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# SIMULATION OF FRACTIONAL BROWNIAN MOTION WITH GIVEN RELIABILITY AND ACCURACY IN $\mathbf{C}([0,1])$[1]

We present here an application of the results on simulation of weakly self-similar stationary increment $\varphi$-sub-Gaussian processes, obtained by Kozachenko, Sottinen and Vasylyk in [1], to the process of fractional Brownian motion.

## 1. Introduction

In this paper we consider simulation of fractional Brownian motion defined on the interval $[0,1]$ with given reliability and accuracy in space $\mathbf{C}([0,1])$.

We apply results obtained in [1] for centred second order $\mathrm{Sub}_\varphi(\Omega)$-processes defined on the interval $[0,1]$ with covariance function

$$R(t,s) = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right).$$

The parameter $H$ takes values in the interval $(0,1)$.

In order to construct a model of such process we used a series expansion approach based on a series representation proved by Dzaparidze and van Zanten [2] for the fractional Brownian motion $B$:

$$B_t = \sum_{n=1}^{\infty} \frac{\sin(x_n t)}{x_n} X_n + \sum_{n=1}^{\infty} \frac{1-\cos(y_n t)}{y_n} Y_n, \quad t \in [0,1]. \tag{1}$$

Here the $X_n$'s and the $Y_n$'s are independent zero mean Gaussian random variables with certain variances depending on $H$ and $n$. The $x_n$'s are the positive real zeros of the Bessel function $J_{-H}$ of the first kind and the $y_n$'s are the positive real zeros of the Bessel function $J_{1-H}$. The series in (1) converge in mean square as well as uniformly on $[0,1]$ with probability 1.

Replacing the $X_n$'s and $Y_n$'s by independent random variables from the space $\mathrm{Sub}_\varphi(\Omega)$ we got series representation for $\varphi$-subGaussian random processes with covariance function $R$. This representation was used for simulation of such processes with given reliability and accuracy in $\mathbf{C}([0,1])$. Processes of fractional Brownian motion belong to the space $\mathrm{Sub}_\varphi(\Omega)$ with

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$\varphi(x) = x^2/2$. So, in this paper we present some examples of simulation of fractional Brownian motion with different values of parameter $H$.

## 2. Space $\mathrm{Sub}_\varphi(\Omega)$

We need the following facts about the space $\mathrm{Sub}_\varphi(\Omega)$ of *$\varphi$-sub-Gaussian* (or *generalised sub-Gaussian*) random variables.

**Definition 2.1** ([3])**.** A continuous even convex function $u = \{u(x), x \in \mathbb{R}\}$ is an *Orlicz N-function* if it is strictly increasing for $x > 0$, $u(0) = 0$,

$$\frac{u(x)}{x} \to 0 \quad \text{as} \quad x \to 0$$

and

$$\frac{u(x)}{x} \to \infty \quad \text{as} \quad x \to \infty.$$

**Proposition 2.2** ([3])**.** *The function $u$ is an Orlicz N-function if and only if*

$$u(x) = \int_0^{|x|} l(u)\,\mathrm{d}u, \quad x \in \mathbb{R},$$

*where the* density function *$l$ is nondecreasing, right continuous, $l(u) > 0$ as $u > 0$, $l(0) = 0$ and $l(u) \to \infty$ as $u \to \infty$.*

**Definition 2.3.** Let $u$ be an Orlicz N-function. The even function $u^* = \{u^*(x), x \in \mathbb{R}\}$ defined by the formula

$$u^*(x) = \sup_{y>0} \big(xy - u(y)\big),\ x \geq 0,$$

is the *Young-Fenchel transformation* of the function $u$.

**Proposition 2.4** ([3])**.** *The function $u^*$ is an Orlicz N-function and for $x > 0$*

$$u^*(x) = xy_0 - u(y_0) \qquad \textit{if} \quad y_0 = l^{-1}(x).$$

*Here $l^{-1}$ is the* generalised inverse function *of $l$, i.e.*

$$l^{-1}(x) := \sup\{v \geq 0 : l(v) \leq x\}.$$

**Definition 2.5.** An Orlicz N-function $\varphi$ satisfies *assumption $Q$* if $\varphi$ is quadratic around the origin, i.e. there exist such constants $x_0 > 0$ and $C > 0$ that $\varphi(x) = Cx^2$ for $|x| \leq x_0$.

**Definition 2.6.** A zero mean random variable $\xi$ belongs to the space $\mathrm{Sub}_\varphi(\Omega)$, the space of *$\varphi$-sub-Gaussian random variables*, if there exists a positive and finite constant $a$ such that the inequality

$$\mathbf{E}\exp\{\lambda\xi\} \leq \exp\{\varphi(a\lambda)\}$$

holds for all $\lambda \in \mathbb{R}$.

The space $\mathrm{Sub}_\varphi(\Omega)$ is a Banach space with respect to the norm

$$\tau_\varphi(\xi) = \inf\Big\{a \ge 0 : \mathbf{E}\exp\{\lambda\xi\} \le \exp\big\{\varphi(a\lambda)\big\},\ \lambda \in \mathbb{R}\Big\}.$$

**Definition 2.7.** A stochastic process $X = (X_t)_{t\in[0,1]}$ is a $\mathrm{Sub}_\varphi(\Omega)$-*process* if it is a bounded family of $\mathrm{Sub}_\varphi(\Omega)$ random variables: $X_t \in \mathrm{Sub}_\varphi(\Omega)$ for all $t \in [0,1]$ and

$$\sup_{t\in[0,1]} \tau_\varphi(X_t) < \infty.$$

The properties of random variables from the spaces $\mathrm{Sub}_\varphi(\Omega)$ were studied in the book [4].

*Remark* 2.8. When $\varphi(x) = \frac{x^2}{2}$ the space $\mathrm{Sub}_\varphi(\Omega)$ is called the space of *sub-Gaussian* random variables and is denoted by $\mathrm{Sub}(\Omega)$. Centred Gaussian random variable $\xi$ belongs to the space $\mathrm{Sub}(\Omega)$, and in this case $\tau_\varphi(\xi)$ is just the standard deviation: $(\mathbf{E}\xi^2)^{1/2}$. Also, if $\xi$ is bounded, i.e. $|\xi| \le c$ a.s. then $\xi \in \mathrm{Sub}(\Omega)$ and $\tau_\varphi(\xi) \le c$.

## 3. Simulation of $\mathrm{Sub}_\varphi(\Omega)$-processes

Define a process $Z = (Z_t)_{t\in[0,1]}$ by the expansion

$$Z_t = \sum_{n=1}^{\infty} c_n \sin(x_n t)\,\xi_n + \sum_{n=1}^{\infty} d_n\big(1-\cos(y_n t)\big)\,\eta_n, \tag{2}$$

where

$$c_n = \frac{\pi^H\sqrt{2c}}{x_n^{H+1} J_{1-H}(x_n)}, \qquad n = 1,2,\dots, \tag{3}$$

$$d_n = \frac{\pi^H\sqrt{2c}}{y_n^{H+1} J_{-H}(y_n)}, \qquad n = 1,2,\dots, \tag{4}$$

$$c = \frac{\Gamma(2H+1)\sin(\pi H)}{\pi^{2H+1}}, \tag{5}$$

$\xi_n$, $\eta_n$, $n = 1,2,\dots$, are independent identically distributed centred random variables from the space $\mathrm{Sub}_\varphi(\Omega)$ with

$$\mathbf{E}\xi_n^2 = \mathbf{E}\eta_n^2 = 1$$

and

$$\tau_\varphi(\xi_n) = \tau_\varphi(\eta_n) =: a_\varphi, \qquad n = 1,2,\dots;$$

$x_n$ is the $n$th positive real zero of the Bessel function $J_{-H}$; $y_n$ is the $n$th positive real zero of $J_{1-H}$,

$$J_\nu(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (x/2)^{\nu+2n}}{\Gamma(n+1)\Gamma(\nu+n+1)}.$$

Here $x > 0$, $\nu \neq -1, -2, \dots$ and $\Gamma$ denotes the Euler Gamma function

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, \mathrm{d}t.$$

We shall assume that the function $\varphi(\sqrt{\cdot})$ is convex.

Since $\varphi$-sub-Gaussian random variables are square integrable we have the following direct consequence of the series representation (1) for fractional Brownian motion.

**Proposition 3.1.** *The series (2) converges in mean square and the covariance function of the process $Z$ is $R$.*

**Theorem 3.2** ([1])**.** *The series (2) converges uniformly with probability one and the process $Z$ is almost surely continuous on* $[0,1]$. *Moreover, if $Z$ is strongly self-similar with stationary increments then it is $\beta$-Hölder continuous with any index $\beta < H$.*

Consider the space $C([0,1])$ equipped with the usual sup-norm.

**Definition 3.3.** The model $\tilde{Z}$ approximates the process $Z$ with given *reliability* $1-\nu$, $0 < \nu < 1$, and *accuracy* $\delta > 0$ in $C([0,1])$ if

$$\mathbf{P}\left( \sup_{t\in[0,1]} |Z_t - \tilde{Z}_t| > \delta \right) \leq \nu.$$

Let $\tilde{c}_n$ and $\tilde{d}_n$ be the approximated values of the $c_n$ and $d_n$, respectively. Let

$$|\tilde{c}_n - c_n| \leq \gamma_n^c,$$
$$|\tilde{d}_n - d_n| \leq \gamma_n^d,$$
$$n = 1, \dots .$$

The errors $\gamma_n^c$ and $\gamma_n^d$ are assumed to be known. Let $\tilde{x}_n$ and $\tilde{y}_n$ be approximations of the corresponding zeros $x_n$ and $y_n$ with error bounds

$$|\tilde{x}_n - x_n| \leq \gamma_n^x,$$
$$|\tilde{y}_n - y_n| \leq \gamma_n^y.$$

The error bounds $\gamma_n^x$ and $\gamma_n^y$ are also assumed to be known.

Then, the model of the process $Z$ we define as follows

$$\tilde{Z}_t = \sum_{n=1}^{N} \Big( \tilde{c}_n \sin(\tilde{x}_n t)\, \xi_n \; + \; \tilde{d}_n \big(1 - \cos(\tilde{y}_n t)\big)\, \eta_n \Big). \tag{6}$$

The following theorem contains the main result of the paper [1].

**Theorem 3.4** ([1]). *Let $b$ and $\alpha$ be such that $0 < b < \alpha < H$. Denote*

$$\begin{aligned}
\gamma_0 &= \sqrt{\gamma^{\mathrm{appr}} + \gamma^{\mathrm{cut}}},\\
\gamma_\alpha &= \sqrt{\gamma_\alpha^{\mathrm{appr}} + \gamma_\alpha^{\mathrm{cut}}},\\
\beta &= \min\left\{\gamma_0, \frac{\gamma_\alpha}{2^\alpha}\right\},
\end{aligned}$$

*where*

$$\begin{aligned}
\gamma^{\mathrm{cut}} &= a_\varphi^2 \sum_{n=N+1}^{\infty} \left(c_n^2 + 4d_n^2\right),\\
\gamma^{\mathrm{appr}} &= a_\varphi^2 \sum_{n=1}^{N} \left\{\left(c_n\gamma_n^x + \gamma_n^c\right)^2 + \left(d_n\gamma_n^y + 2\gamma_n^d\right)^2\right\},
\end{aligned}$$

$$\begin{aligned}
\gamma_\alpha^{\mathrm{cut}} &= 2^{2-2\alpha} a_\varphi^2 \sum_{n=N+1}^{\infty} \left(c_n^2 x_n^{2\alpha} + d_n^2 y_n^{2\alpha}\right),\\
\gamma_\alpha^{\mathrm{appr}} &= 2^{3-2\alpha} a_\varphi^2 \sum_{n=1}^{N} \Bigg\{ x_n^{2\alpha}(\gamma_n^c)^2 + y_n^{2\alpha}(\gamma_n^d)^2\\
&+ 2^{3-2\alpha}\left((\tilde{c}_n)^2(\gamma_n^x)^{2\alpha}\left(\frac{(x_n+\tilde{x}_n)^{2\alpha}}{2^{2\alpha}} + 1\right) +\right.\\
&+ \left.(\tilde{d}_n)^2(\gamma_n^y)^{2\alpha}\left(\frac{(y_n+\tilde{y}_n)^{2\alpha}}{2^{2\alpha}} + 1\right)\right)\Bigg\}.
\end{aligned}$$

*Let $l$ be the density of $\varphi$.*

*The model $\tilde{Z}$, defined by (6), approximates the separable process $Z$, defined by (2), with given reliability*

$$1 - \nu, \qquad 0 < \nu < 1,$$

*and accuracy $\delta > 0$ in $C([0,1])$ if the following three inequalities are satisfied:*

$$\gamma_0 < \delta, \tag{7}$$

$$\frac{\beta\gamma_0}{\gamma_\alpha} < \frac{\delta}{2^\alpha(\exp\{\varphi(1)\} - 1)^\alpha}, \tag{8}$$

$$2\exp\left\{-\varphi^*\left(\frac{\delta}{\gamma_0} - 1\right)\right\}\left(\frac{1}{2^b(1-\frac{b}{\alpha})}\left(\frac{\gamma_\alpha\delta}{\beta\gamma_0}\right)^{\frac{b}{\alpha}} l^{-1}\left(\frac{\delta}{\gamma_0} - 1\right) + 1\right)^{\frac{2}{b}} \le \nu. \tag{9}$$

## 4. Simulation of fractional Brownian motion

Let us assume that the constants $c_n$ and $d_n$ and the zeros $x_n$ and $y_n$ are correctly calculated. In case of sub-Gaussian random processes we have the following corollary from the Theorem 3.4.

**Corollary 4.1.** *Suppose that there is no approximation error, i.e.*

$$\gamma_n^c = \gamma_n^d = \gamma_n^x = \gamma_n^y = 0.$$

*If the process $Z$ is sub-Gaussian then conditions of the Theorem 3.4 are satisfied if*

$$N \geq \max\left\{\left(\frac{a_\varphi}{\delta}\sqrt{\frac{5c}{2H}}\right)^{1/H} + 1; \quad \frac{2^{2-\frac{4}{H}}5^{\frac{1}{H}}}{\pi}\right\} \tag{10}$$

*and*

$$2\mu \exp\left\{-\frac{1}{2}\left(\frac{\delta N^H}{a_\varphi\sqrt{\frac{5c}{2H}}} - 1\right)^2\right\} N^{14} \leq \nu, \tag{11}$$

*where*

$$\mu = \pi^2 2^{\frac{22}{H}-4} 5^{-\frac{8}{H}} \left(\frac{H}{c}\right)^{\frac{6}{H}} \left(\frac{\delta}{a_\varphi}\right)^{\frac{12}{H}}.$$

Recall that in the sub-Gaussian case we have $\varphi(x) = \frac{x^2}{2}$ and that centered Gaussian random variables belong to the space $\mathrm{Sub}(\Omega)$. Parameters $\alpha$ and $b$ have to be optimized, but here $\alpha = \frac{H}{2}$ and $b = \frac{H}{4}$.

If in the series representation (2) the $\xi_n$ and $\eta_n$, $n = 1, 2, \dots$, are independent identically distributed centered Gaussian random variables with

$$\mathbf{E}\xi_n^2 = \mathbf{E}\eta_n^2 = 1, \qquad n = 1, 2, \dots,$$

then $Z$ is a process of fractional Brownian motion. In this case $a_\varphi = 1$.

Using Corollary 4.1 we construct a model $\tilde{Z}$ of the fractional Brownian motion $Z$, such that $\tilde{Z}$ approximates $Z$ with given reliability $1 - \nu = 0.99$ and accuracy $\delta = 0.01$ in $C([0, 1])$. In this paper we present examples of such models for three values of parameter $H$.

- $H_1 = \frac{3}{4}$. In this case we have: $c = 0.0537337$; $\mu = 97700.7$.
  From condition (10) follows that

  $$N \geq \max\{101.5, 0.270008\}.$$

  And condition (11) gives us $N \geq 6832.5$.

  So, we take $N = 6833$. As a result of simulation we have the model, which approximates fractional Brownian motion with parameter $H = \frac{3}{4}$ with given reliability 0.99 and accuracy 0.01 in $C([0, 1])$ (see Figure 1).

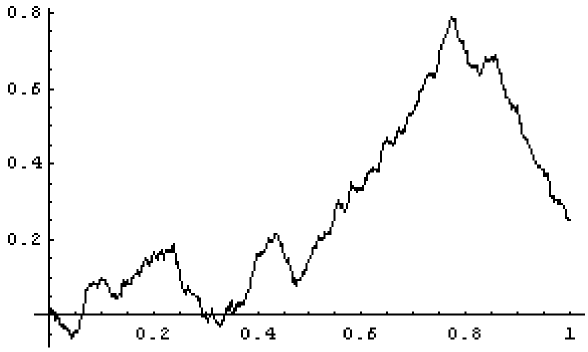


*Figure 1. Model of fractional Brownian motion with parameter* $H = \frac{3}{4}$

- $H_2 = \frac{7}{8}$. In this case $c = 0.0264278$; $\mu = 50832.8$.
  From conditions (10) and (11) follows:

$$N \geq \max\{57.8064,\ 0.336974\}$$

and $N \geq 1054.22$.
  The model with $N = 1055$ for fractional Brownian motion with parameter $H = \frac{7}{8}$ is presented on Figure 2.

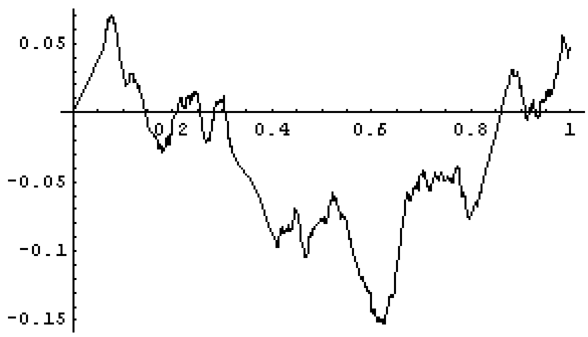


*Figure 2. Model of fractional Brownian motion with parameter* $H = \frac{7}{8}$

- $H_3 = \frac{8}{9}$. In this case we have: $c = 0.0234107$ and $\mu = 47369.5$.
  From condition (10) follows that

$$N \geq \max\{54.8061, 0.344046\}.$$

  From condition (11) we have $N \geq 863.771$.
  For $N = 864$ we obtained the model presented on Figure 3.

We can see that, as it was expected, $N$ decreases when value of $H$ increases, so the closer is $H$ to 1 the smoother curve we get.

All calculations and simulation were made using software *Mathematica*.

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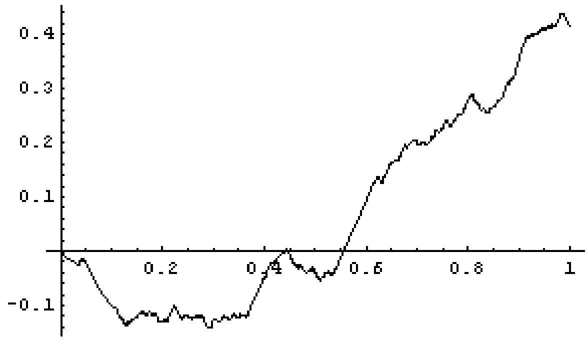


*Figure 3. Model of fractional Brownian motion with parameter* $H = \frac{8}{9}$

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