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N. S. AIUBOVA

LIMIT THEOREMS FOR ONE STATISTIC OF FBM IN THE MODEL OF REAL OBSERVATIONS

In this article the central limit theorem as Hurst index $H \in \left(0, \frac{3}{4}\right]$ and the non-central limit theorem as Hurst index $H \in \left(\frac{3}{4}, 1\right)$ for statistics of fraction Brownian motion in the model of real observations are obtained.

1. Introduction

A centered Gaussian random process $\{\xi_H(t), t \in \mathbb{R}\}$ with the covariance function

$$B_H(s,t) = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H}), t, s \in \mathbb{R}$$

is called a fraction Brownian motion (FBM) with Hurst parameter $H \in (0,1)$. For $H = \frac{1}{2}$ an FBM is a standard Wiener process. For $H > \frac{1}{2}$ an FBM has a property of long-range dependence and for $H < \frac{1}{2}$ it is short-range dependent. An FBM is widely used in the contemporary models of a hydrology, meteorology, finance mathematics and other sciences. The problem of the statistical estimation of the Hurst parameter was considered by the many authors. An overview of several methods to estimate the Hurst parameter is given in [1]. The most popular methods are based on the Baxter sums [2], [3], [4]. The Levy-Baxter theorems provide for consistence of those estimations. The Baxter sums methods permit to obtain the non-asymptotic confidence regions for the estimated parameters.

Recently, interest in problem of estimating by observations with errors has increased. For example, in the monograph [5] the application of regression models with measurement errors to radiation risks assessment is considered. The estimation of the Hurst parameter of fractional Brownian motion by observations with errors was investigated in articles [6],[7].

2. Model of real observations

The real observation of the value of a random process at a point is carried out by a device that has a certain inertia. Therefore, when measuring the value of a random process $\xi(\cdot)$ at a moment t, the device gives the value of the integral $\int_{O(t)} \xi(s)\varphi(s)ds$, where $O(t)$ a certain neighborhood of the point t, $\varphi(s)$ is a function that characterizes the device [8].

Let $\delta \in (0, \frac{1}{3})$, $\Delta > 1 + 2\delta$; $\varphi \in L_1([-\delta, \delta])$ is known nonnegative function, $\int_{-\delta}^{\delta} \varphi(s)ds = 1$.

Let

$$\eta_{k,H} = \int_{k\Delta-\delta}^{k\Delta+\delta} \xi_H(s)\varphi(s - k\Delta)ds, k \in \mathbb{Z}.$$

$$\xi_{k,H} = \eta_{k+1,H} - \eta_{k,H}, k \in \mathbb{Z};$$

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In article [9] with statistics

$$S_{N,H} = \sum_{k=0}^{N-1} \xi_{k,H}^2, N \geq 1 \tag{1}$$

consistent estimate of the Hurst parameter of the fractional Brownian motion is constructed. Note that $(\xi_{k,H})$ is stationary Gaussian random sequence, $E\xi_{k,H} = 0$; the function $\kappa(H) = E\xi_{0,H}^2$, $H \in (0,1)$ is nondecreasing and continuous on the interval $(0,1)$, $\kappa(0+) = 0, \kappa(1-) = \Delta^2$ [9].

In this paper for statistics $S_{N,H}, N \geq 1$ the central limit theorem as $H \in \left(0, \frac{3}{4}\right]$ and the non-central limit theorem as $H \in \left(\frac{3}{4}, 1\right)$ are obtained.

3. Asymptotic normality of $S_{N,H}$ as $H \in \left(0, \frac{3}{4}\right]$

Let $r(n) = E\xi_{0,H}\xi_{n,H}, n \geq 0$. In article [9] it is proved that

$$r(n) = \frac{1}{2}\int_A (((n-1)\Delta+s-t)^{2H} - 2(n\Delta+s-t)^{2H} + ((n+1)\Delta+s-t)^{2H})\varphi(s)\varphi(t)dsdt, \tag{2}$$

where $H \in (0,1)$, $A = [-\delta,\delta]^2, n \geq 2$;

$$r^2(n) \leq \frac{(2H(2H-1))^2}{4((n-2)+1)^{4-4H}}\Delta^4, n \geq 3 \tag{3}$$

Then apply the results of the article [10] in the case of $J = 1$; $X_n, n \in \mathbb{Z}$ is a stationary Gaussian sequence with zero mean and unit variance. Let $\widetilde{r}(l) = EX_0X_l$, $l \in \mathbb{Z}$; $H(x) = H_2(x) = x^2 - 1$, $x \in \mathbb{R}$ is a second-degree Hermite polynomial; the degree of the polynomial is $k = 2$, $c_1 = 0$, $c_2 = 1$, $c_m = 0$, $m \geq 3$; $Z_0^N = \frac{1}{A_N}\sum_{j=1}^N H_2(X_j)$, where A_N are corresponding norming constants. In this case, Theorem 3.1 and Theorem 3.2 respectively follow from Theorem 1 and Theorem 1 '[10].

Theorem 3.1. *Let stationary Gaussian sequence $X_n, n \in \mathbb{Z}$ satisfies the condition*

$$\sum_{N\in\mathbb{Z}} \widetilde{r}^2(n) < \infty.$$

Let $A_N = \sqrt{N}$. Then exists

$$\lim_{N\to\infty} \frac{2}{N}\sum_{i,j=1}^N \widetilde{r}^2(i-j) = 2\sigma_2^2, \sigma^2 = 2\sigma_2^2,$$

and Z_0^N converges weakly to σZ_0^*, where Z_0^* is a standard Gaussian random variable.

Theorem 3.2. *Let the correlation function of a stationary Gaussian sequence $X_n, n \in \mathbb{Z}$ satisfies the condition*

$$\sum_{j=1}^N \widetilde{r}^2(l) = L(N)$$

and exists

$$\lim_{N\to\infty} \frac{1}{L(N)}\sum_{j=1}^N \widetilde{r}^l(j)$$

for all $l \geq k$, where function $L(N)$ is the function of slow change. Let $A_N = \sqrt{NL(N)}$. Then exists

$$\lim_{N\to\infty} \frac{2}{NL(N)}\sum_{i,j=1}^N \widetilde{r}^2(i-j) = 2\sigma_2^2, \sigma^2 = 2\sigma_2^2,$$

and Z_0^N converges weakly to σZ_0^*, where Z_0^* is a standard Gaussian random variable.

Let $X_j = \frac{\xi_{j-1,H}}{\sqrt{\kappa(H)}}$, $j \in \mathbb{Z}$. Note that $X_j, j \in \mathbb{Z}$. is a stationary Gaussian sequence with zero mean, unit variance and correlation function

$$\widetilde{r}(l) = EX_0 X_l = \frac{1}{\kappa(H)} E\xi_{-1,H}\xi_{l-1,H} = \frac{r(l)}{\kappa(H)}, l \in \mathbb{Z}$$

If $H \in (0, \frac{3}{4})$ then $4 - 4H > 1$. Due to inequality (3)

$$\sum_{l=1}^{\infty} \widetilde{r}^2(l) = \frac{1}{\kappa^2(H)} \sum_{l=1}^{\infty} r^2(l) < \infty.$$

From the Theorem 3.1 for $H \in (0, \frac{3}{4})$ it follows

Theorem 3.3. *Let $H \in (0, \frac{3}{4})$. Then a sequence of random variables*

$$\frac{1}{\sqrt{N}} \sum_{j=1}^{N} (X_j^2 - 1) = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} \left(\frac{\xi_{j,H}^2}{\kappa(H)} - 1 \right) = \frac{1}{\kappa(H)\sqrt{N}} (S_{N,H} - ES_{N,H})$$

is asymptotically normal with zero mean and variance

$$\sigma^2 = \lim_{N \to \infty} \frac{2}{N} \sum_{i,j=1}^{N} \widetilde{r}^2(i-j).$$

Let's proof the Lemma:

Lemma 3.1. *For any $H \in \left[\frac{3}{4}, 1\right)$ there exists a positive constant $C(H)$ such that*

$$r^2(n) \sim \frac{C(H)}{n^{4-4H}}, n \to \infty. \tag{4}$$

Proof. Note that the expression

$$B_n = ((n-1)\Delta + s - t)^{2H} - 2(n\Delta + s - t)^{2H} + ((n+1)\Delta + s - t)^{2H},$$

under the integral sign on the right side of equality (3), is equal to the second order increment of the function $f(x) = x^{2H}$ on an interval $[(n-1)\Delta + s - t, (n+1)\Delta + s - t]$.

Let $z = (n-1)\Delta + s - t$, $(s,t) \in A = [-\delta, \delta]^2$. Then

$$B_n = f(z) - 2f(z + \Delta) + f(z + 2\Delta) = \int_z^{z+\Delta} ds \int_s^{s+\Delta} f''(u)du.$$

Since $-2\delta \le s - t \le 2\delta$, then

$$\frac{2H(2H-1)}{((n+1)\Delta + 2\delta)^{2-2H}} \le f''(u) = \frac{2H(2H-1)}{u^{2-2H}} \le \frac{2H(2H-1)}{((n-1)\Delta - 2\delta)^{2-2H}}.$$

For any $v \in [z, z + \Delta]$ let's integrate term-by-term this double inequality in an interval $[v, v+\Delta]$ with respect to variable u, then integrate term-by-term in an interval $[z, z+\Delta]$ with respect to variable v. We obtain double inequality

$$\frac{2H(2H-1)\Delta^2}{((n+1)\Delta + 2\delta)^{2-2H}} \le \int_z^{z+\Delta} dv \int_v^{v+\Delta} \frac{2H(2H-1)}{u^{2-2H}} du \le \frac{2H(2H-1)\Delta^2}{((n-1)\Delta - 2\delta)^{2-2H}}.$$

Let's multiply all parts of this inequality by $\frac{1}{2}\varphi(s)\varphi(t)$, $(s,t) \in A$ and integrate on a set A. Taking into account that $\int_A \varphi(s)\varphi(t)dsdt = 1$ and applying an equality (3), we obtain that

$$\frac{H(2H-1)\Delta^2}{((n+1)\Delta + 2\delta)^{2-2H}} \le r(n) \le \frac{H(2H-1)\Delta^2}{((n-1)\Delta - 2\delta)^{2-2H}},$$

it follows that

$$\frac{H^2(2H-1)^2\Delta^4}{((n+1)\Delta+2\delta)^{4-4H}} \le r^2(n) \le \frac{H^2(2H-1)^2\Delta^4}{((n-1)\Delta-2\delta)^{4-4H}}.$$

From the last double inequality it follows that the relation of equivalence (4) holds as $C(H) = H^2(2H-1)^2\Delta^{4H}$.

The lemma is proven. □

Let $H = \frac{3}{4}$. Let's prove that the function $L(N) = \sum_{l=1}^{N} \tilde{r}^2(l) = \frac{1}{\kappa(H)} \sum_{l=1}^{N} r^2(l)$ is a function of slow change. To find the limit

$$\lim_{N\to\infty} \frac{\sum_{l=1}^{N} r^2(l)}{\ln N}$$

Stolz theorem applies:

$$\lim_{N\to\infty} \frac{\sum_{j=1}^{N} r^2(j)}{\ln N} = \lim_{N\to\infty} \frac{r^2(N+1)}{\ln(N+1)-\ln N} = \lim_{N\to\infty} N r^2(N+1) = C\left(\frac{3}{4}\right),$$

due to Lemma 3.1. So, $L(N) \sim C(\frac{3}{4}) \ln N$, $N \to \infty$, whence it follows that a function $L(N), N \ge 1$ is a function of slow change.

It is similarly proved that for $l > 2$

$$\lim_{N\to\infty} \frac{1}{L(N)} \sum_{j=1}^{N} r^l(j) = \lim_{N\to\infty} \frac{1}{C(\frac{3}{4}) \ln N} \sum_{j=1}^{N} r^l(j) = 0.$$

Now from Theorem 3.2 it follows

Theorem 3.4. *Let $H = \frac{3}{4}$. Then a sequence of random variables*

$$\frac{1}{\sqrt{C(\frac{3}{4}) N \ln N}} \sum_{j=1}^{N} \left(X_j^2 - 1\right) = \frac{1}{\sqrt{C(\frac{3}{4}) N \ln N}} \sum_{j=0}^{N-1} \left(\frac{\xi_{j,H}^2}{\kappa(H)} - 1\right) =$$

$$\frac{1}{\kappa(H)\sqrt{C(\frac{3}{4}) N \ln N}} (S_{N,H} - ES_{N,H})$$

is asymptotically normal with zero mean and variance

$$\sigma^2 = \lim_{N\to\infty} \frac{2}{C(\frac{3}{4}) N \ln N} \sum_{i,j=1}^{N} \tilde{r}^2(i-j).$$

4. Non-central limit theorem for $S_{N,H}$ as $H \in (\frac{3}{4}, 1)$

In the article [13], in the case of a slow decrease to zero of the covariance function, a non-central limit theorem for nonlinear functions from Discrete-time Gaussian stationary processes is obtained. Using Theorem 1 from this article we get

Theorem 4.1. *For $H \in (\frac{3}{4}, 1)$ the sequence of random variables*

$$\frac{1}{N^{2H-1}} \cdot \frac{S_{N,H} - ES_{N,H}}{\kappa(H)}, N \ge 1$$

is weakly convergent to a

$$\frac{1}{D} \int_{R^2} e^{i(x_1+x_2)} \frac{e^{i(x_1+x_2)} - 1}{i(x_1+x_2)} |x_1|^{\frac{\alpha-1}{2}} |x_2|^{\frac{\alpha-1}{2}} dW(x_1) dW(x_2),$$

the Wiener-Ito multiple integer relative to a random spectral measure W of the white noise process, where $D = 2\Gamma(\alpha)\cos(\frac{\pi\alpha}{2})$, $\alpha = 2 - 2H$.

Proof. In Theorem 1 of [13] let $k = 2$, $H(x) = H_2(x) = x^2 - 1$, $x \in \mathbb{R}$. Lemma 1 ensures that the conditions of Theorem 1 of [13] are fulfilled. Indeed, for $H \in (\frac{3}{4}, 1)$ constant $\alpha = 2 - 2H < \frac{1}{k} = \frac{1}{2}$,

$$r(N) \sim \frac{H(2H-1)\Delta^{2H}}{N^\alpha}, N \to \infty.$$

It follows, from Theorem 1 of [13], that the sequence of random variables

$$\frac{1}{N^{2H-1}} \cdot \frac{S_{N,H} - ES_{N,H}}{\kappa(H)}, N \geq 1$$

converges in distribution to a random variable

$$\frac{1}{D} \int e^{i(x_1+x_2)} \frac{e^{i(x_1+x_2)} - 1}{i(x_1+x_2)} |x_1|^{\frac{\alpha-1}{2}} |x_2|^{\frac{\alpha-1}{2}} dW(x_1) dW(x_2), \tag{5}$$

where $D = 2\Gamma(\alpha)\cos(\frac{\pi\alpha}{2})$. □

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Kyiv National Taras Shevchenko University, Mechanics and Mathematics Faculty, Department of Mathematical Analysis, Volodymyrs'ka Street 64, 01601, Kyiv, Ukraine

Current address: Kyiv National Taras Shevchenko University, Mechanics and Mathematics Faculty, Department of Mathematical Analysis, Volodymyrs'ka Street 64, 01601, Kyiv, Ukraine

E-mail address: n.aiubova@gmail.com