\sqrt{2\log n}\right\}_+ = \frac{\sqrt{\pi/2}}{a+1}.$$

Theorem 6.2. *Let $a > -1$, then*

$$\lim_{\epsilon \searrow \frac{\sqrt{a+1}}{2}} \left(1 - \frac{a+1}{4\epsilon^2}\right)^{1/2} \sum_{n=1}^{+\infty} n^a \mathbb{E}\left\{\|\beta_{n,\mathscr{W}}\| - \epsilon\sqrt{2\log n}\right\}_+ = \frac{\sqrt{\pi/2}}{a+1},$$

and

$$\lim_{\epsilon \searrow \frac{\sqrt{a+1}}{2}} \left(1 - \frac{a+1}{4\epsilon^2}\right)^{1/2} \sum_{n=1}^{+\infty} \frac{(\log n)^a}{n} \mathbb{E}\left\{\|\beta_{n,\mathscr{W}}\| - \epsilon\sqrt{2\log n}\right\}_+ = \frac{\sqrt{\pi/2}}{a+1}.$$

7. Approximations of the bootstrapped Kac empirical processes

Let us define

$$\widetilde{A}_n(t)=\sum_{i=1}^{n}\left(Z_i-\overline{Z}_n\right)\mathbb{1}_{\{X_i\leq t\}},\ \text{for}\ -\infty<t<\infty,$$

where $\overline{Z}_n$ denotes the empirical mean. Let us remark that

$$\begin{aligned}\alpha_{n,\mathscr{W}}(t) &= \frac{n}{T_n}\left\{n^{-1/2}\sum_{i=1}^{n}(Z_i-\overline{Z}_n)\mathbb{1}_{\{U_i\leq t\}}\right\}\\ &= \left(\frac{n}{T_n}\right)n^{-1/2}A_n(t).\end{aligned}$$

In this section, we are mainly concerned with the strong approximation of the following process

$$\widetilde{A}_{\nu_n}(t)=\sum_{i=1}^{\nu_n}\left(Z_i-\overline{Z}_n\right)\mathbb{1}_{\{X_i\leq t\}},\ \text{for}\ -\infty<t<\infty,$$

where ν_n denotes a Poisson random variable with mean n, independent of Z_i's and X_i's by a sequence of Brownian bridges.

7.1. Some useful results.

Consider now the version of $\mathbb{L}_{\nu_n}(\cdot)$ on $(0,1)$, let

$$\mathbb{L}_{\nu_n}(t):=\sum_{i=1}^{\nu_n}\mathbb{1}_{\{U_i\leq t\}},\quad \text{for}\quad 0<t<1,$$

where the random variables U_i's are i.i.d. uniformly distributed on $(0,1)$. Let us introduce the Poisson bridges defined, for each $n\in\mathbb{N}^*$, by

$$\mathbb{N}_{\nu_n}(t):=\frac{1}{\sqrt{n}}\big(\mathbb{L}_{\nu_n}(t)-\nu_n t\big)\quad \text{for}\quad 0<t<1.$$

The inequality of [33] stipulates that there exists a positive constant c_{13} such that, for any $x>0$ and any $n\in\mathbb{N}^*$,

$$\mathbb{P}\left\{\sup_{t\in\mathbb{R}}|\mathbb{F}_n(t)-F(t)|\geq\frac{x}{\sqrt{n}}\right\}\leq c_{13}\ \exp(-2x^2). \tag{7.1}$$

Actually (7.1) simply reads, by means of $\alpha_n(\cdot)$, for any $x>0$ and any $n\in\mathbb{N}^*$, as

$$\mathbb{P}\bigg\{\sup_{u\in[0,1]}|\alpha_n(u)|\geq x\bigg\}\leq c_{13}\ \exp(-2x^2).$$

We also mention some bounds that we will use further. By appealing to Chung's law of the iterated logarithm for the empirical process, see [18], which stipulates that

$$\limsup_{n\to\infty}\frac{\sup_{t\in\mathbb{R}}|\widetilde{\alpha}_n(t)|}{\sqrt{\log_{(2)}n}}=\frac{1}{\sqrt{2}}\quad \text{a.s.},$$

we see that, with probability 1, as $n\to\infty$,

$$\sup_{t\in\mathbb{R}}|\widetilde{\alpha}_n(t)|=\mathcal{O}\Big(\sqrt{\log_{(2)}n}\Big). \tag{7.2}$$

Moreover, by [57], on a suitable probability space, we can define the uniform empirical process $\{\alpha_n:n\in\mathbb{N}^*\}$, in combination with a sequence of Brownian bridges $\{B_n:n\in\mathbb{N}^*\}$ together with a Kiefer process $\{K(s,u):s\geq 0,u\in[0,1]\}$, such that, with probability 1, as $n\to\infty$,

$$\sup_{u\in[0,1]}|\alpha_n(u)-B_n(u)|=\mathcal{O}\left(\frac{\log n}{\sqrt{n}}\right), \tag{7.3}$$

and

$$\max_{1\le k\le n}\sup_{u\in[0,1]}\left|\sqrt{k}\,\alpha_k(u)-K(k,u)\right|=\mathcal{O}\big((\log n)^2\big)\,,$$

from which we extract, with probability 1, as $n\to\infty$,

$$\sup_{u\in[0,1]}\left|\alpha_n(u)-\frac{1}{\sqrt{n}}\,K(n,u)\right|=\mathcal{O}\bigg(\frac{(\log n)^2}{\sqrt{n}}\bigg). \tag{7.4}$$

As a result, by putting (7.2) into (7.3) and (7.4), one derives the following bounds: with probability 1, as $n\to\infty$,

$$\sup_{u\in[0,1]}|B_n(u)|=\mathcal{O}\Big(\sqrt{\log_{(2)}n}\Big)\quad\text{and}\quad\sup_{u\in[0,1]}|K(n,u)|=\mathcal{O}\Big(\sqrt{n\log_{(2)}n}\Big). \tag{7.5}$$

Notice that the second bound in (7.5) comes also from the law of the iterated logarithm for the Kiefer process; see [20], p. 81.

7.2. **Poisson index.** Recall that, letting $\{\nu_n : n\in\mathbb{N}^*\}$ be a sequence of Poisson r.v.'s independent of the sequence $\{X_i : i\in\mathbb{N}^*\}$ such that $\mathbb{E}(\nu_n)=n$ and, for each $n\in\mathbb{N}^*$, set

$$\mathbb{L}_{\nu_n}(t):=\sum_{i=1}^{\nu_n}\mathbb{1}_{\{X_i\le t\}},\quad\text{for}\quad t\in\mathbb{R}.$$

It is easy to check that for any $n\in\mathbb{N}^*$, $\mathbb{L}_{\nu_n}=\{\mathbb{L}_{\nu_n}(t):t\in\mathbb{R}\}$ is a Poisson process with intensity $\mathbb{E}(\mathbb{L}_{\nu_n}(t))=nF(t)$, see [38]. Let $\widetilde{\nu}_\lambda$ be a Poisson random variable with mean $\lambda>0$, and let $U_1,U_2,\ldots$ be independent real random variables with law $U(0,1)$ independent of $\widetilde{\nu}_\lambda$. [54] defines the modified empirical process by

$$\widetilde{\alpha}_\lambda(t)=\sqrt{\lambda}\left(\lambda^{-1}\sum_{i=1}^{\widetilde{\nu}_\lambda}\mathbb{1}_{\{U_i\le t\}}-t\right),\quad\text{for}\quad 0\le t\le 1,$$

where the sum is taken to be zero if $\widetilde{\nu}_\lambda=0$. Some important properties of this process such that $\{\widetilde{\alpha}_\lambda(t):0\le t\le 1\}$ is an independent increment process, $\sqrt{\frac{\widetilde{\nu}_\lambda}{\lambda}}\widetilde{\alpha}_{\widetilde{\nu}_\lambda}(\cdot)\to_{\mathscr{D}}B(\cdot)$, can be found in chapter 7 of [20], also in the general case when $\widetilde{\nu}_\lambda$ is replaced by a sequence $\{\widetilde{\nu}_n\}$ of positive integer value random variables defined on the same probability space. Without loss of generality we will consider that $X_i=F^{-1}(U_i)$ and we will replace $\widetilde{A}_{\nu_n}(t)$ for $t\in\mathbb{R}$ by $A_{\nu_n}(t)$ for $t\in[0,1]$.

Theorem 7.1. *If conditions (5.2), (5.3) and (5.4) hold, consider also the random variable ν_n with Poisson law with mean n, then there exists a sequence of Brownian bridges $\{B_n(t),\ 0\le t\le 1\}$ such that*

$$\mathbb{P}\left\{\sup_{0\le t\le 1}|n^{-1/2}A_{\nu_n}(t)-B_n(t)|>n^{-1/2}(c_{14}\log n+x)\right\}\le c_{15}\exp(-c_{16}x), \tag{7.6}$$

for all $x\ge 0$, where c_{14}, c_{15} and c_{16} are positive universal constants.

8. Proofs

This section is devoted to the proofs of our results. The previously presented notation continues to be used in the following.

PROOF OF THEOREM 3.2.

We will follows the proof of [24]. Given any $0 < \varepsilon < 1/2$, by (3.9) we get

$$\int_{\varepsilon}^{1/2} \frac{||\beta_{m,n}(s)|^p - |\widetilde{B}_m(s)|^p|}{q(s)} ds = o_{\widetilde{\mathbb{P}}}(1). \tag{8.1}$$

By Markov's inequality we obtain

$$\lim_{\epsilon\to 0} \limsup_{m\to\infty} \widetilde{\mathbb{P}} \left\{ \int_{1/(m+1)}^{\varepsilon} \frac{|\widetilde{B}_m(s)|^p}{q(s)} ds > \delta \right\} = 0, \tag{8.2}$$

for all $\delta > 0$. Now making use of (3.9) with $\nu = 0$, we get

$$\begin{aligned} \int_{1/(m+1)}^{\varepsilon} \frac{|\beta_{m,n}(s)|^p}{q(s)} ds &\leq 2^p \int_{1/(m+1)}^{\varepsilon} \frac{|\beta_{m,n}(s) - \widetilde{B}_m(s)|^p}{q(s)} ds + 2^p \int_{1/(m+1)}^{\varepsilon} \frac{|\widetilde{B}_m(s)|^p}{q(s)} ds \\ &= O_{\widetilde{\mathbb{P}}}(1) \int_0^{\varepsilon} s^{p/2}/q(s) ds + 2^p \int_{1/(m+1)}^{\varepsilon} \frac{|\widetilde{B}_m(s)|^p}{q(s)} ds. \qquad (8.3) \end{aligned}$$

Therefore

$$\lim_{\epsilon\to 0} \limsup_{m\to\infty} \widetilde{\mathbb{P}} \left\{ \int_{1/(m+1)}^{\varepsilon} \frac{|\beta_{m,n}(s)|^p}{q(s)} ds > \delta \right\} = 0, \tag{8.4}$$

for all $\delta > 0$. Hence (3.18) and (3.19) are proven. The proofs of (3.20) and (3.21) are similar, and hence omitted. □

PROOF OF THEOREM 3.3.

First we note that it follows by Markov's inequality that

$$n^{-\nu} \int_{1/(m+1)}^{m/(m+1)} |\widetilde{B}_m(s)|^{p-1}/(s(1-s))^{p/2+1/2+\nu} ds = O_{\widetilde{\mathbb{P}}}(1), \tag{8.5}$$

for $\nu > 0$. Note that for $p \geq 1$, we have

$$||a|^p - |b|^p| \leq p2^{p-1}|a-b|^p + p2^{p-1}|b|^{p-1}|a-b|. \tag{8.6}$$

The last equation when combined with (3.9) implies, for $\nu > 0$, that

$$\begin{aligned} &\int_{1/(m+1)}^{m/(m+1)} \frac{||\beta_{m,n}(s)|^p - |B_m^*(s)|^p|}{(s(1-s))^{p/2+1}} ds \\ &\leq p2^{p-1} \int_{1/(m+1)}^{m/(m+1)} \frac{|\beta_{m,n}(s) - B_m^*(s)|^p}{(s(1-s))^{p/2+1}} ds \\ &\quad + p2^{p-1} \int_{1/(m+1)}^{m/(m+1)} \frac{|\beta_{m,n}(s) - B_m^*(s)||B_m^*(s)|^{p-1}}{(s(1-s))^{p/2+1}} ds \\ &= O_{\widetilde{\mathbb{P}}}(1). \qquad (8.7) \end{aligned}$$

This completes the proof of (3.22). In a similar way we get (3.23). By combining (3.23) and Theorem 3.2 of [24] (refer to [74]) we obtain (3.24). □

PROOF OF THEOREM 3.4.

As in [24] in the proof of Theorem 2.2, we have, for $1 + q/2 - \tau > 0$, $\tau < 1 + q/2$, that

$$\limsup_{m\to\infty} \widetilde{\mathbb{P}} \left\{ \left(\frac{\zeta_{1,m}}{m}\right)^{\tau-(q/2+1)} L\left(\frac{\zeta_{1,m}}{m}\right) \int_0^{\zeta_{1,m}/m} \frac{|W(t)|^q}{t^\tau L(t)} dt > K \right\} \leq \mu(q) \frac{1 + q/2 - \tau}{K},$$

for all $K > 0$. Consequently,

$$\left(\frac{\zeta_{1,m}}{m}\right)^{\tau-(q/2+1)} L\left(\frac{\zeta_{1,m}}{m}\right) \int_0^{\zeta_{1,m}/m} \frac{|W(t)|^q}{t^\tau L(t)} dt = O_{\widetilde{\mathbb{P}}}(1), \tag{8.8}$$

and similar argument also yields

$$\left(\frac{\zeta_{1,m}}{m}\right)^{\tau-(q/2+1)} L\left(\frac{\zeta_{1,m}}{m}\right) \int_0^{\zeta_{1,m}/m} \frac{|B(t)|^q}{t^\tau L(t)} dt = O_{\widetilde{\mathbb{P}}}(1). \tag{8.9}$$

Let $0 < \eta < 1/2$ so that

$$\nu < p/2 + 1 - p\eta.$$

By combining (3.9) with (8.6), we infer that

$$\begin{aligned}
&\left(\frac{\zeta_{1,m}}{m}\right)^{\nu-(q/2+1)} L\left(\frac{\zeta_{1,m}}{m}\right) \int_{1/(m+1)}^{\zeta_{1,m}/m} \frac{||\beta_{m,n}(s)|^p - |\widetilde{B}_m(s)|^p|}{s^\nu L(s)} ds \\
\leq\ & p2^{p-1}\left(\frac{\zeta_{1,m}}{m}\right)^{\nu-(q/2+1)} L\left(\frac{\zeta_{1,m}}{m}\right) \int_{1/(m+1)}^{\zeta_{1,m}/m} \frac{|\beta_{m,n}(s) - \widetilde{B}_m(s)|^p}{s^\nu L(s)} ds \\
& + p2^{p-1}\left(\frac{\zeta_{1,m}}{m}\right)^{\nu-(q/2+1)} L\left(\frac{\zeta_{1,m}}{m}\right) \int_{1/(m+1)}^{\zeta_{1,m}/m} \frac{|\beta_{m,n}(s) - \widetilde{B}_m(s)||\widetilde{B}_m(s)|^{p-1}}{s^\nu L(s)} ds \\
=\ & O_{\widetilde{\mathbb{P}}}(1)\left(\frac{\zeta_{1,m}}{m}\right)^{\nu-(q/2+1)} L\left(\frac{\zeta_{1,m}}{m}\right) n^{-p\eta} \int_{1/(m+1)}^{\zeta_{1,m}/m} \frac{s^{p/2-p\eta-\nu}}{L(s)} ds \\
& + O_{\widetilde{\mathbb{P}}}(1)\left(\frac{\zeta_{1,m}}{m}\right)^{\nu-(q/2+1)} L\left(\frac{\zeta_{1,m}}{m}\right) n^{-\eta} \int_{1/(m+1)}^{\zeta_{1,m}/m} \frac{\widetilde{B}_m^{p-1}(s)}{s^{\nu-1/2+\eta} L(s)} ds \\
=\ & o_{\widetilde{\mathbb{P}}}(1).
\end{aligned}$$

By using (4.9) of [24], we infer that

$$\left(\frac{\zeta_{1,m}}{m}\right)^{\nu-(q/2+1)} L\left(\frac{\zeta_{1,m}}{m}\right) \int_{1/(m+1)}^{\zeta_{1,m}/m} \frac{|\widetilde{B}_m(s)|^p}{s^\nu L(s)} ds \to_{\mathscr{D}} \int_0^1 s^{-\nu} |W(s)|^p ds.$$

This suffices for the proof of (3.26). In a similar way, we have

$$\left(\frac{\zeta_{1,m}}{m}\right)^{\nu-(p/2+1)} L\left(\frac{\zeta_{1,m}}{m}\right) \int_{U_{m,n}(1/(m+1))}^{U_{m,n}(\zeta_{1,m}/m)} \frac{||\alpha_{m,n}(s)|^p - |\widetilde{B}^*_m(s)|^p|}{s^\nu L(s)} ds = o_{\widetilde{\mathbb{P}}}(1).$$

By using (4.17) of [24], we infer that

$$\left(\frac{\zeta_{1,m}}{m}\right)^{\nu-(p/2+1)} L\left(\frac{\zeta_{1,m}}{m}\right) \int_{U_{m,n}(1/(m+1))}^{U_{m,n}(\zeta_{1,m}/m)} \frac{|\widetilde{B}^*_m(s)|^p}{s^\nu L(s)} ds \to_{\mathscr{D}} \int_0^1 s^{-\nu} |W(s)|^p ds. \tag{8.10}$$

This suffices for the proof of (3.27). Hence the proof of Theorem 3.4 is complete. □

Proof of Corollary 3.1.

The proof follows the same line of the proof of Corollary 2.1 of [24]. □

Proof of Corollary 3.2.

Let $1 \leq r_m \leq m - m/(m+1)$. By Theorems 3.2 and 3.3 of [24], we have

$$\int_{U_{m,n}(rm/m)\vee(rm/m)}^{U_{m,n}(rm/m)\wedge(rm/m)} \frac{|\widetilde{B}^*_m(s)|^p}{(s(1-s))^{p/2+1}} ds = O_{\widetilde{\mathbb{P}}}(1), \tag{8.11}$$

$$\int_{U_{m,m}}^{1} \frac{|\alpha_{m,n}(s)|^p}{(s(1-s))^{p/2+1}} ds = O_{\widetilde{\mathbb{P}}}(1), \tag{8.12}$$

and

$$\int_{0}^{U_{1,m}} \frac{|\alpha_{m,n}(s)|^p}{(s(1-s))^{p/2+1}} ds = O_{\widetilde{\mathbb{P}}}(1). \tag{8.13}$$

Moreover, from [71] we have that from a normalized Brownian bridge, we can obtain an Ornstein-Uhlenbeck process denoted by $V(\cdot)$ given by

$$\frac{B(t)}{\sqrt{t(1-t)}} = V\left(\frac{1}{2}\log\left(\frac{t}{1-t}\right)\right), \tag{8.14}$$

where $B(\cdot)$ denotes a brownian bridge. Now from Theorem 3.3 and 3.4 of [24], jointly with (3.24) and (8.14) we obtain (3.29). Hence the proof of Corollary 3.2 is complete. □

Proof of Theorem 4.2

We use the notation $\vartheta_m = \vartheta_{m,n}$. We have the following easy-to-check representation

$$\begin{aligned} L_{m,n}(t) - L_n(t) &= \frac{1}{\vartheta_m}\int_0^t \left(\mathbb{U}_{m,n}^{-1}(s) - \mathbb{U}_n^{-1}(s)\right) ds - \frac{\vartheta_m - \vartheta_n}{\vartheta_m} L_n(t) \\ &= \frac{1}{\vartheta_m}\int_0^t \left(\mathbb{U}_{m,n}^{-1}(s) - \mathbb{U}_n^{-1}(s)\right) ds \\ &\quad - \frac{L_n(t)}{\vartheta_m}\int_0^1 \left(\mathbb{U}_{m,n}^{-1}(s) - \mathbb{U}_n^{-1}(s)\right) ds. \end{aligned} \tag{8.15}$$

In order to work in the same spirit of [78], let us define

$$\begin{aligned} \lambda_{m,n}(t) &= \frac{1 - L_n(t)}{\vartheta_n}\int_0^t \left(\mathbb{U}_{m,n}^{-1}(s) - \mathbb{U}_n^{-1}(s)\right) ds \\ &\quad - \frac{L_n(t)}{\vartheta_n}\int_t^1 \left(\mathbb{U}_{m,n}^{-1}(s) - \mathbb{U}_n^{-1}(s)\right) ds, \\ \Lambda_{m,n}(t) &= -\frac{1 - L_n(t)}{\vartheta_n}\int_0^t \left(\mathbb{U}_{m,n}(s) - \mathbb{U}_n(s)\right) ds \\ &\quad + \frac{L_n(t)}{\vartheta_n}\int_t^1 \left(\mathbb{U}_{m,n}(s) - \mathbb{U}_n(s)\right) ds, \end{aligned} \tag{8.16}$$

$$\begin{aligned} \mathbb{V}_{m,n}(t) &= \int_0^t \left(\mathbb{U}_{m,n}^{-1}(s) - \mathbb{U}_n^{-1}(s)\right) ds \\ &\quad + \int_0^t \left(\mathbb{U}_{m,n}(s) - \mathbb{U}_n(s)\right) ds - 2t^2. \end{aligned} \tag{8.17}$$

In similar spirt of [28], notice that we have

$$L_{m,n}(t) - L_n(t) = \frac{\vartheta_n}{\vartheta_m}\lambda_{m,n}(t)$$

where

$$\lambda_{m,n} = \Lambda_{m,n} + \vartheta_n^{-1}\mathbb{V}_{m,n}. \tag{8.18}$$

The relation (8.18) is obtained by combing change of variables and integration by parts with the following relation

$$-\int_0^1 \left(\mathbb{U}_{m,n}^{-1}(s) - \mathbb{U}_n^{-1}(s)\right) ds = \int_0^1 \left(\mathbb{U}_{m,n}(s) - \mathbb{U}_n(s)\right) ds.$$

We can also obtain, for $t > \max\{\mathbb{U}_{m,n}(t),\ \mathbb{U}_n(t)\}$ in other word $\mathbb{U}_{m,n}^{-1}(t) > t$ and $\mathbb{U}_n^{-1}(t) > t$,

$$\begin{aligned} \mathbb{V}_{m,n}(t) &= \int_t^{\min\left(\mathbb{U}_{m,n}^{-1}(t),\ \mathbb{U}_n^{-1}(t)\right)} \gamma_{m,n}(s)ds \\ &\quad + \int_{\min\left(\mathbb{U}_{m,n}^{-1}(t),\ \mathbb{U}_n^{-1}(t)\right)}^{\max\left(\mathbb{U}_{m,n}^{-1}(t),\ \mathbb{U}_n^{-1}(t)\right)} \left(t - \min\left(\mathbb{U}_{m,n}(s),\ \mathbb{U}_n(s)\right)\right) ds, \end{aligned} \tag{8.19}$$

where

$$\gamma_{m,n}(t) = \mathbb{U}_{m,n}(t) - \mathbb{U}_n(t).$$

We now prove that $\mathbb{V}_{m,n}$ is asymptotically negligible in the same way as in [78] to handle $\mathbb{V}_n$, refer to the equation (3.1) therein. Let us introduce the following definitions

$$\begin{aligned} a(t) &= \max\left(\mathbb{U}_{m,n}(t),\ \mathbb{U}_n(t)\right), \\ a^{-1}(t) &= \max\left(\mathbb{U}_{m,n}^{-1}(t),\ \mathbb{U}_n^{-1}(t)\right) \end{aligned}$$

and

$$\begin{aligned} b(t) &= \min\left(\mathbb{U}_{m,n}(t),\ \mathbb{U}_n(t)\right), \\ b^{-1}(t) &= \min\left(\mathbb{U}_{m,n}^{-1}(t),\ \mathbb{U}_n^{-1}(t)\right). \end{aligned}$$

Lemma 8.1. *We have, as $m = m(n) \longrightarrow \infty$*

$$\limsup_{m\to\infty} \delta_m^{-1/2} \|\mathbb{V}_{m,n}\| = 0,\ \ a.s., \tag{8.20}$$

where

$$\delta_m = m^{-1}\log_{(2)} m.$$

Proof of Lemma 8.1.

We have the inequalities

$$\begin{aligned} |\mathbb{V}_{m,n}(t)| &\leq \left|\int_t^{b^{-1}(t)} \gamma_{m,n}(s)ds\right| + \left|\int_{b^{-1}(t)}^{a^{-1}(t)} (t - b(s))\, ds\right| \\ &\leq \sup_{s\in[t,\ b^{-1}(t)]} |\gamma_{m,n}(s)| \left(b^{-1}(t) - t\right) \\ &\quad + \sup_{s\in[b^{-1}(t),\ a^{-1}(t)]} |t - b(s)| \left(a^{-1}(t) - b^{-1}(t)\right) \\ &\leq |t - E_m(t)| \left(b^{-1}(t) - t\right) \\ &\quad + \sup_{s\in[b^{-1}(t),\ a^{-1}(t)]} |t - b(s)| \left(a^{-1}(t) - b^{-1}(t)\right) \\ &\leq \|E_m - I\| \left(a^{-1}(t) - t\right). \end{aligned}$$

We therefore obtain readily that

$$\delta_m^{-1/2}|\mathbb{V}_{m,n}(t)| \leq \delta_m^{-1/2}\left(\frac{\|E_m - I\|}{q_\delta} \times q_\delta \max\left(\|\mathbb{U}_n^{-1} - I\|,\ \|\mathbb{U}_{m,n}^{-1} - I\|\right)\right),$$

where

$$q_\delta(t) = t^{1/2-\delta}(1-t)^{1/2-\delta}.$$

We obtain from [53] that

$$\limsup_{n\longrightarrow 0} \delta_m^{-1/2} \frac{\|E_m - I\|}{q_\delta} = 0,\ \ a.s. \tag{8.21}$$

From exercise 5 on page 651 of [71], we can refer also to [64], we have that

$$\max\left(\|q_\delta(\mathbb{U}_n^{-1}-I)\|,\ \|q_\delta(\mathbb{U}_{m,n}^{-1}-I)\|\right)=o(1),\ a.s. \tag{8.22}$$

By combining (8.21) and (8.22), readily implies (8.20). □

Making use of (8.18) and Lemma 8.1 completes the proof of Theorem 4.2. □

Proof of Theorem 4.3.

Let us collect some known results that are needed in the proof of the Theorem 4.2. From [21] p.159, we infer that we have

$$\sup_{0\leq t\leq 1}|\mathbb{U}_n^{-1}(t))-t|=\sup_{0\leq t\leq 1}|\mathbb{U}_n(t))-t|. \tag{8.23}$$

An application of the law of iterated logarithm, refer to Theorem 5.3.1 in [20], implies, with probability one, that

$$\sup_{0\leq t\leq 1}|\mathbb{U}_n^{-1}(t))-t|=O\left(n^{-1/2}(\log_{(2)}n)^{1/2}\right). \tag{8.24}$$

Let us remark that we have

$$\begin{aligned}\mathbb{U}_n^{-1}(E_m^{-1}(t))-\mathbb{U}_n^{-1}(t) &= \mathbb{U}_n^{-1}(E_m^{-1}(t))-E_m^{-1}(t)\\ &\quad -(\mathbb{U}_n^{-1}(t)-b^{-1}(t))+E_m^{-1}(t)-t.\end{aligned}$$

This when combined with (8.23), implies that

$$\mathbb{U}_{m,n}^{-1}(t)-\mathbb{U}_n^{-1}(t)\ \text{and}\ -(\mathbb{U}_n(t)-\mathbb{U}_{m,n}(t)), \tag{8.25}$$

are asymptotically equivalent. From [30], we have

$$\begin{aligned}\mathbb{U}_n^{-1}(E_m^{-1}(t))-\mathbb{U}_n^{-1}(t) &= \frac{B_{m,n}^*(t)}{\sqrt{m}}+\frac{B_n^*(t)}{\sqrt{n}}\\ &\quad +O(m^{-1/2}(l(m)\vee l(n)))+E_m^{-1}(t)-t,\end{aligned} \tag{8.26}$$

where we recall

$$l(m)=m^{-1/4}(\log m)^{1/2}(\log_{(2)}m)^{1/4}.$$

Recall the following inequality, for $c_{17}>0$ and $x>0$, see for instance p.2463 of [26],

$$\mathbb{P}\left(\sup_{0\leq t\leq 1}|B_n^*(t)|>x^{1/2}\right)\leq c_{17}\exp(-x/2), \tag{8.27}$$

Making use of [72] about the oscillation of the empirical processes, we infer that

$$\begin{aligned}-(\mathbb{U}_n(t)-\mathbb{U}_{m,n}(t)) &= E_m(\mathbb{U}_n(t))-\mathbb{U}_n(t)\\ &= E_m(t)-t+O\left(\frac{n^{1/4}}{m^{1/2}}(\log n)^{-1/2}(\log_{(2)}n)^{-1/4}\right).\end{aligned}$$

Making use of [5] and [55, 56] results, we infer that

$$\begin{aligned}\mathbb{U}_{m,n}^{-1}(t)-\mathbb{U}_n^{-1}(t) &= E_m(t)-t\\ &\quad +O\left(\frac{n^{1/4}}{m^{1/2}}(\log n)^{-1/2}(\log_{(2)}n)^{-1/4}\right)\end{aligned} \tag{8.28}$$

We have

$$\sup_{0\leq t\leq 1}|t-b(t)|\leq\begin{cases}\sup_{0\leq t\leq 1}|t-U_n(t)|+\sup_{0\leq t\leq 1}|t-E_m(t)| & \text{if}\quad b(t)=U_{m,n}(t),\\ \sup_{0\leq t\leq 1}|t-U_n(t)| & \text{if}\quad b(t)=U_n(t).\end{cases}$$

By the law of iterated logarithm, we readily infer that
(8.29)
$$\sup_{0\le t\le 1}|t-b(t)|\le\begin{cases} O\left(\left(\frac{\log_{(2)} n}{n}\right)^{1/2}\right)+O\left(\left(\frac{\log_{(2)} m}{m}\right)^{1/2}\right) & \text{if} \quad b(t)=U_{m,n}(t),\\ O\left(\left(\frac{\log_{(2)} n}{n}\right)^{1/2}\right) & \text{if} \quad b(t)=U_{n}(t).\end{cases}$$
In a similar way, we obtain
$$\sup_{0\le t\le 1}|t-b^{-1}(t)|\le\begin{cases} \sup_{0\le t\le 1}|t-U_{m,n}^{-1}(t)| & \text{if} \quad b^{-1}(t)=U_{m,n}^{-1}(t),\\ \sup_{0\le t\le 1}|t-U_{n}^{-1}(t)| & \text{if} \quad b^{-1}(t)=U_{n}^{-1}(t).\end{cases}$$
Making use of the last equation in combination with [21] and Theorem 5.3.1 in [20], we infer that
(8.30)
$$\sup_{0\le t\le 1}|t-b^{-1}(t)|\le\begin{cases} O\left(\{l(m)\vee l(n)\}\right)+O\left(\left(\frac{\log_{(2)} n}{n}\right)^{1/2}\right) & \text{if} \quad b(t)=U_{m,n}(t),\\ O\left(\left(\frac{\log_{(2)} n}{n}\right)^{1/2}\right) & \text{if} \quad b(t)=U_{n}(t).\end{cases}$$
Recall that
$$\mathbb{V}_{m,n}(t)=\int_t^{b^{-1}(t)}\gamma_{m,n}(s)ds+\int_{b^{-1}(t)}^{a^{-1}(t)}(t-b(s))\,ds. \tag{8.31}$$
It is noteworthy that we have
$$\begin{aligned}\mathbb{V}_{m,n}(t) &= -\frac{1}{2}(U_n^{-1}(t)-t)^2+\int_t^{b^{-1}(t)}\{\gamma_{m,n}(s)-\gamma_{m,n}(t)\}\,ds\\ &\quad+\int_{b^{-1}(t)}^{a^{-1}(t)}(t-b(s))ds\\ &\quad+(b^{-1}(t)-t)\gamma_{m,n}(t)-t(a^{-1}(t)-b^{-1}(t)).\end{aligned}$$
An application of Lemma 5.4 of [21], gives
$$\limsup_{n\to\infty}(\log n)^{-1/2}\sup_{0\le t\le 1-h}|\widetilde{B}_{m,n}(t+h)-\widetilde{B}_{m,n}(t)|\le(2h)^{1/2}.$$
By using the last equation and choosing $h(t)=(b^{-1}(t)-t)$ which implies $h=(l(m)\vee l(n))$, we have
$$\begin{aligned}\int_t^{b^{-1}(t)}\{\gamma_{m,n}(s)-\gamma_{m,n}(t)\}\,ds &= \frac{1}{\sqrt{m}}\int_t^{b^{-1}(t)}\{\alpha_{m,n}(t)-\alpha_{m,n}(s)\}\,ds\\ &= \frac{1}{\sqrt{m}}\int_t^{b^{-1}(t)}\left(\widetilde{B}_{m,n}(s)-\widetilde{B}_{m,n}(t)\right)ds\\ &\quad+O\left(m^{-1/2}(l(m)\vee l(n))\right)\\ &= O\left(m^{-1/4}(l(m)\vee l(n))^{1/2}\right).\end{aligned} \tag{8.32}$$
From (8.29), we obtain
$$\int_{b^{-1}(t)}^{a^{-1}(t)}(t-b(s))ds=O\left((\log_{(2)} n/n)^{1/2}\vee\ (\log_{(2)} m/m)^{1/2}\right). \tag{8.33}$$
Using once more (8.29), we have
$$|(b^{-1}(t)-t)\gamma_{m,n}(t)|=O\left((l(n)\vee l(m))/\sqrt{m}\right). \tag{8.34}$$

A combination of (8.28) with (8.29) yields to

$$t(a^{-1}(t)-b^{-1}(t))=o(1). \tag{8.35}$$

We infer readily from (8.32), (8.33), (8.34) and (8.35) that

$$\mathbb{V}_{m,n}(t)=-\frac{1}{2}(U_n^{-1}(t)-t)^2.$$

Therefore the proof is complete. □

PROOF OF THEOREM 5.1.

On one hand, one can write

$$\beta_{n,\mathscr{W}}(t)=\beta_n(t)-\beta_n(W_n^{-1}(t))-\sqrt{n}\{W_n^{-1}(t)-t\}, \tag{8.36}$$

where, for $0\leq t\leq 1$,

$$\begin{aligned} W_n(t) &= \sum_{i\leq nt} W_{i:n}=\frac{1}{T_n}\sum_{i\leq nt} Z_i\\ W_n^{-1}(t) &= \inf\{u: W_n(u)\geq t, u\in[0,1]\}. \end{aligned}$$

On the other hand, the process $\alpha_{n,\mathscr{W}}(t)$ can be expressed as

$$\alpha_{n,\mathscr{W}}(t)=\sqrt{n}\left(\frac{1}{T_n}-\frac{1}{n}\right)\sum_{i=1}^n Z_i\mathbb{1}_{\{U_i\leq t\}}+\frac{\sqrt{n}}{n}\sum_{i=1}^n(Z_i-1)\,\mathbb{1}_{\{U_i\leq t\}}.$$

In view of the last equation, we obtain readily that

$$\begin{aligned} &\mathbb{P}\left(\sup_{0\leq t\leq 1}|R_{n,\mathscr{W}}(t)-R_n(t)|>n^{-1/2}(c_{18}\log n+x)\right) \\ &=\mathbb{P}\left(\sup_{0\leq t\leq 1}|\alpha_{n,\mathscr{W}}(t)-\alpha_n(t)+\beta_{n,\mathscr{W}}(t)-\beta_n(t)|>n^{-1/2}(c_{18}\log n+x)\right)\\ &\leq\mathbb{P}\left(\sup_{0\leq t\leq 1}|\alpha_{n,\mathscr{W}}(t)-\alpha_n(t)|+\sup_{0\leq t\leq 1}|\beta_{n,\mathscr{W}}(t)-\beta_n(t)|>n^{-1/2}(c_{18}\log n+x)\right)\\ &\leq\mathbb{P}\left(\sup_{0\leq t\leq 1}|\alpha_{n,\mathscr{W}}(t)-\alpha_n(t)|>2^{-1}n^{-1/2}(c_{18}\log n+x)\right)\\ &+\mathbb{P}\left(\sup_{0\leq t\leq 1}|\beta_{n,\mathscr{W}}(t)-\beta_n(t)|>2^{-1}n^{-1/2}(c_{18}\log n+x)\right)\\ &=I_{n,1}+I_{n,2}. \end{aligned} \tag{8.37}$$

We can write

$$\begin{aligned} I_{n,2} &= \mathbb{P}\left(\sup_{0\leq t\leq 1}|\beta_n(W_n^{-1}(t))+\sqrt{n}\{W_n^{-1}(t)-t\}|>n^{-1/2}(c_{19}\log n+x)\right)\\ &\leq \mathbb{P}\left(\sup_{0\leq t\leq 1}|\beta_n(W_n^{-1}(t))-B_n^{(2)}(W_n^{-1}(t))|>n^{-1/2}(c_{20}\log n+x/3)\right)\\ &+\mathbb{P}\left(\sup_{0\leq t\leq 1}|B_n^{(2)}(W_n^{-1}(t))-B_n^{(2)}(t)|>n^{-1/2}(c_{21}\log n+x/3)\right)\\ &+\mathbb{P}\left(\sup_{0\leq t\leq 1}|\sqrt{n}\{W_n^{-1}(t)-t\}+B_n^{(2)}(t)|>n^{-1/2}(c_{22}\log n+x/3)\right). \end{aligned}$$

Making use of Theorem 4, one has

$$\mathbb{P}\left(\sup_{0\leq t\leq 1}|\beta_n(W_n^{-1}(t))-B_n^{(2)}(W_n^{-1}(t))|>n^{-1/2}(c_{10}\log n+x/3)\right)\leq c_{11}\exp\left(-\frac{c_{12}}{3}x\right).$$

The two others probabilities can be handled analogously to (6.8) and (6.9) in [1]. Hence, we conclude that, with probability one, we have

$$\sup_{0\le t\le 1}|\beta_{n,\mathscr{W}}(t)-\beta_n(t)| = O\left(n^{-1/2}\log n\right).$$

We next evaluate the first term $I_{n,1}$ in the right side of (8.37). We can therefore write the following chain of inequalities

$$\begin{aligned}
&\mathbb{P}\left(\sup_{0\le t\le 1}|\alpha_{n,\mathscr{W}}(t)-\alpha_n(t)| > n^{-1/2}(c_{23}\log n + x)\right)\\
&\le \mathbb{P}\left(\sup_{0\le t\le 1}|\alpha_{n,\mathscr{W}}(t)| + \sup_{0\le t\le 1}|\alpha_n(t)| > n^{-1/2}(c_{23}\log n + x)\right)\\
&\le \mathbb{P}\left(\sup_{0\le t\le 1}|\alpha_{n,\mathscr{W}}(t)| > 2^{-1}n^{-1/2}(c_{23}\log n + x)\right)\\
&\quad +\mathbb{P}\left(\sup_{0\le t\le 1}|\alpha_n(t)| > 2^{-1}n^{-1/2}(c_{23}\log n + x)\right),
\end{aligned}\tag{8.38}$$

for some positive constant c_{23}. The first term in the right hand-side of (8.38) can be evaluated by

$$\begin{aligned}
&\mathbb{P}\left(\sup_{0\le t\le 1}|\alpha_{n,\mathscr{W}}(t)| > n^{-1/2}(c_{24}\log n + x)\right)\\
&\le \mathbb{P}\left(\sup_{0\le t\le 1}\left|\sqrt{n}\left(\frac{1}{T_n}-\frac{1}{n}\right)\sum_{i=1}^n Z_i\mathbb{1}_{\{U_i\le t\}}\right|\right.\\
&\quad \left. + \sup_{0\le t\le 1}\left|\frac{\sqrt{n}}{n}\sum_{i=1}^n (Z_i-1)\,\mathbb{1}_{\{U_i\le t\}}\right| > n^{-1/2}(c_{24}\log n + x)\right)\\
&= \mathbb{P}\left(\sup_{0\le t\le 1}\left|\left(\frac{n}{T_n}-1\right)\frac{1}{\sqrt{n}}\sum_{i=1}^n Z_i\mathbb{1}_{\{U_i\le t\}}\right|\right.\\
&\quad \left. + \sup_{0\le t\le 1}\left|\frac{\sqrt{n}}{n}\sum_{i=1}^n (Z_i-1)\,\mathbb{1}_{\{U_i\le t\}}\right| > n^{-1/2}(c_{24}\log n + x)\right).
\end{aligned}$$

[50, p.3] constructed a sequence of Wiener processes $W_n(t)$ in such a way that

$$\mathbb{P}\left(\sup_{0\le t\le 1}\left|\frac{1}{\sqrt{n}}\sum_{i=1}^n (Z_i-1)\mathbb{1}_{\{U_i\le t\}} - W_n(t)\right| > n^{-1/2}(c_{25}\log n + x)\right) \le c_{26}\exp(-c_{27}x),$$

where $c_{25}, c_{26}, c_{27} > 0$ are universal constants. Moreover from [1] (6.6)

$$\mathbb{P}\left(\left|\frac{n}{T_n}-1\right|\ge 1\right)\le c_{28}\exp(-c_{29}x),$$

where $c_{28}, c_{29} > 0$ are universal constants and from [1] (6.3) and (6.4)

$$\begin{aligned}
&\mathbb{P}\left(\sup_{0\le t\le 1}\left|\frac{1}{\sqrt{n}}\sum_{i=1}^n Z_i\mathbb{1}_{\{U_i\le t\}}\right| > 2^{-1}n^{-1/2}(c_{24}\log n + x)\right)\\
&\le \mathbb{P}\left(\left|\sum_{i=1}^n Z_i\right| > x/2\right)\le c_{30}\exp(-c_{31}x),
\end{aligned}$$

where $c_{30}, c_{31} > 0$ are universal constants. The second term in the right hand-side of (8.38), can be evaluated by using the results of [57]. More precisely, one can construct a sequence of Brownian bridges in such a way that

$$\begin{aligned}
&\mathbb{P}\left(\sup_{0\leq t\leq 1}|\alpha_n(t)| > 2^{-1}n^{-1/2}(c_{23}\log n + x)\right)\\
&\leq \mathbb{P}\left(\sup_{0\leq t\leq 1}|\alpha_n(t) - B_n^{(1)}(t)| > 2^{-2}n^{-1/2}(c_{23}\log n + x)\right)\\
&\quad +\mathbb{P}\left(\sup_{0\leq t\leq 1}|B_n^{(1)}(t)| > 2^{-2}n^{-1/2}(c_{23}\log n + x)\right).
\end{aligned}$$

From [62], see Theorem 1, we have, for $x > 0$,

$$\mathbb{P}\left\{\sup_{0\leq t\leq 1}|\alpha_n(t) - B_n^{(1)}(t)| \geq n^{-1/2}(c_1\log n + x)\right\} \leq c_2\exp(-c_3 x). \tag{8.39}$$

Making use of similar arguments as in (6.5) and (6.6) of [1] and keeping in mind equation (8.27), we have

$$\begin{aligned}
&\mathbb{P}\left(\sup_{0\leq t\leq 1}|B_n^{(1)}(t)| > 2^{-2}n^{-1/2}(c_{23}\log n + x)\right)\\
&\leq \mathbb{P}\left(\sup_{0\leq t\leq 1}|B_n^{(1)}(t)| > 2^{-2}n^{-1/2}x\right)\\
&\leq \exp(-c_{32}x),
\end{aligned}$$

for some positive constant $c_{32} > 0$. We conclude that

$$\sup_{0\leq t\leq 1}|\alpha_{n,\mathscr{W}}(t) - \alpha_n(t)| = O\left(n^{-1/2}\log n\right).$$

Hence the proof is complete. □

Proof of Theorem 5.2

By the law of the iterated logarithm for empirical process the results of [55, 56] immediately implies

$$\limsup_{n\to\infty} n^{1/4}(\log n)^{-1/2}(\log_{(2)} n)^{-1/4}\|R_n\| = 2^{-1/4} \tag{8.40}$$

$$\liminf_{n\to\infty} n^{1/4}(\log n)^{-1/2}(\log_{(2)} n)^{1/4}\|R_n\| = \frac{\pi^{1/2}}{8^{1/4}}. \tag{8.41}$$

We may refer to [19, 847] for more details and references on these relations. By combining relation (8.40) with Theorem 5.1, we infer that

$$\begin{aligned}
&\limsup_{n\to\infty} n^{1/4}(\log n)^{-1/2}(\log_{(2)} n)^{-1/4}\|R_{n,\mathscr{W}}\|\\
&\leq \limsup_{n\to\infty} n^{1/4}(\log n)^{-1/2}(\log_{(2)} n)^{-1/4}\|R_{n,\mathscr{W}} - R_n\|\\
&\quad + n^{1/4}(\log n)^{-1/2}(\log_{(2)} n)^{-1/4}\|R_n\|\\
&= O\left(n^{1/4}(\log n)^{-1/2}(\log_{(2)} n)^{-1/4}\frac{\log n}{n^{1/2}}\right)\\
&\quad + \limsup_{n\to\infty} n^{1/4}(\log n)^{-1/2}(\log_{(2)} n)^{-1/4}\|R_n\|\\
&= O\left(n^{-1/4}(\log n)^{1/2}(\log_{(2)} n)^{-1/4}\right)\\
&\quad + \limsup_{n\to\infty} n^{1/4}(\log n)^{-1/2}(\log_{(2)} n)^{-1/4}\|R_n\|\\
&= 2^{-1/4},\ a.s.
\end{aligned}$$

By using Theorem 5.1 in connection with (8.41), equation (5.13) can be obtained analogously. Therefore the proof is complete. □

Proof of Corollary 5.1

Making use of the triangle inequality, we obtain readily that

$$\|V_{n,\mathscr{W}} - n^{-1/2}\alpha_n^2\| \leq \|V_{n,\mathscr{W}} - V_n\| + \|V_n - n^{-1/2}\alpha_n^2\|.$$

An application of Theorem 4.1 of [78] gives

$$\|V_n - n^{-1/2}\alpha_n^2\| = o(n^{-1}\log_{(2)} n). \tag{8.42}$$

Now, we infer from Theorem 5.1 that

$$\|V_{n,\mathscr{W}} - V_n\| = O(n^{-1/2}\log n). \tag{8.43}$$

Theorem 5.1 is consequence of (8.42) and (8.43). □

Proof of Theorem 6.1

In a similar way as in the paper by [15], we infer, for $p < -1/2$, that

$$\begin{aligned}
\mathbb{E}\left[\left\|B_{n,\mathscr{W}}^{(1)}\right\| - \left(\epsilon\sqrt{2\log n} + (\log n)^p\right)\right]_+ - \mathbb{E}\left[\sup_{0\leq t\leq 1}\left|\alpha_{n,\mathscr{W}}(t) - B_{n,\mathscr{W}}^{(1)}(t)\right| - (\log n)^p\right]_+ \\
\leq \mathbb{E}\left[\|\alpha_{n,\mathscr{W}}\| - \epsilon\sqrt{2\log n}\right]_+ \\
\leq \mathbb{E}\left[\left\|B_{n,\mathscr{W}}^{(1)}\right\| - \left(\epsilon\sqrt{2\log n} + (\log n)^p\right)\right]_+ + \mathbb{E}\left[\sup_{0\leq t\leq 1}\left|\alpha_{n,\mathscr{W}}(t) - B_{n,\mathscr{W}}^{(1)}(t)\right| - (\log n)^p\right]_+ .
\end{aligned}$$

Making use of Theorem 3 in connection with the following relation

$$\mathbb{E}\left[\|B_{n,\mathscr{W}}^{(1)}\| - t\right]_+ = \int_0^\infty \mathbb{P}\left(\|B_{n,\mathscr{W}}^{(1)}\| \geq t + x\right)dx,$$

it follows in a similar fashion as in [15], p.9089, that

$$\begin{aligned}
&\sum_{n=1}^\infty n^a\mathbb{E}\left[\sup_{0\leq t\leq 1}|\alpha_{n,\mathscr{W}}(t) - B_{n,\mathscr{W}}^{(1)}(t)| - (\log n)^p\right]_+ \\
&= \sum_{n=1}^\infty n^a\int_0^\infty \mathbb{P}\left(\sup_{0\leq t\leq 1}\left|\beta_{n,\mathscr{W}}(t) - B_{n,\mathscr{W}}^{(1)}(t)\right| \geq (\log n)^p + x\right)dx \\
&\leq \sum_{n=1}^\infty \frac{c_2 n^a\exp(c_3c_1\log n)}{\exp(c_3\sqrt{n}(\log n)^p)}\int_0^\infty \exp(-c_3\sqrt{n}x)dx \\
&< \infty.
\end{aligned}$$

We then obtain

$$\lim_{\epsilon\searrow\frac{\sqrt{a+1}}{2}}\left(1 - \frac{a+1}{4\epsilon^2}\right)^{1/2}\sum_{n=1}^\infty n^a\mathbb{E}\left[\sup_{0\leq t\leq 1}\left|\alpha_{n,\mathscr{W}}(t) - B_{n,\mathscr{W}}^{(1)}(t)\right| - (\log n)^p\right]_+ = 0.$$

The rest of proof of being similar the proof of Theorem 1.1 of [15], and therefore, omitted. □

PROOF OF THEOREM 6.2

An application of Theorem 4 in combination with

$$\mathbb{E}\left[\left\|B^{(2)}_{n,\mathscr{W}}\right\| - t\right]_{+} = \int_0^\infty \mathbb{P}\left(\left\|B^{(2)}_{n,\mathscr{W}}\right\| \geq t + x\right) dx,$$

implies, in the same way as in [15], p.9089, that

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^a \mathbb{E}\left[\sup_{0\leq t\leq 1}\left|\beta_{n,\mathscr{W}}(t) - B^{(2)}_{n,\mathscr{W}}(t)\right| - (\log n)^p\right]_{+} \\
&= \sum_{n=1}^{\infty} n^a \int_0^\infty \mathbb{P}\left(\sup_{0\leq t\leq 1}\left|\beta_{n,\mathscr{W}}(t) - B^{(2)}_{n,\mathscr{W}}(t)\right| \geq (\log n)^p + x\right) dx \\
&\leq \sum_{n=1}^{\infty} \frac{c_2 n^a \exp(c_3 c_1 \log n)}{\exp(c_3\sqrt{n}(\log n)^p)} \int_0^\infty \exp(-c_3\sqrt{n}x)dx \\
&< \infty.
\end{aligned}$$

We infer that

$$\lim_{\epsilon\searrow\frac{\sqrt{a+1}}{2}} \left(1 - \frac{a+1}{4\epsilon^2}\right)^{1/2} \sum_{n=1}^{\infty} n^a \mathbb{E}\left[\sup_{0\leq t\leq 1}\left|\beta_{n,\mathscr{W}}(t) - B^{(2)}_{n,\mathscr{W}}(t)\right| - (\log n)^p\right]_{+} = 0.$$

The proof is achieved by using similar arguments as in the proof of Theorem 1.1 of [15], and therefore, omitted. □

PROOF OF THEOREM 7.1

Let $\epsilon_i = Z_i - 1$. First of all, remark that we have

$$\begin{aligned}
n^{-1/2}A_{\nu_n}(t) - B_n(t) &= n^{-1/2}A_{\nu_n}(t) - n^{-1/2}A_n(t) \\
&\quad + n^{-1/2}A_n(t) - B_n(t).
\end{aligned} \tag{8.44}$$

[50] constructed a sequence of Wiener processes $\{W_n(t),\ 0 \leq t < \infty\}$ such that, for all $x \geq 0$,

$$\mathbb{P}\left(\sup_{0\leq t\leq 1}\left|n^{-1/2}\sum_{i=1}^{n}\epsilon_i \mathbb{1}_{\{U_i\leq t\}} - W_n(t)\right| > n^{-1/2}(c_{33}\log n + x)\right) \leq c_{34}\exp(-c_{35}x),$$

where $c_{33}, c_{34}, c_{35} > 0$ are universal constants. From the last equation, we readily infer that

$$\mathbb{P}\left(\sup_{0\leq t\leq 1}\left|n^{-1/2}\left(\sum_{i=1}^{n}\epsilon_i \mathbb{1}_{\{U_i\leq t\}} - t\sum_{i=1}^{n}\epsilon_i\right) - (W_n(t) - tW_n(1))\right| > n^{-1/2}(c_{36}\log n + x)\right) \leq c_{37}\exp(-c_{38}x),$$

where $c_{36}, c_{37}, c_{38} > 0$ are universal constants. This implies that

$$\mathbb{P}\left(\sup_{0\leq t\leq 1}|n^{-1/2}A_n(t) - B_n(t)| > n^{-1/2}(c_{36}\log n + x)\right) \leq c_{37}\exp(-c_{38}x),$$

where

$$B_n(t) = W_n(t) - tW_n(1).$$

Hence, the main term to be investigated in the relation (8.44) is $A_{\nu_n}(t) - A_n(t)$. Notice that we have

$$\begin{aligned} A_{\nu_n}(t) - A_n(t) &= \sum_{i=1}^{\nu_n} \epsilon_i \mathbb{1}_{\{U_i \leq t\}} - \sum_{i=1}^{n} \epsilon_i \mathbb{1}_{\{U_i \leq t\}} \\ &\quad -\bar{\epsilon}_n \left(\sum_{i=1}^{\nu_n} \mathbb{1}_{\{U_i \leq t\}} - \sum_{i=1}^{n} \mathbb{1}_{\{U_i \leq t\}} \right). \\ &= \sum_{i=1}^{\nu_n} \epsilon_i \mathbb{1}_{\{U_i \leq t\}} - \sum_{i=1}^{n} \epsilon_i \mathbb{1}_{\{U_i \leq t\}} \\ &\quad -\bar{\epsilon}_n \left(\mathbb{L}_{\nu_n}(t) - nU_n(t) \right). \end{aligned}$$

For any $t \in [0,1]$, we obtain

$$|n\mathbb{U}_n(t) - \mathbb{L}_{\nu_n}(t)| = |n\left(\mathbb{U}_n(t) - t\right) - \left(\mathbb{L}_{\nu_n}(t) - \nu_n t\right) + \left(nt - \nu_n t\right)|,$$

from which we deduce that

$$|n\mathbb{U}_n(t) - \mathbb{L}_{\nu_n}(t)| = |\sqrt{n}\left(\alpha_n(t) - \mathbb{N}_{\nu_n}(t)\right) + t\left(n - \nu_n\right)|.$$

[14], Theorem 2, showed the following result related to the corresponding empirical uniform process.

Theorem 7. *For all $x > 0$ and $n \in \mathbb{N}\backslash\{0,1\}$,*

$$\mathbb{P}\left\{ \sup_{u \in [0,1]} \left|\alpha_n(u) - \mathbb{N}_{\nu_n}(u)\right| \geq \frac{x}{n^{1/4}} \right\} \leq 8\exp(-x). \tag{8.45}$$

Applying Theorem 2 of [14] (see the preceding Theorem 7), we obtain

$$\mathbb{P}\left(\sup_{0 \leq t \leq 1} |\alpha_n(t) - \mathbb{N}\nu_n(t)| \geq xn^{-1/4} \right) \leq 8\exp(-x),$$

for all $x > 0$, and for $c_{39} > 0$,

$$\mathbb{P}\left(|\nu_n - n| > \sqrt{nx}/2\right) \leq \exp(-c_{39}n) \leq \exp(-c_{39}x)$$

if $0 \leq x \leq n$, where we have used (3.12) of [50], also see Theorem 2.6 of [69]. The case where $x > n$. Let us define

$$S_n = \sum_{i=1}^{n} Y_i,$$

where $Y_i = \delta_i - 1$ are i.i.d. and centered random variables, with δ_i of same Poisson with mean one law, then

$$\begin{aligned} \mathbb{E}[\exp(tY_1)] &= \exp(-t)\mathbb{E}[\exp(t\delta_1)] \\ &= \exp(-(t+1))\exp(\exp(t)). \end{aligned}$$

Notice that for $0 < t < 1$, we have $\mathbb{E}[\exp(tY_1)] = \exp(t^2/2 + o(1))$, and from Theorem 2.6 of [69], we have that, for $c_{40} > 0$,

$$\mathbb{P}\left(|\nu_n - n| > \sqrt{nx}/2\right) \leq \exp(-c_{40}x), \text{ for } x > c'n. \tag{8.46}$$

Then, for all $x > 0$, we have, for some positive constant c_{41},

$$\begin{aligned} &\mathbb{P}\left(\sup_{0 \leq t \leq 1} |n\mathbb{U}_n(t) - \mathbb{L}_{\nu_n}(t)| \geq n^{1/4}x + \sqrt{nx}/2 \right) \\ &\leq \mathbb{P}\left(\sup_{0 \leq t \leq 1} |\alpha_n(t) - \mathbb{N}_{\nu_n}(t)| \geq xn^{-1/4} \right) \\ &\quad + \mathbb{P}\left(|n - \nu_n| > \sqrt{nx}/2\right) \\ &\leq 8\exp(-x) + \exp(-c_{41}x). \end{aligned}$$

Hence, we obtain

$$\begin{aligned}
&\mathbb{P}\left(\sup_{0\le t\le 1}|\bar{\epsilon}_n\left(n\mathbb{U}_n(t)-\mathbb{L}_{\nu_n}(t)\right)|\ge n^{1/4}x+\sqrt{nx}/2\right)\\
&\le \quad \mathbb{P}\left(\sup_{0\le t\le 1}|n\mathbb{U}_n(t)-\mathbb{L}_{\nu_n}(t)|\ge n^{1/4}x+\sqrt{nx}/2\right)\\
&\quad +\mathbb{P}\left(\left|\sum_{i=1}^{n}\epsilon_i\right|\ge\sqrt{nx}\right)\\
&\le \quad 8\exp(-x)+\exp(-c_{41}x)+\mathbb{P}\left(\left|\sum_{i=1}^{n}\epsilon_i\right|\ge\sqrt{nx}\right).
\end{aligned}$$

From (3.3) of [51], we obtain, for $0\le x\le n$,

$$\mathbb{P}\left(\left|\sum_{i=1}^{n}\epsilon_i\right|\ge\sqrt{nx}\right)\le c_{42}\exp(-c_{43}n)\le c_{42}\exp(-c_{43}x).$$

where $c_{42},c_{43}>0$ are universal constants. Then, for all $x>0$, we have

$$\mathbb{P}\left(\sup_{0\le t\le 1}|\bar{\epsilon}_n\left(n\mathbb{U}_n(t)-\mathbb{L}_{\nu_n}(t)\right)|\ge n^{1/4}x+\sqrt{nx}/2\right)\le c_{44}\exp(-c_{45}x),$$

where $c_{44},c_{45}>0$ are universal constants. Now, in order to obtain a bound of $A_{\nu_n}(t)-A_n(t)$, we have to deal with

$$\begin{aligned}
&\mathbb{P}\left(\sup_{0\le t\le 1}\left|\sum_{i=1}^{\nu_n}\epsilon_i\mathbb{1}_{\{U_i\le t\}}-\sum_{i=1}^{n}\epsilon_i\mathbb{1}_{\{U_i\le t\}}\right|\ge\sqrt{xn}\right)\\
&\le \quad \mathbb{P}\left(\sup_{0\le t\le 1}\left|\sum_{i=\inf(\nu_n,\lfloor n/2\rfloor)}^{\sup(\nu_n,3n/2)}\epsilon_i\mathbb{1}_{\{U_i\le t\}}\right|\ge\sqrt{xn}\right)\\
&\quad +\mathbb{P}\left(|\nu_n-n|\ge\sqrt{nx}/2\right),
\end{aligned}$$

for the first term in the right hand-side, we have by using (3.3) of [51] that

$$\mathbb{P}\left(\left|\sum_{i=1}^{\lceil 3n/2\rceil}\epsilon_i\right|\ge\sqrt{xn}\right)\le c_{46}\exp(-c_{47}x),$$

where $c_{46},c_{47}>0$ are universal constants. This when combined with (8.46), implies that

$$\mathbb{P}\left(\sup_{0\le t\le 1}|A_{\nu_n}(t)-A_n(t)|\ge\sqrt{nx}\right)\le c_{46}\exp(-c_{47}x)+\exp(-c_{41}x),$$

for all $x>0$. Hence the proof is complete. □

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References

1. Sergio Alvarez-Andrade and Salim Bouzebda, *Strong approximations for weighted bootstrap of empirical and quantile processes with applications*, Stat. Methodol. **11** (2013), 36–52.
2. Sergio Alvarez-Andrade and Salim Bouzebda, *Asymptotic results for hybrids of empirical and partial sums processes*, Statist. Papers **55** (2014), no. 4, 1121–1143.
3. Sergio Alvarez-Andrade and Salim Bouzebda, *On the hybrids of k-spacing empirical and partial sum processes*, Rev. Mat. Complut. **30** (2017), no. 1, 185–216.
4. Sergio Alvarez-Andrade and Salim Bouzebda, *Almost sure central limit theorem for the hybrid process*, Statistics **52** (2018), no. 3, 519–532.
5. R. R. Bahadur, *A note on quantiles in large samples*, Ann. Math. Statist. **37** (1966), 577–580.
6. Philippe Barbe and Patrice Bertail, *The weighted bootstrap*, Lecture Notes in Statistics, vol. 98, Springer-Verlag, New York, 1995.
7. Leonard E. Baum and Melvin Katz, *Convergence rates in the law of large numbers*, Trans. Amer. Math. Soc. **120** (1965), 108–123.
8. Luisa Beghin, *Weak convergence of some randomly indexed empirical processes*, Probab. Math. Statist. **22** (2002), no. 2, Acta Univ. Wratislav. No. 2470, 333–342.
9. R. J. Beran, L. Le Cam, and P. W. Millar, *Convergence of stochastic empirical measures*, J. Multivariate Anal. **23** (1987), no. 1, 159–168.
10. Peter J. Bickel and David A. Freedman, *Some asymptotic theory for the bootstrap*, Ann. Statist. **9** (1981), no. 6, 1196–1217.
11. S. Bouzebda, *On the strong approximation of bootstrapped empirical copula processes with applications*, Math. Methods Statist. **21** (2012), no. 3, 153–188.
12. Salim Bouzebda, *Kac's representation for empirical copula process from an asymptotic viewpoint*, Statist. Probab. Lett. **123** (2017), 107–113.
13. Salim Bouzebda and Mohamed Cherfi, *General bootstrap for dual ϕ-divergence estimates*, J. Probab. Stat. (2012), Art. ID 834107, 33.
14. J. Bretagnolle and P. Massart, *Hungarian constructions from the nonasymptotic viewpoint*, Ann. Probab. **17** (1989), no. 1, 239–256.
15. Youyou Chen and Tailong Li, *Moment convergence rates for the uniform empirical process and the uniform sample quantile process*, Comm. Statist. Theory Methods **46** (2017), no. 18, 9086–9091.
16. Michael R. Chernick, *Bootstrap methods*, Wiley Series in Probability and Statistics: Applied Probability and Statistics, John Wiley & Sons, Inc., New York, 1999, A practitioner's guide, A Wiley-Interscience Publication.
17. Michael R. Chernick, *Bootstrap methods: a guide for practitioners and researchers*, second ed., Wiley Series in Probability and Statistics, Wiley-Interscience [John Wiley & Sons], Hoboken, NJ, 2008.
18. Kai-Lai Chung, *An estimate concerning the Kolmogoroff limit distribution*, Trans. Amer. Math. Soc. **67** (1949), 36–50.
19. Endre Csáki, Miklós Csörgő, Antónia Földes, Zhan Shi, and Ričardas Zitikis, *Pointwise and uniform asymptotics of the Vervaat error process*, J. Theoret. Probab. **15** (2002), no. 4, 845–875.
20. M. Csörgő and P. Révész, *Strong approximations in probability and statistics*, Probability and Mathematical Statistics, Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York-London, 1981.
21. Miklós Csörgő, Sándor Csörgő, and Lajos Horváth, *An asymptotic theory for empirical reliability and concentration processes*, Lecture Notes in Statistics, vol. 33, Springer-Verlag, Berlin, 1986.
22. Miklós Csörgő, Sándor Csörgő, Lajos Horváth, and David M. Mason, *Weighted empirical and quantile processes*, Ann. Probab. **14** (1986), no. 1, 31–85.
23. Miklós Csörgő, Sándor Csörgő, Lajos Horváth, David M. Mason, and Brian S. Yandell, *Asymptotic theory of some bootstrapped empirical processes*, Technical report, Laboratory for Research in Statistical Probability, Carleton University, Ottawa, Canada. (1986).
24. Miklós Csörgő and Lajos Horváth, *On the distributions of L_p norms of weighted uniform empirical and quantile processes*, Ann. Probab. **16** (1988), no. 1, 142–161.
25. Miklós Csörgő and Lajos Horváth, *Weighted approximations in probability and statistics*, Wiley Series in Probability and Mathematical Statistics: Probability and Mathematical Statistics, John Wiley & Sons, Ltd., Chichester, 1993, With a foreword by David Kendall.
26. Miklós Csörgő, Lajos Horváth, and Piotr Kokoszka, *Approximation for bootstrapped empirical processes*, Proc. Amer. Math. Soc. **128** (2000), no. 8, 2457–2464.

27. Miklós Csörgő and Pál Révész, *Strong approximations of the quantile process*, Ann. Statist. **6** (1978), no. 4, 882–894.
28. Miklós Csörgő and Ričardas Zitikis, *Strassen's LIL for the Lorenz curve*, J. Multivariate Anal. **59** (1996), no. 1, 1–12.
29. Miklós Csörgő and Ričardas Zitikis, *On the rate of strong consistency of Lorenz curves*, Statist. Probab. Lett. **34** (1997), no. 2, 113–121.
30. Sándor Csörgő and David M. Mason, *Bootstrapping empirical functions*, Ann. Statist. **17** (1989), no. 4, 1447–1471.
31. A. C. Davison and D. V. Hinkley, *Bootstrap methods and their application*, Cambridge Series in Statistical and Probabilistic Mathematics, vol. 1, Cambridge University Press, Cambridge, 1997, With 1 IBM-PC floppy disk (3.5 inch; HD).
32. Monroe D. Donsker, *Justification and extension of Doob's heuristic approach to the Komogorov-Smirnov theorems*, Ann. Math. Statistics **23** (1952), 277–281.
33. A. Dvoretzky, J. Kiefer, and J. Wolfowitz, *Asymptotic minimax character of the sample distribution function and of the classical multinomial estimator*, Ann. Math. Statist. **27** (1956), 642–669.
34. B. Efron, *Bootstrap methods: another look at the jackknife*, Ann. Statist. **7** (1979), no. 1, 1–26.
35. Bradley Efron and Robert J. Tibshirani, *An introduction to the bootstrap*, Monographs on Statistics and Applied Probability, vol. 57, Chapman and Hall, New York, 1993.
36. Helen Finkelstein, *The law of the iterated logarithm for empirical distributions*, Ann. Math. Statist. **42** (1971), 607–615.
37. P. Gaenssler and W. Stute, *Seminar on empirical processes*, DMV Seminar, vol. 9, Birkhäuser Verlag, Basel, 1987.
38. Peter Gänssler, *Empirical processes*, Institute of Mathematical Statistics Lecture Notes—Monograph Series, vol. 3, Institute of Mathematical Statistics, Hayward, CA, 1983.
39. Evarist Giné and Joel Zinn, *Bootstrapping general empirical measures*, Ann. Probab. **18** (1990), no. 2, 851–869.
40. Phillip Good, *Permutation, parametric and bootstrap tests of hypotheses*, third ed., Springer Series in Statistics, Springer-Verlag, New York, 2005.
41. A. Gut and J. Steinebach, *Precise asymptotics—a general approach*, Acta Math. Hungar. **138** (2013), no. 4, 365–385.
42. Allan Gut and Aurel Spătaru, *Precise asymptotics in the Baum-Katz and Davis laws of large numbers*, J. Math. Anal. Appl. **248** (2000), no. 1, 233–246.
43. Allan Gut and Aurel Spătaru, *Precise asymptotics in the law of the iterated logarithm*, Ann. Probab. **28** (2000), no. 4, 1870–1883.
44. Allan Gut and Ulrich Stadtmüller, *On the Hsu-Robbins-Erdős-Spitzer-Baum-Katz theorem for random fields*, J. Math. Anal. Appl. **387** (2012), no. 1, 447–463.
45. Allan Gut and Josef Steinebach, *Convergence rates in precise asymptotics*, J. Math. Anal. Appl. **390** (2012), no. 1, 1–14.
46. Allan Gut and Josef Steinebach, *Convergence rates in precise asymptotics II*, Ann. Univ. Sci. Budapest. Sect. Comput. **39** (2013), 95–110.
47. Peter Hall, *The bootstrap and Edgeworth expansion*, Springer Series in Statistics, Springer-Verlag, New York, 1992.
48. Lajos Horváth, *A note on the rate of Poisson approximation of empirical processes*, Ann. Probab. **18** (1990), no. 2, 724–726.
49. Lajos Horváth, *Kac's representation from an asymptotic viewpoint*, J. Statist. Plann. Inference **46** (1995), no. 1, 27–45.
50. Lajos Horváth, *Approximations for hybrids of empirical and partial sums processes*, J. Statist. Plann. Inference **88** (2000), no. 1, 1–18.
51. Lajos Horváth, Piotr Kokoszka, and Josef Steinebach, *Approximations for weighted bootstrap processes with an application*, Statist. Probab. Lett. **48** (2000), no. 1, 59–70.
52. P. L. Hsu and Herbert Robbins, *Complete convergence and the law of large numbers*, Proc. Nat. Acad. Sci. U.S.A. **33** (1947), 25–31.
53. Barry R. James, *A functional law of the iterated logarithm for weighted empirical distributions*, Ann. Probability **3** (1975), no. 5, 762–772.
54. M. Kac, *On deviations between theoretical and empirical distributions*, Proc. Nat. Acad. Sci. U.S.A. **35** (1949), 252–257.
55. J. Kiefer, *On Bahadur's representation of sample quantiles*, Ann. Math. Statist. **38** (1967), 1323–1342.
56. J. Kiefer, *Deviations between the sample quantile process and the sample* df, Nonparametric Techniques in Statistical Inference (Proc. Sympos., Indiana Univ., Bloomington, Ind., 1969), Cambridge Univ. Press, London, 1970, pp. 299–319.

57. J. Komlós, P. Major, and G. Tusnády, *An approximation of partial sums of independent* RV*'s and the sample* DF*. I*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **32** (1975), 111–131.
58. J. Komlós, P. Major, and G. Tusnády, *An approximation of partial sums of independent RV's, and the sample DF. II*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **34** (1976), no. 1, 33–58.
59. Michael R. Kosorok, *Introduction to empirical processes and semiparametric inference*, Springer Series in Statistics, Springer, New York, 2008.
60. De Li Li, Fu Xing Zhang, and Andrew Rosalsky, *A supplement to the Baum-Katz-Spitzer complete convergence theorem*, Acta Math. Sin. (Engl. Ser.) **23** (2007), no. 3, 557–562.
61. Péter Major, *A note on the approximation of the uniform empirical process*, Ann. Probab. **18** (1990), no. 1, 129–139.
62. Péter Major, *The approximation of the normalized empirical distribution function by a brownian bridge*, Notes available from http://www.renyi.hu/~major/probability/empir.html (2000).
63. Bryan F. J. Manly, *Randomization, bootstrap and Monte Carlo methods in biology*, third ed., Chapman & Hall/CRC Texts in Statistical Science Series, Chapman & Hall/CRC, Boca Raton, FL, 2007.
64. David M. Mason, *Some characterizations of almost sure bounds for weighted multidimensional empirical distributions and a Glivenko-Cantelli theorem for sample quantiles*, Z. Wahrsch. Verw. Gebiete **59** (1982), no. 4, 505–513.
65. David M. Mason and Michael A. Newton, *A rank statistics approach to the consistency of a general bootstrap*, Ann. Statist. **20** (1992), no. 3, 1611–1624.
66. David M. Mason and Willem R. van Zwet, *A refinement of the KMT inequality for the uniform empirical process*, Ann. Probab. **15** (1987), no. 3, 871–884.
67. Yan-Jiao Meng, *General laws of precise asymptotics for sums of random variables*, J. Korean Math. Soc. **49** (2012), no. 4, 795–804.
68. Ja. Ju. Nikitin, *Limit distributions and comparative asymptotic efficiency of the Kolmogorov-Smirnov statistics with a random index*, Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **55** (1976), 185–194, 202, Problems in the theory of probability distributions, III.
69. Valentin V. Petrov, *Limit theorems of probability theory*, Oxford Studies in Probability, vol. 4, The Clarendon Press, Oxford University Press, New York, 1995, Sequences of independent random variables, Oxford Science Publications.
70. Galen R. Shorack, *Bootstrapping robust regression*, Comm. Statist. A—Theory Methods **11** (1982), no. 9, 961–972.
71. Galen R. Shorack and Jon A. Wellner, *Empirical processes with applications to statistics*, Wiley Series in Probability and Mathematical Statistics: Probability and Mathematical Statistics, John Wiley & Sons, Inc., New York, 1986.
72. Winfried Stute, *The oscillation behavior of empirical processes*, Ann. Probab. **10** (1982), no. 1, 86–107.
73. Aad W. van der Vaart and Jon A. Wellner, *Weak convergence and empirical processes*, Springer Series in Statistics, Springer-Verlag, New York, 1996, With applications to statistics.
74. Jon A. Wellner, *Limit theorems for the ratio of the empirical distribution function to the true distribution function*, Z. Wahrsch. Verw. Gebiete **45** (1978), no. 1, 73–88.
75. Qing-pei Zang, *Precise asymptotics for complete moment convergence of* U*-statistics of i.i.d. random variables*, Appl. Math. Lett. **25** (2012), no. 2, 120–127.
76. Qing Pei Zang and Wei Huang, *A general law of moment convergence rates for uniform empirical process*, Acta Math. Sin. (Engl. Ser.) **27** (2011), no. 10, 1941–1948.
77. Yong Zhang and Xiao-Yun Yang, *Precise asymptotics in the law of the iterated logarithm and the complete convergence for uniform empirical process*, Statist. Probab. Lett. **78** (2008), no. 9, 1051–1055.
78. Ričardas Zitikis, *The Vervaat process*, Asymptotic methods in probability and statistics (Ottawa, ON, 1997), North-Holland, Amsterdam, 1998, pp. 667–694.

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