Theory of Stochastic Processes
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YURIY KOZACHENKO AND IRYNA ROZORA

APPLICATION OF THE THEORY OF SQUARE-GAUSSIAN PROCESSES TO SIMULATION OF STOCHASTIC PROCESSES[1]

In the paper the simulation of stochastic processes is considered. For this purpose the estimation for distribution of supremum of square-Gaussian processes is found. The theorems are proved that give the conditions under which the constructed model approximates stochastic process in Banach space with given accuracy and reliability. The obtained results can be widely used in actuarial science and financial mathematics.

1. Introduction

We'll consider some properties of square-Gaussian stochastic processes. Namely, a theorem about large deviation of supremum of the process is proved. This result is used for simulation of Gaussian stochastic process, that is entered the system (filter) as input process, taking into account output process, with given reliability and accuracy in Banach space $C([0,T]^d)$. The particular case in which the output process is equal to the derivative of input process is also considered.

The theory of simulation of stochastic process is extensively used in theory of actuarial science [7]. The obtained results can be also used in such way.

The paper consists of four parts. The first part is introduction, in the second one the main definitions and properties of square-Gaussian process are given. The third part is devoted to the distribution of supremum of square-gaussian processes. In last part we construct the model for Gaussian stochastic process which is entered the system and give conditions under which this model approximates the process, taking into account output process, with give accuracy and reliability in Banach space.

2. Square-Gaussian processes

Let $(\Omega, \mathcal{F}, P)$ be probability space and $(\mathbf{T}, \rho)$ be a compact metric space with metric ρ.

Use the definitions which were given in the paper [6] .

Definition 1.[6] Let $\Xi = \{\xi_t,\, t \in \mathbf{T}\}$ be a family of jointed Gaussian random variables, $\mathbf{E}\xi_t = 0$ (for example, let ξ_t, $t \in \mathbf{T}$, be a Gaussian random process).

[1]Invited lecture.

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A space $SG_\Xi(\Omega)$ is called the space of square-Gaussian random variables if any $\eta \in SG_\Xi(\Omega)$ can be represented in such way

$$\eta = \bar{\xi}^T A\bar{\xi} - \mathbf{E}\bar{\xi}^T A\bar{\xi}, \tag{1}$$

where $\bar{\xi}^T = (\xi_1, \xi_2, \dots, \xi_n)$, $\xi_k \in \Xi$, $k = 1, \dots, n$, A is real-valued matrix,

or $\eta \in SG_\Xi(\Omega)$ can be represented as a limit in mean square of the sequence of random variables from (1)

$$\eta = l.i.m._{n\to\infty}(\bar{\xi}_n^T A\bar{\xi}_n - \mathbf{E}\bar{\xi}_n^T A\bar{\xi}_n).$$

Definition 2.[6] A stochastic process $X = \{X(t),\, t \in \mathbf{T}\}$ is called square-Gaussian if for any $t \in \mathbf{T}$ a random variable $X(t)$ belongs to the space $SG_\Xi(\Omega)$.

Example 1. Consider a family of Gaussian centered stochastic processes $\xi_1(t),\, \xi_2(t), \dots, \xi_n(t)$, $t \in \mathbf{T}$. Let a matrix $A(t)$ be symmetric. Then

$$X(t) = \bar{\xi}^T(t)A(t)\bar{\xi}(t) - \mathbf{E}\bar{\xi}^T(t)A(t)\bar{\xi}(t),$$

where $\bar{\xi}^T(t) = (\xi_1(t), \xi_2(t), \dots, \xi_n(t))$, is square-Gaussian stochastic process.

Some properties of square-Gaussian stochastic processes can be found in papers [3,4].

Under $N(u)$ we denote the least numbers of closed balls of radius u covering the set $\mathbf{T}$ with respect to metric ρ.

Let $X = \{X(t,\, t \in \mathbf{T})\}$ be a square-Gaussian process. A function $\sigma(h)$, $h > 0$, is monotonically increasing continuous $\sigma(h) \to 0$ as $h \to 0$ and

$$\sup_{\rho(t,s)\le h} (\mathbf{Var}\,(X(t) - X(s)))^{\frac{1}{2}} \le \sigma(h).$$

Define

$$\varepsilon_0 = \inf_{t\in\mathbf{T}} \sup_{s\in\mathbf{T}} \rho(t,s), \qquad t_0 = \sigma(\varepsilon_0),$$

$$\gamma_0 = \sup_{t\in\mathbf{T}} (\mathbf{Var}\, X(t))^{1/2},$$

Under $\sigma^{(-1)}(h)$ we will understand a general inverse function to the function $\sigma(h)$.

Theorem 1.[3] *Let $X(t) = \{X(t), t \in \boldsymbol{T}\}$ be separable square-Gaussian stochastic process. Consider increasing function $r(u) \ge 0, u \ge 1$ such that $r(u) \to \infty$ as $u \to \infty$ and the function $r(\exp\{t\})$ is convex. If the condition*

$$\int_0^{t_0} r(N(\sigma^{(-1)}(u)))du < \infty,$$

holds true then for all integer $M = 1, 2, \dots,$ $\quad 0 < p < 1$ and u such that

$$0 < u < \frac{1-p}{\sqrt{2}} \min\left\{\frac{1}{\gamma_0}, \frac{1}{t_0 p^{M-1}}\right\}, \tag{2}$$

the following inequality is satisfied

$$\mathbf{P}\left\{\sup_{t\in \boldsymbol{T}}|X(t)|>x\right\}\le W(p,x), \tag{3}$$

where

$$\begin{aligned} W(p,x) &= 2\left(R\Big(\frac{u\sqrt{2}\gamma_0}{1-p}\Big)\right)^{1-p}\cdot A(p) \\ &\times \Big(1-\frac{p^{M-1}u\sqrt{2}t_0}{1-p}\Big)^{-p/2}\exp\Big\{-\frac{p^M u\sqrt{2}t_0}{2(1-p)}-ux\Big\}, \end{aligned}$$

the function $R(s)=(1-|s|)^{-\frac{1}{2}}\exp\{-\frac{|s|}{2}\}$ *and*

$$A(p)=r^{(-1)}\left(\frac{1}{t_0p^M}\int_0^{t_0p^M} r(N(\sigma^{(-1)}(v)))\,dv\right).$$

3. An estimation of supremum distribution of square-Gaussian stochastic processes

We consider the space $\mathbf{T}=[0,T]^d$, $d\ge 1$, with respect to metric $\rho(t,s)=\max\limits_{1\le i\le d}|t_i-s_i|$. Let $X=\{X(t),\ t\in\mathbf{T}\}$ be square-Gaussian stochastic process.

In the case when $\sigma(h)=C\cdot h^\alpha$, $\alpha\in(0,1]$, where $C>0$, the constants ε_0 and t_0 are equal to

$$\varepsilon_0=\inf_{t\in\mathbf{T}}\sup_{s\in\mathbf{T}}\rho(t,s)=\frac{T}{2},\qquad t_0=\sigma(\varepsilon_0)=C\left(\frac{T}{2}\right)^\alpha.$$

The following theorem holds true.

Theorem 2. *Let* $X(t),t\in[0,T]^d$, *be separable square-Gaussian stochastic process and*

$$\sup_{\rho(t,s)\le h}(\mathbf{Var}\,(X(t)-X(s)))^{\frac{1}{2}}\le\sigma(h)=C\cdot h^\alpha,\qquad \alpha\in(0,1],C>0.$$

If for integer $M>1$, $x>0$

$$x>\frac{\sqrt{2}\gamma_0 Md}{\alpha}\max\Big\{1;\left(\Big(\frac{T}{2}\Big)^\alpha C\frac{1}{\gamma_0}\right)^{\frac{1}{M-1}}\Big\}, \tag{4}$$

then the next estimation holds

$$\begin{aligned} \mathbf{P}\Big\{\sup_{t\in\boldsymbol{T}}|X(t)|>x\Big\} &\le 2^{1+d}e^{\frac{(M+1)d}{\alpha}}\exp\Big\{-\frac{x}{\sqrt{2}\gamma_0}\Big\} \\ &\times \Big(\frac{\alpha x}{\sqrt{2}\gamma_0 Md}\Big)^{Md/\alpha}\Big(1+\frac{2x}{\sqrt{2}\gamma_0}\Big)^{1/2}, \end{aligned} \tag{5}$$

where $\gamma_0=\sup\limits_{t\in\mathbf{T}}(\mathbf{Var}\,X(t))^{\frac{1}{2}}$.

Proof. The proof of this theorem follows from theorem 1.

Notice that since $\sigma(h) = C \cdot h^\alpha$, then $\sigma^{(-1)}(h) = \left(\frac{h}{C}\right)^{1/\alpha}$.

It was proved in [2] that $N(u)$ on $[0,T]^d$ is satisfied the inequality

$$N(\sigma^{(-1)}(u)) \leq \left(\frac{T}{2\sigma^{(-1)}(u)} + 1\right)^d = \left(\frac{T}{2}\Big(\frac{C}{u}\Big)^{1/\alpha} + 1\right)^d.$$

Let's consider a function $r(u) = u^\beta - 1, \quad \beta \in (0, \frac{\alpha}{d})$ for which all conditions of theorem 1 are satisfied. It's easy to see that $r^{(-1)}(u) = (u+1)^{1/\beta}$.

Since $0 < p < 1$ and $t_0 = C\left(\frac{T}{2}\right)^\alpha$ then $\frac{T}{2}\left(\frac{C}{p^M t_0}\right)^{1/\alpha} > 1$. Hence, as $0 < u < t_0 p^M$

$$N(\sigma^{(-1)}(u)) \leq \left(T\Big(\frac{C}{u}\Big)^{1/\alpha}\right)^d.$$

We estimate now $A(p)$ in such way:

$$\begin{aligned} A(p) &= \Big(\frac{1}{t_0 p^M}\int_0^{t_0 p^M} [(\frac{T}{2}\Big(\frac{C}{u}\Big)^{1/\alpha} + 1\big)^{d\beta} - 1] du + 1\Big)^{1/\beta} \\ &\leq \Big(\frac{1}{t_0 p^M}\int_0^{t_0 p^M} \big[T\Big(\frac{C}{u}\Big)^{1/\alpha}\big]^{d\beta} du\Big)^{1/\beta} \\ &= 2^d\big(\frac{\alpha}{\alpha - d\beta}\big)^{1/\beta} p^{-Md/\alpha}. \end{aligned}$$

Let's find minimum of $A(p)$ with respect to β.

$$\min_{\beta \in (0, \frac{\alpha}{d})} \big(\frac{\alpha}{\alpha - d\beta}\big)^{1/\beta} = \lim_{\beta \to 0} \big(\frac{1}{1 - d\beta/\alpha}\big)^{1/\beta} = e^{d/\alpha}.$$

Therefore, from (3) of theorem 1 follows the inequality

$$\begin{aligned} W(p,x) &\leq 2^{1+d} e^{d/\alpha} p^{-Md/\alpha} \left(R\Big(\frac{u\sqrt{2}\gamma_0}{1-p}\Big)\right)^{1-p} \\ &\times \left[\Big(1 - \frac{p^{M-1} u\sqrt{2} t_0}{1-p}\Big)^{-\frac{1}{2}} \exp\Big\{-\frac{p^{M-1} u\sqrt{2} t_0}{2(1-p)}\Big\}\right]^p e^{-ux}. \qquad (6) \end{aligned}$$

We remind that increasing function $R(s)$ is equal to

$$R(s) = (1 - |s|)^{-1/2} \exp\{-\frac{|s|}{2}\}.$$

If $t_0 p^{M-1} < \gamma_0$ then from (6) follows that

$$W(p,x) \leq 2^{1+d} e^{d/\alpha} p^{-Md/\alpha} R\Big(\frac{u\sqrt{2}\gamma_0}{1-p}\Big) \exp\{-ux\}.$$

The minimum of right side with respect to u is reached at the point

$$u_{min} = \frac{1}{z} - \frac{1}{z + 2x}, \quad \text{where } z = \frac{\sqrt{2}\gamma_0}{1-p}.$$

The point u_{min} is satisfied condition (2).

If substitute u_{min} in $W(p,x)$ we obtain

$$W(p,x) \le 2^{1+d} e^{d/\alpha} p^{-Md/\alpha} \exp\left\{-\frac{x(1-p)}{\sqrt{2}\gamma_0}\right\} \left(1+\frac{2x(1-p)}{\sqrt{2}\gamma_0}\right)^{1/2}.$$

Since $0<p<1$ then

$$W(p,x) \le 2^{1+d} e^{d/\alpha} p^{-Md/\alpha} \exp\left\{-\frac{x(1-p)}{\sqrt{2}\gamma_0}\right\} \left(1+\frac{2x}{\sqrt{2}\gamma_0}\right)^{1/2}.$$

Let's find minimum of right expression with respect to p. We obtain

$$p = \frac{\sqrt{2}\gamma_0 Md}{x\alpha}.$$

From $p\in(0,1)$ follows that

$$x > \frac{\sqrt{2}\gamma_0 Md}{\alpha},$$

and this is holds true from conditions of theorem. Then

$$\begin{aligned}\mathbf{P}\left\{\sup_{t\in\mathbf{T}} |X(t)| > x\right\} &\le 2^{1+d} e^{d/\alpha} \exp\left\{-\frac{x}{\sqrt{2}\gamma_0} + \frac{Md}{\alpha}\right\} \\ &\times \left(\frac{\alpha x}{\sqrt{2}\gamma_0 Md}\right)^{\frac{Md}{\alpha}} \left(1+\frac{2x}{\sqrt{2}\gamma_0}\right)^{1/2}.\end{aligned}$$

In the proof of the theorem we suppose that $t_0 p^{M-1} < \gamma_0$, but it follows from relationship (4). Then the theorem is proved.

In the case when $\mathbf{T}=[0,T]$ the following corollary is carried out.

Corollary 1. *Let $X(t), t\in[0,T]$, be separable square-Gaussian stochastic process for which*

$$\sup_{\rho(t,s)\le h} (\mathbf{Var}\,(X(t)-X(s)))^{\frac{1}{2}} \le \sigma(h) = C\cdot h^{\alpha}, \qquad \alpha\in(0,1], C>0.$$

If for integer $M>1$ and $x>0$

$$x > \frac{\sqrt{2}\gamma_0 M}{\alpha} \max\Big\{1; \left((\frac{T}{2})^{\alpha} C \frac{1}{\gamma_0}\right)^{\frac{1}{M-1}}\Big\},$$

then

$$\begin{aligned}\mathbf{P}\Big\{\sup_{t\in \boldsymbol{T}} |X(t)| > x\Big\} &\le 4e^{\frac{M+1}{\alpha}} \exp\Big\{-\frac{x}{\sqrt{2}\gamma_0}\Big\} \\ &\times \left(\frac{\alpha x}{\sqrt{2}\gamma_0 M}\right)^{M/\alpha} \left(1+\frac{2x}{\sqrt{2}\gamma_0}\right)^{1/2},\end{aligned}$$

where $\gamma_0 = \sup_{t\in[0,T]} (\mathbf{Var}\,X(t))^{\frac{1}{2}}$.

4. Accuracy and reliability of simulation of Gaussian stochastic processes which can be represented in the form of series

Consider the same space $\mathbf{T} = [0, T]^d$, $d > 1$ with metric $\rho(t, s) = \max_{1\le i\le d} |t_i - s_i|$, where t, s are vectors from $\mathbf{T}$. Let $\xi = \{\xi(t), t \in \mathbf{T}\}$ be centered Gaussian stochastic process and

$$\xi(t) = \sum_{n=0}^{\infty} \xi_n f_n(t), \tag{7}$$

where the functions $f_n(t)$, $n \ge 0$, are continuous and such that for all $t \in \mathbf{T}$

$$\sum_{n=0}^{\infty} f_n^2(t) < \infty,$$

ξ_n, $n = 0, 1, 2, \ldots$, are independent Gaussian random variables, $\mathbf{E}\xi_n = 0$, $E\xi_n^2 = 1$. Since

$$\mathbf{E}\xi^2(t) = \sum_{n=0}^{\infty} f_n^2(t) < \infty,$$

then the series $\sum_{n=0}^{\infty} \xi_n f_n(t)$ converges with probability one (see, for example, [8]).

Consider such situation: Let Σ be some system(filter, device), which is intended for transformation of signals (functions) $f_n(t)$. The function which has to be transformed is called input function on system; the transformed function is called output function or reaction on input function. Under $g_n(t)$ we will define output function. More information about filter can be found in [9].

Remark 1. In particular case $g_n(t) = z_n \cdot f_n(t)$. It means that transformation doesn't change the shape of signal.

In particular case can be also considered the situation when $g_n(t) = f_n'(t)$.

If input process on the system Σ is $\xi(t) = \sum_{n=0}^{\infty} \xi_n f_n(t)$, then output process is $\eta(t) = \sum_{n=0}^{\infty} \xi_n g_n(t)$. Suppose that for all $t \in \mathbf{T}$ the series $\sum_{n=0}^{\infty} g_n^2(t)$ converges. i's sufficient condition for convergence with probability one of the series $\eta(t) = \sum_{n=0}^{\infty} \xi_n g_n(t)$.

Definition 3. The process $\tilde{\xi}_N(t)$ is called the model of the process $\xi(t)$, $t \in \mathbf{T}$ if

$$\tilde{\xi}_N(t) = \sum_{k=0}^{N} \xi_k f_k(t), \quad t \in \mathbf{T}.$$

Let's define the difference between the process and the model under

$$\xi_N(t) = \xi(t) - \tilde{\xi}_N(t) = \sum_{k=N+1}^{\infty} \xi_k f_k(t), \quad t \in \mathbf{T}.$$

In the same way $\eta_N(t)$ can be defined:

$$\eta_N(t) = \sum_{k=N+1}^{\infty} \xi_k g_k(t), \quad t \in \mathbf{T}.$$

We'll investigate conditions under which the model $\tilde{\xi}_N(t)$ approximates $\xi(t)$ with given accuracy and reliability in Banach space $C([0,T]^d)$ taking into account the process $\eta(t)$. For this purpose the relationship $\xi^2(t)+\eta^2(t)$ can be analyzed. If generalize this case we can consider a semi-positive quadratic form

$$X(x,y) = a \cdot x^2 + 2c \cdot x \cdot y + b \cdot y^2,$$

where a, b, c are such that $a > 0$, $ab - c^2 > 0$.

For convenience,under $X_N(t)$ we'll define a quadratic form which is defined on the processes $\xi_N(t)$, $\eta_N(t)$:

$$X_N(t) = X(\xi_N(t), \eta_N(t)) = a \cdot (\xi_N(t))^2 + 2c \cdot \xi_N(t) \cdot \eta_N(t) + b \cdot (\eta_N(t))^2.$$

Stochastic process $X_N(t)$ is equal to

$$X_N(t) = \sum_{k=N+1}^{\infty} \sum_{n=N+1}^{\infty} \xi_k \xi_n \phi_{kn}(t), \tag{8}$$

where

$$\phi_{kn}(t) = a f_k(t) f_n(t) + c(f_k(t) g_n(t) + g_k(t) f_n(t)) + b g_k(t) g_n(t). \tag{9}$$

the function $\phi_{kn}(t)$ is symmetric with respect to k and n. Hence, $\phi_{kn}(t) = \phi_{nk}(t)$.

Denote

$$\bar{X}_N(t) = X_N(t) - \mathbf{E} X_N(t).$$

Definition 4. The model $\tilde{\xi}_N(t)$ approximates stochastic process $\xi(t)$ on input of the system, taking into account output process, with given reliability $1-\nu$, $\nu \in (0,1)$ and accuracy $\delta > 0$ in Banach space $C([0,T]^d)$, if

$$\mathbf{P}\left\{ \sup_{t \in \mathbf{T}} |\bar{X}_N(t)| > \delta \right\} \le \nu.$$

Notice that $X_N(t) - \mathbf{E} X_N(t) = \bar{X}_N(t)$, $t \in [0,T]^d$, is square-Gaussian stochastic process.

The next additional assertion is proved.

Lemma 1. *Let the series* $\sum_{k,n=N+1}^{\infty} \phi_{kn}^2(t)$ *be convergent for any* $t \in \mathbf{T}$. *Define*

$$\Delta_{kn}(t,s) = \phi_{kn}(t) - \phi_{kn}(s).$$

Then for the processes $\bar{X}_N(t)$, $X_N(t)$ *the following relationships hold true*

$$\mathbf{E}X_N(t) = \sum_{k=N+1}^{\infty} \phi_{kk}(t),$$

$$\mathbf{Var}\,\bar{X}_N(t) = 2\sum_{k=N+1}^{\infty}\sum_{n=N+1}^{\infty} \phi_{kn}^2(t), \tag{10}$$

$$\mathbf{Var}\,(X_N(t) - X_N(s)) = 2\sum_{k=N+1}^{\infty}\sum_{n=N+1}^{\infty} \Delta_{kn}^2(t,s), \tag{11}$$

where the functions $\phi_{kn}(t)$ *are from* (9).

Proof. From (8) it follows that

$$X_N(t) = \sum_{k=N+1}^{\infty}\sum_{n=N+1}^{\infty} \phi_{kn}(t)\xi_k\xi_n,$$

ξ_k, $k \geq 0$, are independent Gaussian random variables with zero expectation and variance 1. Then

$$\mathbf{E}X_N(t) = \sum_{k=N+1}^{\infty} \phi_{kk}(t).$$

Find now $\mathbf{E}(X_N(t))^2$.

$$\mathbf{E}(X_N(t))^2 = \sum_{k=N+1}^{\infty}\sum_{n=N+1}^{\infty}\sum_{k'=N+1}^{\infty}\sum_{n'=N+1}^{\infty} \phi_{kn}(t)\phi_{k'n'}(t)\mathbf{E}\xi_k\xi_n\xi_{k'}\xi_{n'}$$

From equality

$$\mathbf{E}\xi_k\xi_n\xi_{k'}\xi_{n'} = \mathbf{E}\xi_k\xi_n\mathbf{E}\xi_{k'}\xi_{n'} + \mathbf{E}\xi_k\xi_{k'}\mathbf{E}\xi_n\xi_{n'} + \mathbf{E}\xi_k\xi_{n'}\mathbf{E}\xi_{k'}\xi_n,$$

and from $\phi_{kn}(t) = \phi_{nk}(t)$ follows that

$$\mathbf{E}(X_N(t))^2 = \sum_{k=N+1}^{\infty}\sum_{n=N+1}^{\infty} \left(\phi_{kk}(t)\phi_{nn}(t) + 2\phi_{kn}^2(t)\right).$$

The relationship (10) we obtain from $\mathbf{Var}\,X_N(t) = \mathbf{Var}\,(\bar{X}_N(t))$.

Let's find $\mathbf{E}X_N(t)X_N(s)$.

$$\mathbf{E}X_N(t)X_N(s) = \sum_{k=N+1}^{\infty}\sum_{n=N+1}^{\infty} \left(\phi_{kk}(t)\phi_{nn}(s) + 2\phi_{kn}(t)\phi_{kn}(s)\right).$$

The formula (11) is obtained from

$$\begin{aligned}\mathbf{Var}\,(\bar{X}_N(t)-\bar{X}_N(s)) &= \mathbf{Var}\,\bar{X}_N(t)+\mathbf{Var}\,\bar{X}_N(s)\\ &- 2\mathbf{E}X_N(t)X_N(s)+2\mathbf{E}X_N(t)\mathbf{E}X_N(s).\end{aligned}$$

The lemma is proved.

Denote $s_{kn}=\sup_{t\in\mathbf{T}}|\phi_{kn}(t)|$, then

$$\sup_{t\in\mathbf{T}}(\mathbf{Var}\,\bar{X}_N(t))^{\frac{1}{2}}\leq\left(2\sum_{k=N+1}^{\infty}\sum_{n=N+1}^{\infty}s_{kn}^2\right)^{\frac{1}{2}}. \tag{12}$$

The following theorem holds true.

Theorem 3. *Let $\xi(t),t\in[0,T]^d$, be separable Gaussian stochastic process for which*

$$\sup_{\rho(t,s)\leq h}|\phi_{kn}(t)-\phi_{kn}(s)|\leq d_{kn}h^{\alpha},\qquad \alpha\in(0,1], \tag{13}$$

and

$$2\sum_{k=N+1}^{\infty}\sum_{n=N+1}^{\infty}d_{kn}^2=C^2(N)<\infty,$$

where$\phi_{kn}(t)$ are from (9).

The model $\tilde{\xi}_N(t)$ approximates separable Gaussian process $\xi(t)$, taking into accout output process, with given accuracy $\delta>0$ and reliability $1-\nu$, $\nu\in(0,1)$, if for $N\geq1$ the conditions are fulfilled

$$\delta>\frac{\sqrt{2}\gamma_0(N)Md}{\alpha}\max\Big\{1;\left(\Big(\frac{T}{2}\Big)^{\alpha}\frac{C(N)}{\gamma_0(N)}\right)^{\frac{1}{M-1}}\Big\},$$

$$2^{1+d}e^{\frac{(M+1)d}{\alpha}}\exp\Big\{-\frac{\delta}{\sqrt{2}\gamma_0(N)}\Big\}\Big(\frac{\alpha\delta}{\sqrt{2}\gamma_0(N)Md}\Big)^{Md/\alpha}\Big(1+\frac{2\delta}{\sqrt{2}\gamma_0(N)}\Big)^{1/2}<\nu,$$

where $M>1$ is arbitrary integer number, $\gamma_0(N)=\left(2\sum_{k=N+1}^{\infty}\sum_{n=N+1}^{\infty}s_{kn}^2\right)^{\frac{1}{2}}$.

Proof. Since the process $\bar{X}_N(t)$ is square-Gaussian then it can be used the results of theorem 2. From theorem condition (13) and from (11) follows that

$$\sup_{\rho(t,s)\le h}\left(\mathbf{Var}\left(X(t)-X(s)\right)\right)^{\frac{1}{2}} = \sup_{\rho(t,s)\le h}\left(2\sum_{k=N+1}^{\infty}\sum_{n=N+1}^{\infty}\left(\phi_{kn}(t)-\phi_{kn}(s)\right)^2\right)^{\frac{1}{2}}$$
$$\le \left(2\sum_{k=N+1}^{\infty}\sum_{n=N+1}^{\infty} d_{kn}^2\right)^{\frac{1}{2}} h^{\alpha}$$
$$= C(N)h^{\alpha} = \sigma_N(h), \quad \alpha\in(0,1].$$

From (12) follows that

$$\sup_{t\in\mathbf{T}}(\mathbf{Var}\,\bar{X}_N(t))^{\frac{1}{2}} \le \left(2\sum_{k=N+1}^{\infty}\sum_{n=N+1}^{\infty} d_{kn}^2\right)^{\frac{1}{2}} = \gamma_0(N).$$

If we substitute the obtained relationships in inequalities from theorem 2, the theorem will be proved.

In particular case $\mathbf{T}=[0,T]$ the following corollary holds true.

Corollary 2. *Let $\xi(t), t\in[0,T]$, be separable Gaussian stochastic process for which*

$$\sup_{|t-s|\le h}|\phi_{kn}(t)-\phi_{kn}(s)| \le d_{kn}h^{\alpha}, \qquad \alpha\in(0,1],$$

and

$$2\sum_{k=N+1}^{\infty}\sum_{n=N+1}^{\infty} d_{kn}^2 = C^2(N) < \infty,$$

where $\phi_{kn}(t)$ are from (9).

The model $\tilde{\xi}_N(t)$ approximates separable Gaussian process $\xi(t)$, taking into account output process, with given accuracy $\delta>0$ and reliability $1-\nu$, $\nu\in(0,1)$, if for N the next inequalities are satisfied

$$\delta > \frac{\sqrt{2}\gamma_0(N)M}{\alpha}\max\Big\{1; \Big(\big(\frac{T}{2}\big)^{\alpha}\frac{C(N)}{\gamma_0(N)}\Big)^{\frac{1}{M-1}}\Big\},$$

$$4e^{\frac{(M+1)}{\alpha}}\exp\Big\{-\frac{\delta}{\sqrt{2}\gamma_0(N)}\Big\}\Big(\frac{\alpha\delta}{\sqrt{2}\gamma_0(N)M}\Big)^{M/\alpha}\Big(1+\frac{2\delta}{\sqrt{2}\gamma_0(N)}\Big)^{1/2} < \nu,$$

where $M>1$ is arbitrary integer number, $\gamma_0(N) = \left(2\sum\limits_{k=N+1}^{\infty}\sum\limits_{n=N+1}^{\infty} s_{kn}^2\right)^{\frac{1}{2}}$.

Example 2. Let $\xi=\{\xi(t), t\in[0,T]\}$, be centered Gaussian process which can be represented in the form (7), where the functions $f_n(t)$, $n\ge 0$, are continuously differentiable and for all $t \in [0,T]$ $\sum\limits_{n=0}^{\infty}(f_n'(t))^2 < \infty$ and $\sum\limits_{n=0}^{\infty}|f_n'(t)| < \infty$.

Consider the case when output process is equal to $\eta(t) = \xi'(t), t \in [0,T]$. There exists derivative of stochastic process $\xi'(t) = \sum\limits_{n=0}^{\infty} f_n'(t)\xi_n$ in mean square.

The difference between the process and the model is

$$\xi_N(t) = \xi(t) - \tilde{\xi}_N(t) = \sum_{k=N+1}^{\infty} \xi_k f_k(t), \quad t \in \mathbf{T}$$

and the process $\eta_N(t)$ is equal to

$$\eta_N(t) = \sum_{k=N+1}^{\infty} \xi_k f_k'(t), \quad t \in \mathbf{T}.$$

Let's construct a semi-positive quadratic form $X_N(t)$, which is defined on the processes $\xi_N(t)$, $\eta_N(t)$

$$X_N(t) = X(\xi_N(t), \eta_N(t)) = a \cdot (\xi_N(t))^2 + 2c \cdot \xi_N(t) \cdot \eta_N(t) + b \cdot (\eta_N(t))^2.$$

The process $X_N(t)$ can be represented in the form

$$X_N(t) = \sum_{k=N+1}^{\infty} \sum_{n=N+1}^{\infty} \xi_k \xi_n \phi_{kn}(t), \tag{14}$$

where

$$\phi_{kn}(t) = af_k(t)f_n(t) + c(f_k(t)f_n'(t) + f_k'(t)f_n(t)) + bf_k'(t)f_n'(t). \tag{15}$$

Then it can be used corollary 2 for stochastic process (14) which gives the conditions under which the model approximates separable Gaussian process, taking into account its derivative, with given accuracy and reliability. It's shown in the next theorem.

Theorem 4. *Let $\xi(t), t \in [0,T]$, be separable Gaussian stochastic process for which*

$$\sup_{|t-s|\le h} |\phi_{kn}(t) - \phi_{kn}(s)| \le d_{kn} h^{\alpha}, \qquad \alpha \in (0,1], \tag{16}$$

and

$$2 \sum_{k=N+1}^{\infty} \sum_{n=N+1}^{\infty} d_{kn}^2 = C^2(N) < \infty,$$

where $\phi_{kn}(t)$ are from (15).

The model $\tilde{\xi}_N(t)$ approximates separable Gaussian process $\xi(t)$, taking into account output process, with given accuracy $\delta > 0$ and reliability $1 - \nu$, $\nu \in (0,1)$, if for N the next inequalities are satisfied

$$\delta > \frac{\sqrt{2}\gamma_0(N)M}{\alpha} \max\Big\{1; \Big(\big(\frac{T}{2}\big)^{\alpha} \frac{C(N)}{\gamma_0(N)} \Big)^{\frac{1}{M-1}} \Big\},$$

$$4e^{\frac{(M+1)}{\alpha}} \exp\Big\{ -\frac{\delta}{\sqrt{2}\gamma_0(N)} \Big\} \Big(\frac{\alpha\delta}{\sqrt{2}\gamma_0(N)M} \Big)^{M/\alpha} \Big(1 + \frac{2\delta}{\sqrt{2}\gamma_0(N)} \Big)^{1/2} < \nu,$$

where $M > 1$is arbitrary integer number, $\gamma_0(N) = \left(2 \sum_{k=N+1}^{\infty} \sum_{n=N+1}^{\infty} s_{kn}^2 \right)^{\frac{1}{2}}$.

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Department of Probability Theory and Mathematical Statistics, Kyiv National Taras Shevchenko University, Kyiv, Ukraine

E-mail address: yvk@univ.kiev.ua

Department of Applied Statistics, Faculty of Cybernetics, Kyiv National Taras Shevchenko University, Kyiv, Ukraine

E-mail address: irozora@bigmir.net