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BERRY-ESSEEN BOUNDS FOR DRIFT PARAMETER ESTIMATION OF DISCRETELY OBSERVED FRACTIONAL VASICEK-TYPE PROCESS

In this paper, we study statistical estimation problems of drift parameters of Vasicek-type processes driven by fractional Brownian motion. Based on fixed-time-step observations and using Malliavin calculus combined with the recent Nourdin-Peccati analysis, we provide estimators of the drift parameters and analyze their asymptotic behaviors. More precisely, we study the strong consistency and the asymptotic distribution of the estimators and we give the rate of their convergence in law.

1. Introduction

In this work, we are concerned about solving a parameter estimation problem for the Vasicek-type process X, defined as the unique solution to the following stochastic differential equation

$$dX_t = a(b - X_t)dt + dB_t^H, \quad t \geq 0, \qquad X_0 = 0, \tag{1}$$

where B^H is a fractional Brownian motion of Hurst parameter $H \in (0,1)$. The drift parameters $a > 0$ and $b \in \mathbb{R}$ in (1) are assumed to be unknown, we aim to estimate those parameters based on discrete time observations of the process X.
Similar question was studied recently by Nourdin and Tran [14], where they consider that the driving noise in equation (1) is an Hermite process and the process is observed continuously in time. They proved the strong consistency and the asymptotic distribution of the estimators of a and b.
Practically, it is not easy to observe a process continuously in time, this reason motivated us to solve this parameter estimation problem using discrete time observations with no in fill assumptions on the data when the noise of equation (1) is a fractional Brownian motion.
The estimators are constructed based on empirical moments and this work continues the line of research of the papers [7] and [3]. In [7], using stationary property, this type of estimation was performed where no constraints on the mesh of the data are made. In those papers a variety of parameter estimation problems are studied for a range of fractional-driven noise processes with stationary and non-stationary covariance structures.
The tools used in [7], [3] and in the present paper are related to the analysis on Wiener space through Malliavin Calculus and the so-called optimal fourth moment theorem, see [13].

Using those techniques among others, we were able to prove that the estimators of the drift parameters for the Vasicek process (1) are strongly consistent for all $H \in (0,1)$ and asymptotically normal and we gave the speed of their convergence in law in the Wasserstein metric.

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Recently, several researchers have been interested in studying statistical estimation problems for (1). Let us mention some works in this direction: in the case when $a > 0$, the statistical estimation for the parameters a and b based on continuous-time observations of $\{X_t,\ t \in [0,T]\}$ as $T \to \infty$, has been studied by several papers, for instance [2, 1, 14, 17] and the references therein. When $b = 0$ in (1), the estimation of a has been investigated by using least squares method as follows: the case of ergodic fractional Ornstein-Uhlenbeck processes, corresponding to $a > 0$, has been considered in [8, 5, 9], and the case non-ergodic fractional Ornstein-Uhlenbeck processes has been studied in [4, 6].
On the other hand, using Malliavin-calculus advances (see [12]), the work [7] provided new techniques to statistical inference for stochastic differential equations related to stationary Gaussian processes, and its result has been used to study drift parameter estimation problems for some stochastic differential equations driven by fractional Brownian motion with fixed-time-step observations, in particular for the fractional Ornstein-Uhlenbeck given in (1), where $b = 0$ and $a > 0$. Similarly, in [3] the authors studied an estimator problem for the parameter a in (1), where the fractional Brownian motion is replaced with a general Gaussian process.

The paper is organized as follows. In Section 2 we recall some properties of the driving noise of equation (1), which is the fractional Brownian motion. We also recall some elements of Malliavin Calculus with respect to fractional Brownian motion. In Section 3 we present the estimators chosen to estimate the drift parameters a and b respectively. We show their strong consistency and we prove in details how we obtained their speed of convergence in law depending on the values of the Hurst parameter H. Finally, the reader will find in the Appendix the proofs of some auxiliary results.

2. Preliminaries

This section is dedicated to some notions that are required in our study, related mainly to the analysis on Wiener space through Malliavin Calculus. We start by recalling the definition of the fractional Brownian motion. For further details about this process, we refer the reader to [11] and [15].

A fractional Brownian motion (fBm for short) of Hurst parameter $H \in (0,1)$, $B^H = (B_t^H)_{t\geq 0}$ is a centered continuous Gaussian process with covariance function:

$$R_H(t,s) := E(B_t^H B_s^H) = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right) \quad s,t \in \mathbb{R}_+,$$

When $H = 1/2$, $B^{\frac{1}{2}}$ is the standard Brownian motion. Moreover, the fBm has the following main properties:

- (*Self-similarity*) For all $\lambda > 0$, $(\lambda^{-H} B_{\lambda t}^H)_{t\geq 0} \overset{law}{=} (B_t^H)_{t\geq 0}$.
- (*Stationary increments*) For all $h > 0$, $(B_{t+h}^H - B_t^H)_{t\geq 0} \overset{law}{=} (B_t^H)_{t\geq 0}$.

When $H > 1/2$, B^H exhibits also the property of long range dependence, which makes the fBm an important driving noise in modeling different phenomena arising from finance, telecommunication networks, and physics.

Let us now recall some elements of Malliavin Calculus that we will need in our study. The interested reader can find more details in [15, Chapter 1] and [12, Chapter 2]. In the following $\mathcal{H}$ will denote the closure of the set of step functions with respect to the scalar product $\langle \mathbf{1}_{[0,t]}, \mathbf{1}_{[0,s]} \rangle_{\mathcal{H}} = R_H(t,s)$ defined previously and $\mathcal{H}^{\odot q}$ denotes the qth tensor product of $\mathcal{H}$.

- **The Wiener chaos expansion**. For every $q \geq 1$, $\mathcal{H}_q$ denotes the qth Wiener chaos of B^H, defined as the closed linear subspace of $L^2(\Omega)$ generated by the random variables $\{H_q(B^H(h)), h \in \mathcal{H}, \|h\|_{\mathcal{H}} = 1\}$ where H_q is the qth Hermite polynomial.

- **Multiple Wiener integrals**. The mapping $I_q(h^{\otimes q}) := q!H_q(B^H(h))$ is a linear isometry between the symmetric tensor product $\mathcal{H}^{\odot q}$ (equipped with the modified norm $\|.\|_{\mathcal{H}^{\odot q}} = \frac{1}{\sqrt{q!}}\|.\|_{\mathcal{H}^{\otimes q}}$) and $\mathcal{H}_q$. $I_q(h^{\otimes q})$ denotes the multiple Wiener integral of $h^{\otimes q}$ with respect to B^H.
- **Hypercontractivity in Wiener chaos**. For $h \in \mathcal{H}^{\otimes q}$, the multiple Wiener integrals $I_q(h)$, which exhaust the set $\mathcal{H}_q$, satisfy a hypercontractivity property (equivalence in $\mathcal{H}_q$ of all L^p norms for all $p \geq 2$), which implies that for any $F \in \oplus_{l=1}^{q}\mathcal{H}_l$ (i.e. in a fixed sum of Wiener chaoses), we have

$$\left(E\left[|F|^p\right]\right)^{1/p} \leqslant c_{p,q}\left(E\left[|F|^2\right]\right)^{1/2} \quad \text{for any } p \geq 2. \tag{2}$$

- **The isometry property and the product formula** . For every $f, g \in \mathcal{H}^{\odot q}$, the following extended isometry property holds

$$E\left(I_q(f)I_q(g)\right) = q!\langle f, g\rangle_{\mathcal{H}^{\otimes q}}.$$

  We will need the product formula (see [12, Chapter 2]) for $q = 1$: for every $f, g \in \mathcal{H}$,

$$I_1(f)I_1(g) = \frac{1}{2}I_2\left(f \otimes g + g \otimes f\right) + \langle f, g\rangle_{\mathcal{H}}.$$

- **Distances between random variables**. Recall that, if X, Y are two real-valued random variables, then the total variation distance between the law of X and the law of Y is given by

$$d_{TV}\left(X, Y\right) = \sup_{A \in \mathcal{B}(\mathbb{R})} \left|P\left[X \in A\right] - P\left[Y \in A\right]\right|$$

  where the supremum is over all Borel sets. If X, Y are two real-valued integrable random variables, then the Wasserstein distance between the law of X and the law of Y is given by

$$d_W\left(X, Y\right) = \sup_{f \in Lip(1)} \left|Ef(X) - Ef(Y)\right|$$

  where $Lip(1)$ indicates the collection of all Lipschitz functions with Lipschitz constant $\leqslant 1$.
- **The optimal fourth moment theorem**. Let $N \sim \mathcal{N}(0, 1)$ denote the standard normal law. For each integer n, let $X_n \in \mathcal{H}_q$. Assume $Var\left[X_n\right] = 1$ and $\left(X_n\right)_n$ converges in distribution to N. It is known (original proof in [16], known as the *fourth moment theorem*) that this convergence is equivalent to $\lim_n \mathbf{E}\left[X_n^4\right] = 3$. The following optimal estimate for $d_{TV}\left(X_n, N\right)$, known as the optimal fourth moment theorem, was proved in [13]: with the sequence $(X_n)_{n\geq 1}$ as above, assuming convergence, there exist two constant $c, C > 0$ depending only on $(X_n)_{n\geq 1}$ but not on n, such that

$$c\max\left\{E\left[X_n^4\right] - 3, \left|E\left[X_n^3\right]\right|\right\} \leqslant d_{TV}\left(X_n, N\right) \leqslant C\max\left\{E\left[X_n^4\right] - 3, \left|E\left[X_n^3\right]\right|\right\}. \tag{3}$$

3. Asymptotic behavior of the estimators

Note first that the process X of equation (1) has the following explicit form

$$X_t = b(1 - e^{-at}) + \int_0^t e^{-a(t-u)}dB_u^H, \tag{4}$$

where the integral with respect to B^H must be understood in the Wiener sense. As we explained in the introduction, the drift parameters $a > 0$ and $b \in \mathbb{R}$ in (1) and (4) are assumed to be unknown. We propose to estimate them using the following estimators

$$\hat{a}_n := \left[\frac{1}{H\Gamma(2H)}\left(\frac{1}{n}\sum_{k=0}^{n-1}X_i^2 - (\frac{1}{n}\sum_{i=0}^{n-1}X_i)^2\right)\right]^{-1/2H} \text{ and } \hat{b}_n := \frac{1}{n}\sum_{i=0}^{n-1}X_i. \tag{5}$$

In the next section, we prove the strong consistency of the estimators $\hat{a}_n$ and $\hat{b}_n$ given in (5). This study valid for all $H\in (0,1)$.

3.1. **Strong consistency.**

Theorem 3.1. *Let X be given by (1) and (4), where B^H is a fractional Brownian motion with Hurst parameter $H \in (0,1)$, and consider the estimators given in (5), then we have*

$$(\hat{a}_n, \hat{b}_n) \to (a, b) \tag{6}$$

almost surely as $n \to +\infty$.

Proof. The process X given by (1) and (4) can be written as follows

$$X_t = b + Z_t^H + Y_t^H, \quad t \geq 0, \tag{7}$$

where

$$Z_t^H := e^{-at}\int_{-\infty}^{t} e^{as} dB_s^H \text{ and } Y_t^H := -e^{-at}(b + Z_0^H), \quad t \geq 0. \tag{8}$$

Moreover, it is known that Z^H is an ergodic stationary Gaussian process with covariance $\rho_H(k) := E[Z_k^H Z_0^H]$.
Then, using the Ergodic theorem, it's clear that $\hat{b}_n \to b$, a.s. as $n \to +\infty$.
On the other hand, by the Ergodic theorem we also have

$$\frac{1}{n}\sum_{i=0}^{n-1}X_i^2 = \frac{1}{n}\sum_{i=0}^{n-1}(b + Z_i^H + Y_i^H)^2 \longrightarrow \mathbb{E}[(b + Z_0^H)^2] = b^2 + \frac{H\Gamma(2H)}{a^{2H}}$$

almost surely, as $n \to +\infty$. Hence by the expression of the estimator $\hat{a}_n$ given in (5), we get $\hat{a}_n \to a$ almost surely as $n \to +\infty$, which finishes the proof. □

Remark 3.1. Theorem 3.1 has been proved in [10, Theorem 2.3.] for similar estimators for the drift parameters of (1). But, to the best of our knowledge, there is no study for the asymptotic behavior in distribution of the estimators $\hat{a}_n$ and $\hat{b}_n$ given in (5).

3.2. **Asymptotic behavior in distribution.**

In this section we prove that the estimators $\hat{a}_n$ and $\hat{b}_n$ are asymptotically normal. We also give in details the rate of their convergence in law depending on the values of the Hurst parameter H.

3.2.1. *Asymptotic distribution of $\hat{a}_n$.* The estimator $\hat{a}_n$ can be written as follows $\hat{a}_n = f_H^{-1}(V_n)$ where f_H^{-1} is the inverse function of $f_H(x) := \frac{H\Gamma(2H)}{x^{2H}}$, and

$$V_n := \frac{1}{n}\sum_{i=1}^{n}X_i^2 - \left(\frac{1}{n}\sum_{i=1}^{n}X_i\right)^2.$$

Therefore we first study the rates of convergence in law of the sequence V_n. Using the decomposition of X given in (7), we get

$$V_n = \frac{1}{n}\sum_{i=1}^{n}(Z_i^H)^2 + R_H(n),$$

where
(9)

$$R_H(n) := \frac{2}{n}\sum_{i=1}^{n} Y_i^H Z_i^H + \frac{1}{n}\sum_{i=1}^{n}(Y_i^H)^2 - \frac{2}{n^2}\sum_{i,j=1}^{n} Z_i^H Y_j^H - \frac{1}{n^2}\sum_{i,j=1}^{n} Y_i^H Y_j^H - \frac{1}{n^2}\sum_{i,j=1}^{n} Z_i^H Z_j^H.$$

We have $Z_i^H = I_1(\varepsilon_i)$, with $\varepsilon_i(.) := e^{-a(i-.)}\mathbf{1}_{]-\infty,i]}(.)$. Then, using the product formula and the fact that Z^H is a stationary process, we can write

$$V_n = I_2(f_n) + f_H(a) + R_H(n).$$

Hence

$$\sqrt{n}(V_n - f_H(a)) = I_2(\sqrt{n}f_n) + \sqrt{n}R_H(n)$$

where $f_n := \frac{1}{n}\sum_{i=1}^{n}\varepsilon_i^{\otimes 2}$ and $f_H(a) = \frac{H\Gamma(2H)}{a^{2H}} = E[(Z_0^H)^2]$.

We also set $F_n := I_2\left(\frac{\sqrt{n}}{\sqrt{v_n}}f_n\right)$, with $v_n := \mathbb{E}[I_2(\sqrt{n}f_n)^2] = \frac{2}{n}\sum_{k,i=1}^{n}\rho_H(i-k)^2$. Hence, we have

$$\frac{\sqrt{n}}{\sqrt{v_n}}(V_n - f_H(a)) = F_n + \frac{\sqrt{n}R_H(n)}{\sqrt{v_n}}. \tag{10}$$

Now we will study the rate of convergence in law of F_n. From the definition of F_n and estimating the third and the fourth cumulant of the sequence $\{F_n\}_{n\geq 0}$, we get

$$\begin{aligned}
\kappa_3(F_n) &= \frac{8}{(nv_n)^{3/2}}\sum_{i,j,k=1}^{n}\mathbb{E}[Z_i^H Z_k^H]\mathbb{E}[Z_i^H Z_j^H]\mathbb{E}[Z_j^H Z_k^H]\\
&= \frac{1}{(nv_n)^{3/2}}\sum_{i,j,k=1}^{n}\rho_H(i-k)\rho_H(i-j)\rho_H(j-k)\\
&\trianglelefteq \frac{1}{(nv_n)^{3/2}}\sum_{i,j,k=1}^{n}|i-k|^{2H-2}|i-j|^{2H-2}|j-k|^{2H-2}\\
&\trianglelefteq \frac{1}{v_n^{3/2}\sqrt{n}}\left(\sum_{|k|<n}|k|^{3H-3}\right)^2, \qquad (11)
\end{aligned}$$

and

$$\begin{aligned}
\kappa_4(F_n) &= \frac{1}{v_n^2 n^2}\sum_{k,i,j,l=1}^{n}\mathbb{E}[Z_k^H Z_i^H]\mathbb{E}[Z_i^H Z_j^H]\mathbb{E}[Z_j^H Z_l^H]\mathbb{E}[Z_l^H Z_k^H]\\
&= \frac{1}{v_n^2 n^2}\sum_{k,i,j,l=1}^{n}\rho_H(i-k)\rho_H(i-j)\rho_H(j-l)\rho_H(l-k)\\
&\trianglelefteq \frac{1}{v_n^2 n^2}\sum_{k,i,j,l=1}^{n}|i-k|^{2H-2}|i-j|^{2H-2}|j-l|^{2H-2}|l-k|^{2H-2}\\
&\trianglelefteq \frac{1}{v_n^2 n}\left(\sum_{|k|<n}|k|^{\frac{8}{3}(H-1)}\right)^3. \qquad (12)
\end{aligned}$$

Note that, throughout the paper, the symbol $\trianglelefteq$ means that we omit multiplicative universal constants.
The following theorem investigates the rate of convergence in law of the sequence $\{F_n\}_{n\geq 1}$ towards a normal random variable.

Proposition 3.1. *If $0 < H \leqslant \frac{3}{4}$ and $N \sim \mathcal{N}(0,1)$, then*

$$d_{TV}(F_n, N) \trianglelefteq \begin{cases} n^{-\frac{1}{2}} & \text{if } H \in \left(0, \frac{2}{3}\right) \\ n^{-\frac{1}{2}} \log(n)^2 & \text{if } H = \frac{2}{3} \\ n^{6H-\frac{9}{2}} & \text{if } H \in \left(\frac{2}{3}, \frac{3}{4}\right) \\ \log(n)^{-3/2} & \text{if } H = \frac{3}{4}. \end{cases}$$

Proof. A straightforward calculation shows that

$$\sum_{|k|<n} |k|^{3H-3} \trianglelefteq \begin{cases} 1 & \text{if } 0 < H < \frac{2}{3} \\ \log(n) & \text{if } H = \frac{2}{3} \\ n^{3H-2} & \text{if } \frac{2}{3} < H < \frac{3}{4} \\ n^{1/4} & \text{if } H = \frac{3}{4} \end{cases},$$

$$\sum_{|k|<n} |k|^{\frac{8}{3}(H-1)} \trianglelefteq \begin{cases} 1 & \text{if } 0 < H < \frac{5}{8} \\ \log(n) & \text{if } H = \frac{5}{8} \\ n^{\frac{1}{3}(8H-5)} & \text{if } \frac{5}{8} < H < \frac{3}{4} \\ n^{1/3} & \text{if } H = \frac{3}{4}. \end{cases}.$$

Combining these estimates, (11) and (12) together with the fact that the sequence $\{v_n\}_{n\geq 1}$ is convergent (see Proposition 4.1 in the Appendix) and the optimal fourth moment theorem recalled in the preliminaries, we can therefore conclude the desired result. □

Therefore, we obtain the following rates for the normal convergence of $\{V_n\}_{n\geq 1}$.

Theorem 3.2. *If $0 < H < 3/4$ and $N \sim \mathcal{N}(0,1)$, then*

$$d_W\left(\frac{\sqrt{n}}{\sigma_H}\left(V_n - f_H(a)\right), N\right) \trianglelefteq \begin{cases} n^{-\frac{1}{2}} & \text{if } \ 0 < H < 1/2 \\ n^{2H-3/2} & \text{if } \ 1/2 \leq H < 3/4. \end{cases}$$

where $\sigma_H^2 := \lim_n v_n = 2\sum_{i\in\mathbb{Z}} \rho_H(i)^2$, with $\rho_H(k) = E[Z_k^H Z_0^H]$ defined above.
If $H = 3/4$, we have

$$d_W\left(\frac{\sqrt{n}\left(V_n - f_H(a)\right)}{\sigma_{3/4}\sqrt{\log(n)}}, N\right) \trianglelefteq \log(n)^{-1/2}.$$

where $\sigma_{3/4}^2 = \frac{9}{16a^4}$.
Consequently, if $0 < H < 3/4$, then, as $n \to \infty$

$$\sqrt{n}\left(V_n - f_H(a)\right) \xrightarrow{law} \mathcal{N}(0, \sigma_H^2),$$

and if $H = 3/4$, then, as $n \to \infty$

$$\frac{\sqrt{n}\left(V_n - f_{3/4}(a)\right)}{\sqrt{\log(n)}} \xrightarrow{law} \mathcal{N}(0, \sigma_{3/4}^2).$$

Proof. Using a standard properties of the Wasserstein distance (see [7, Lemma 9]), the decomposition (10), Proposition 3.1 and Proposition 4.1, we obtain the following estimates,
if $0 < H < \frac{3}{4}$, then

$$d_W\left(\frac{\sqrt{n}}{\sigma_H}\left(V_n - f_H(a)\right), N\right) \leqslant C\left(\sqrt{n}\|R_H(n)\|_{L^1(\Omega)} + |v_n - \sigma_H^2|\right) + C\begin{cases} n^{-\frac{1}{2}} & \text{if } H \in \left(0, \frac{2}{3}\right) \\ n^{-\frac{1}{2}}\log(n)^2 & \text{if } H = \frac{2}{3} \\ n^{6H-\frac{9}{2}} & \text{if } H \in \left(\frac{2}{3}, \frac{3}{4}\right). \end{cases}$$

Also, if $H = 3/4$, we have

$$d_W\left(\frac{\sqrt{n}}{\sqrt{\log(n)}\sigma_{3/4}}\left(V_n - f_{3/4}(a)\right), N\right) \leqslant C\left(\frac{\sqrt{n}}{\sqrt{\log(n)}}\|R_H(n)\|_{L^1(\Omega)} + \left|\frac{v_n}{\log(n)} - \sigma_{3/4}^2\right|\right) + C\log(n)^{-3/2},$$

which completes the proof. □

The following theorem gives the rates of the convergence in law of the estimator $\hat{a}_n$ based on Theorem 3.2 and the analyze of [[7], Section 5.2.2, page 22] applied to $\hat{a}_n = f_H^{-1}(V_n)$.

Theorem 3.3. *If* $0 < H < 3/4$ *and* $N \sim \mathcal{N}(0,1)$, *then*

$$d_W\left(\frac{2H^2\Gamma(2H)\sqrt{n}}{\sigma_H a^{1+2H}}\left(\hat{a}_n - a\right), N\right) \trianglelefteq \begin{cases} n^{-\frac{1}{2}} & \text{if } \; 0 < H < 1/2 \\ n^{2H-3/2} & \text{if } \; 1/2 \leq H < 3/4. \end{cases}$$

If $H = 3/4$, *we have*

$$d_W\left(\frac{3\Gamma(3/2)\sqrt{n}\left(\hat{a}_n - a\right)}{2\sqrt{a}\sqrt{\log(n)}}, N\right) \trianglelefteq \log(n)^{-1/2}.$$

Consequently, we have, when $0 < H < 3/4$, *as* $n \to \infty$

$$\sqrt{n}\left(\hat{a}_n - a\right) \xrightarrow{law} \mathcal{N}\left(0, \frac{\sigma_H^2 a^{2+4H}}{4H^4\Gamma(2H)^2}\right).$$

Also, if $H = 3/4$, *then, as* $n \to \infty$

$$\frac{\sqrt{n}\left(\hat{a}_n - a\right)}{\sqrt{\log(n)}} \xrightarrow{law} \mathcal{N}\left(0, \frac{4a}{9\Gamma(3/2)^2}\right).$$

3.2.2. *Asymptotic distribution of* $\hat{b}_n$. For the convergence in law of the estimator $\hat{b}_n$ of the parameter b. For this aim we set for $0 < H < 1$ and $n \geq 1$

$$\psi_H(n) = \begin{cases} \sqrt{n} & \text{if } \; 0 < H \leq 1/2 \\ n^{1-H} & \text{if } \; 1/2 < H < 1. \end{cases} \tag{13}$$

From the definition of $\hat{b}_n$, we have

$$\psi_H(n)\left(\hat{b}_n - b\right) = \frac{\psi_H(n)}{n}\sum_{i=1}^{n}\int_0^i e^{-a(i-u)}dB_u^H - \frac{\psi_H(n)}{n}\sum_{i=1}^{n} e^{-ai}$$
$$= I_1\left(g_n^H\right) - \frac{\psi_H(n)}{n}\sum_{i=1}^{n} e^{-ai}$$

where $g_n^H := \frac{\psi_H(n)}{n}\sum_{i=1}^{n} e^{-a(i-.)}\mathbf{1}_{[0,i]}(.)$. Hence $\psi_H(n)\left(\hat{b}_n - b\right)$ is Gaussian. Then using a bound of the Wasserstein metric between the law of two Gaussian random variables see [12] and according to the calculus of the variance of $\psi_H(n)\left(\hat{b}_n - b\right)$ for both cases $0 < H \le 1/2$ and $1/2 < H < 1$ treated separately, we first get

$$d_W\left(I_1(g_n^H), \mathcal{N}(0,\beta_H^2)\right) \le \frac{\sqrt{2/\pi}}{\left(\|I_1(g_n^H)\|_{L^2(\Omega)} \vee \beta_H\right)} \times \left|\mathbb{E}[I_1(g_n^H)^2] - \beta_H^2\right|$$

where $\beta_H^2 := \lim_{n\to+\infty}\mathbb{E}[I_1(g_n^H)^2]$. Therefore, using Lemma 9 in [7], we get the following estimate

$$d_W\left(\psi_H(n)(\hat{b}_n - b), \mathcal{N}(0,\beta_H^2)\right)$$
$$\le \frac{\sqrt{2/\pi}}{\left(\|I_1(g_n^H)\|_{L^2(\Omega)} \vee \beta_H\right)} \times \left|\mathbb{E}[I_1(g_n^H)^2] - \beta_H^2\right| + C\psi_H(n) \times n^{-1}.$$

According the calculus of Lemma 4.1 and Lemma 4.2 in the Appendix, we get the following convergences in law of the estimator $\hat{b}_n$.

Theorem 3.4. *Let $0 < H < 1$ and $N \sim \mathcal{N}(0,1)$. Then, as $n \to \infty$*

$$\psi_H(n)(\hat{b}_n - b) \overset{law}{\longrightarrow} \mathcal{N}(0,\beta_H^2),$$

where $\beta_H^2 = \frac{H\Gamma(2H)}{a^{2H}} + 2\sum_{r\in\mathbb{N}\setminus\{0\}}\rho_H(r)$ if $0 < H \le 1/2$, and $\beta_H^2 = 1/a^2$ if $1/2 < H < 1$. Moreover, if $0 < H \le 1/2$, we have

$$d_W\left(\sqrt{n}(\hat{b}_n - b), \mathcal{N}(0,\beta_H^2)\right) \trianglelefteq \begin{cases} n^{-\frac{1}{2}} & \text{if } \ 0 < H < 1/4 \\ n^{2H-1} & \text{if } \ 1/4 \le H \le 1/2. \end{cases}$$

3.2.3. *Asymptotic distribution of $(\hat{a}_n, \hat{b}_n)$.* Since $f_H(\hat{a}_n) = V_n$ (resp. $\hat{a}_n$) belongs to the second (resp. first) Wiener chaos, we deduce from Theorem 3.2, Theorem 3.3, Theorem 3.4 and the seminal Peccati-Tudor criterion (see, e.g., [12, Theorem 6.2.3]) the following convergences in law for $(\hat{a}_n, \hat{b}_n)$.

Theorem 3.5. *Let $0 < H < 1$. Then, according to the values of the Hurst parameter H, the following convergences in law take place when $n \to +\infty$.*

- $0 < H \le 1/2$

$$\left(\sqrt{n}(\hat{a}_n - a), \sqrt{n}(\hat{b}_n - b)\right) \overset{law}{\to} \left(-\frac{\sigma_H a^{1+2H}}{2H^2\Gamma(2H)}N, \frac{\sigma_H}{\sqrt{2}}N'\right),$$

with $\sigma_H := \left(2\sum_{i\in\mathbb{Z}}\rho_H(i)^2\right)^{1/2}$, where $\rho_H(k) := E[Z_k^H Z_0^H]$, $k \in \mathbb{N}$, Z is the process given in (8) and N, $N' \sim \mathcal{N}(0,1)$ are independent.

- $1/2 < H < 3/4$

$$\left(\sqrt{n}(\hat{a}_n - a), n^{1-H}(\hat{b}_n - b)\right) \overset{law}{\to} \left(-\frac{\sigma_H a^{1+2H}}{2H^2\Gamma(2H)}N, \frac{1}{a}N'\right),$$

where N, $N' \sim \mathcal{N}(0,1)$ are independent.

- $H = \frac{3}{4}$

$$\left(\frac{\sqrt{n}}{\sqrt{\log(n)}}(\hat{a}_n - a), n^{1/4}(\hat{b}_n - b)\right) \overset{law}{\to} \left(-\frac{2\sqrt{a}}{3\Gamma(3/2)}N, \frac{1}{a}N'\right).$$

where N, $N' \sim \mathcal{N}(0,1)$ are independent.

4. Appendix

Lemma 4.1. *Let $0 < H \leq 1/2$, then we have*

$$\left|\mathbb{E}[I_1(g_n^H)^2] - \beta_H^2\right| \trianglelefteq n^{2H-1}. \tag{14}$$

where in this case $\beta_H^2 = \frac{H\Gamma(2H)}{a^{2H}} + 2\sum_{r\in\mathbb{N}\setminus\{0\}} \rho_H(r)$.

Proof. We have $I_1(g_n^H) = \frac{1}{\sqrt{n}}\sum_{i=1}^n \left(Z_i^H - e^{-ai}Z_0^H\right)$, where Z^H is the process defined in (8). Therefore

$$\begin{aligned}
\mathbb{E}[I_1(g_n^H)^2] &= \frac{1}{n}\sum_{i,j=1}^n \mathbb{E}[(Z_i^H - e^{-ai}Z_0^H)(Z_j^H - e^{-aj}Z_0^H)] \\
&= \frac{1}{n}\sum_{i,j=1}^n \mathbb{E}[Z_i^H Z_j^H] - \frac{2}{n}\sum_{i,j=1}^n e^{-aj}\mathbb{E}[Z_i^H Z_0^H] + \frac{\mathbb{E}[(Z_0^H)^2]}{n^{2H}}\left(\sum_{i=1}^n e^{-ai}\right)^2 \\
&= \frac{H\Gamma(2H)}{a^{2H}} + \frac{2}{n}\sum_{i=1}^{n-1}\sum_{j=i+1}^n \rho_H(j-i) - \frac{2}{n}\sum_{i,j=1}^n \rho_H(i)e^{-aj} + \frac{\rho_H(0)}{n}\left(\sum_{i=1}^n e^{-ai}\right)^2.
\end{aligned}$$

On the other hand $\rho_H(0) = \frac{H\Gamma(2H)}{a^{2H}}$. Therefore, we have

$$\begin{aligned}
&\left|\mathbb{E}\left[I_1(g_n^H)^2\right] - \beta_H^2\right| \\
&\leq \left|\frac{2}{n}\sum_{i,j=1}^n \rho_H(j-i) - 2\sum_{r\in\mathbb{N}\setminus\{0\}} \rho_H(r)\right| + \left|\frac{2}{n}\sum_{i,j=1}^n \rho_H(i)e^{-aj}\right| + \left|\frac{\rho_H(0)}{n}\left(\sum_{i=1}^n e^{-ai}\right)^2\right|
\end{aligned}$$

where we set $\beta_H^2 = \frac{H\Gamma(2H)}{a^{2H}} + 2\sum_{r\in\mathbb{N}\setminus\{0\}} \rho_H(r)$. As as $\rho_H(r) \sim \frac{H\Gamma(2H)}{a^2} r^{2H-2}$ as $r \sim +\infty$, we have the following estimate

$$\begin{aligned}
\left|\frac{2}{n}\sum_{i,j=1}^n \rho_H(j-i) - 2\sum_{r\in\mathbb{N}\setminus\{0\}} \rho_H(r)\right| &= \left|\frac{2}{n}\sum_{k=1}^n (n-r)\rho_H(r) - 2\sum_{r\in\mathbb{N}\setminus\{0\}} \rho_H(r)\right| \\
&\leq \left|2\sum_{r=n}^{+\infty} \rho_H(r)\right| + \left|\frac{2}{n}\sum_{r=1}^{n-1} r\rho_H(r)\right| \leq C\sum_{r=n}^{+\infty} r^{2H-2} + \frac{C}{n}\sum_{r=1}^{n-1} r^{2H-1} \trianglelefteq n^{2H-1}.
\end{aligned}$$

On the other hand, as $H < 1/2$, $\sum_r \rho_H(r) < +\infty$. Therefore we have

$$\left|\frac{2}{n}\sum_{i=1}^n \rho_H(i)\sum_{j=1}^n e^{-aj}\right| \trianglelefteq n^{-1}.$$

Finally,

$$\left|\frac{\rho_H(0)}{n}\left(\sum_{i=1}^n e^{-ai}\right)^2\right| \trianglelefteq n^{-1}.$$

which completes the proof. □

Lemma 4.2. *Let $1/2 < H < 1$, then we have in this case*

$$\mathbb{E}\left[I_1(g_n^H)^2\right] \longrightarrow 1/a^2 \quad as \quad n \to +\infty. \tag{15}$$

Proof. We have $I_1(g_n^H) = \frac{1}{n^H}\sum_{i=1}^n \left(Z_i^H - e^{-ai}Z_0^H\right)$, where Z^H is the process defined in (8). Therefore

$$\begin{aligned}\mathbb{E}[I_1(g_n^H)^2] &= \frac{1}{n^{2H}}\sum_{i,j=1}^n \mathbb{E}[(Z_i^H - e^{-ai}Z_0^H)(Z_j^H - e^{-aj}Z_0^H)] \\ &= \frac{1}{n^{2H}}\sum_{i,j=1}^n \mathbb{E}[Z_i^H Z_j^H] - \frac{2}{n^{2H}}\sum_{i,j=1}^n e^{-aj}\mathbb{E}[Z_i^H Z_0^H] + \frac{\mathbb{E}[(Z_0^H)^2]}{n^{2H}}\left(\sum_{i=1}^n e^{-ai}\right)^2\end{aligned}$$

$$= \frac{H\Gamma(2H)}{a^{2H}n^{2H-1}} + \frac{2}{n^{2H}}\sum_{i=1}^{n-1}\sum_{j=i+1}^n \rho_H(j-i) - \frac{2}{n^{2H}}\sum_{i,j=1}^n \rho_H(i)e^{-aj} + \frac{\rho_H(0)}{n^{2H}}\left(\sum_{i=1}^n e^{-ai}\right)^2.$$

On the other hand

$$\begin{aligned}\left|\frac{2}{n^{2H}}\sum_{i=1}^{n-1}\sum_{j=i+1}^n \rho_H(j-i) - 1/a^2\right| &= \left|\frac{2}{n^{2H}}\sum_{r=1}^{n-1}(n-r)\rho_H(r) - \frac{1}{a^2}\right| \\ &= \left|\left(\frac{2}{n^{2H-1}}\sum_{r=1}^{n-1}\rho_H(r) - \frac{2H}{a^2}\right) - \left(\frac{2}{n^{2H}}\sum_{r=1}^{n-1}r\rho_H(r) - \frac{2H-1}{a^2}\right)\right|\end{aligned} \tag{16}$$

we have $\lim_{n\to+\infty}|\frac{2}{n^{2H-1}}\sum_{r=1}^{n-1}\rho_H(r) - \frac{2H}{a^2}| = 0$, indeed as it's shown in [7], $\rho_H(r) \sim \frac{H(2H-1)}{a^2}r^{2H-2}$ for large r and since $H > 1/2$, we have $\sum_{r=1}^n r^{2H-2} \sim \frac{n^{2H-1}}{(2H-1)}$, as $n \sim +\infty$, which gives the desired result. On the other hand, $\lim_{n\to+\infty}|\frac{2}{n^{2H}}\sum_{r=1}^{n-1} r\rho_H(r) - \frac{2H-1}{a^2}| = 0$, in fact as $r\rho_H(r) \sim \frac{H(2H-1)}{a^2}r^{2H-1}$ as $r \sim +\infty$ and using the fact that $\sum_{r=1}^{n-1} r^{2H-1} \sim \frac{n^{2H}}{2H}$ as $n \sim +\infty$ we get the desired limit.

Moreover, using the equivalences above, we get

$$|\frac{2}{n^{2H}}\sum_{i=1}^n \rho_H(i)\sum_{j=1}^n e^{-aj}| \trianglelefteq n^{-1} \to 0$$

as $n \to +\infty$. Finally

$$\frac{1}{n^{2H}}\left(\frac{H\Gamma(2H)}{a^{2H}}\right)\sum_{i,j=1}^n e^{-a(i+j)} \trianglelefteq n^{-2H} \to 0$$

as $n \to +\infty$. □

Proposition 4.1. *Let $0 < H < 3/4$. Define*

$$\sigma_H^2 := 2\sum_{i\in\mathbb{Z}}\rho_H(i)^2 \tag{17}$$

where $\rho_H(k) := E[Z_k^H Z_0^H]$, $k \in \mathbb{N}$ and Z is the process given in (8). Then

$$|v_n - \sigma_H^2| \trianglelefteq \begin{cases} n^{-1} & if \quad 0 < H \le 1/2 \\ n^{4H-3} & if \quad 1/2 < H < 3/4. \end{cases}$$

If $H = 3/4$, we have

$$\left|\frac{v_n}{\log(n)} - \frac{9}{16a^4}\right| \leqslant C\log(n)^{-1}.$$

Proof. From the definition of v_n, we have

$$\begin{aligned} v_n &= \mathbb{E}[(I_2(\sqrt{n}f_n))^2] \\ &= \frac{2}{n}\sum_{i,j}(\mathbb{E}[Z_i^H Z_j^H])^2 = \frac{4}{n}\sum_{i=1}^{n}(n-r)\rho_H^2(r) + 2\rho_H^2(0) \end{aligned}$$

Therefore, as $0 < H < 3/4$ and using the fact that $\rho_H(r) \sim \frac{H(2H-1)}{a^2} r^{2H-2}$ for large r, we get

$$\begin{aligned} \left|v_n - \sigma_H^2\right| &\leq 2\sum_{i=n}^{+\infty} i^{4H-4} + \frac{2}{n}\sum_{i=1}^{n-1} i^{4H-3} \\ &\trianglelefteq \begin{cases} n^{-1} & \text{if } \ 0 < H \leq 1/2 \\ n^{4H-3} & \text{if } \ 1/2 < H < 3/4. \end{cases} \end{aligned}$$

For $H = 3/4$, we use the fact that $\sum_{i=1}^{n} \rho_{3/4}^2(r) \sim \frac{9}{64}\log(n)$ for large n. □

Lemma 4.3. *Consider the random sequence* $\{R_H(n)\}_{n\geq 1}$ *defined in (9), then we have*

$$\sqrt{n}\mathbb{E}[|R_H(n)|] \trianglelefteq \begin{cases} n^{-\frac{1}{2}} & \textit{if } \ 0 < H < 1/2 \\ n^{2H-3/2} & \textit{if } \ 1/2 \leq H < 3/4. \end{cases}$$

If $H = 3/4$*, then*

$$\frac{\sqrt{n}}{\sqrt{\log(n)}}\mathbb{E}[|R_H(n)|] \trianglelefteq \log(n)^{-1/2}.$$

Proof. We have

$$\begin{aligned} \sqrt{n}\mathbb{E}[R_H(n)] = &\frac{2}{\sqrt{n}}\sum_{i=1}^{n}\mathbb{E}[Y_i Z_i] + \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\mathbb{E}[Y_i^2] \\ &- \frac{2}{n^{3/2}}\sum_{i,j=1}^{n}\mathbb{E}[Z_i Y_j] - \frac{1}{n^{3/2}}\sum_{i,j=1}^{n}\mathbb{E}[Y_i Y_j] - \frac{1}{n^{3/2}}\sum_{i,j=1}^{n}\mathbb{E}[Z_i Z_j]. \end{aligned}$$

On the other hand

$$\left|\frac{2}{\sqrt{n}}\sum_{i=1}^{n}\mathbb{E}[Y_i Z_i]\right| = \left|\frac{2}{\sqrt{n}}\sum_{i=1}^{n}\rho_H(i)e^{-ai}\right| \trianglelefteq n^{-1/2},$$

$$\left|\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\mathbb{E}[Y_i^2]\right| = \frac{(b^2+\rho_H(0)^2)}{\sqrt{n}}\sum_{i=1}^{n}e^{-2ai} \trianglelefteq n^{-1/2},$$

$$\begin{aligned} \left|\frac{2}{n^{3/2}}\sum_{i,j=1}^{n}\mathbb{E}[Z_i Y_j]\right| &= \left|\frac{-2}{n^{3/2}}\sum_{i,j=1}^{n}e^{-aj}\rho_H(i)\right| \\ &\trianglelefteq \begin{cases} n^{-\frac{3}{2}} & \text{if } \ 0 < H \leq 1/2 \\ n^{2H-5/2} & \text{if } \ 1/2 < H < 1. \end{cases} \end{aligned}$$

$$\left|\frac{1}{n^{3/2}}\sum_{i,j=1}^{n}\mathbb{E}[Y_j Y_i]\right| = \frac{(b^2+\rho_H(0)^2)}{n^{3/2}}\left(\sum_{i=1}^{n}e^{-aj}\right)^2 \trianglelefteq n^{-3/2}.$$

Finally using the fact that $\rho_H(r) \sim \frac{H(2H-1)}{a^2} r^{2H-2}$ for large r, we get

$$\left| \frac{1}{n^{3/2}} \sum_{i,j=1}^{n} \mathbb{E}[Z_i Z_j] \right| = \left| \frac{1}{n^{3/2}} \sum_{i,j=1}^{n} \rho_H(j-i) \right| = \left| \frac{\rho_H(0)}{\sqrt{n}} + \frac{2}{n^{3/2}} \sum_{r=1}^{n} (n-r)\rho_H(r) \right|$$
$$\leq \begin{cases} n^{-\frac{1}{2}} & \text{if} \quad 0 < H \leq 1/2 \\ n^{2H-3/2} & \text{if} \quad 1/2 < H < 3/4. \end{cases}$$

The upper bound when $H = 3/4$ can be obtained easily using the previous computations and this finishes the proof. □

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