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ON THE CHARACTERIZATION OF PREMIUM PRINCIPLE WITH RESPECT TO POINTWISE COMONOTONICITY[1]

A premium principle is an economic decision rule used by the insurer in order to determine the amount of the net premium for each risk in his portfolio. In this paper we investigate the problem how to determine the premium principle to be used. In Goovaerts & Dhaene (1997), DTEW Research Report 9740, K.U.Leuven, we can see some desirable properties of a premium principle. We consider a premium principle for risks of any sign, and prove a representation of premium principle without some property which involves the distribution of a risk. Later we introduce this property as a corollary.

1. Introduction

Insurance contract can be seen as a risk-exchange between two parties, the insurer and the policyholder. The insurer promises to pay for the financial consequences of the claims produced by the insured risk. In return for this coverage, the policyholder pays a fixed amount, called the premium. Observe that the payments made by the insurer are random, while the payments made by the policyholder are non-random.

The pure premium of the insured risk is defined as the expected value of the claim amounts to be paid by the insurer. In practice the insurer will add a risk loading to this pure premium. The sum of the pure premium and the risk loading is called the net premium. Adding acquisition and administration costs to this premium, one gets the gross premium.

In this paper we will investigate the problem of determining the net premium. We will assume that the insurer adopts some economic decision rule to determine the amount of the net premium for each risk in his portfolio. Such a principle is called a premium principle, see e.g. Kaas et al. (2001). Several premium principles have been presented in the actuarial literature. Wang (1996) introduced a new class of premium principles which, roughly speaking, compute the net premium as the expectation of the risk under an adjusted measure.

In Goovaerts & Dhaene (1997) we can see some desirable properties for a premium principle. They assume that the net premium is not lower than

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the pure premium. We consider premium principle without this property, and than introduce premium principle for risks of any sign. Also we show that Greco's Representation Theorem, see Dennenberg (1996), do not follow from our statements, and vice versa.

2. Basic definitions for non-negative risks

We fix a probability space $(\Omega, \mathcal{F}, P)$. A risk is a non-negative real-valued random variable.

Definition 1. *Two risks X and Y are 1-comonotonic if the set*

$$A_{XY} := \{(X(\omega), Y(\omega)) : \omega \in \Omega\}$$

is comonotonic in $\mathbb{R}^2$.

Let Γ be a set of risks with such properties:

a) $X \in \Gamma, d \geq 0 \Rightarrow \min(X, d) \in \Gamma, (X + d) \in \Gamma$, and $dX \in \Gamma$;

b) $X \in \Gamma, X(\Omega) \subset [0, b], b > 0 \Rightarrow$

for $X_n := \begin{cases} \frac{i}{2^n}b, & \text{if } \frac{i}{2^n}b < X \leq \frac{i+1}{2^n}b, i = \overline{0, 2^n - 1} \\ 0, & \text{if } X = 0 \end{cases}, n = 0, 1, 2, ...,$

holds: $X_n \in \Gamma$.

c) $A \in \mathcal{F} \Rightarrow I_A \in \Gamma$.

Hereafter

$$I_A = I_A(\omega) := \begin{cases} 1, & \text{if } \omega \in A \\ 0, & \text{if } \omega \in \Omega \backslash A, \end{cases}$$

and

$$X(\Omega) := \{X(\omega) : \omega \in \Omega\}.$$

Definition 2. *A premium principle is a functional $H : \Gamma \to [0, \infty]$. For $X \in \Gamma, H(X)$ is called the premium.*

Remark 1. For risks with the same distribution, the premiums can be different.

3. Properties of a premium principle

Property 1 $X, Y \in \Gamma, X \leq Y \Rightarrow H(X) \leq H(Y)$.

Property 2 If $X, Y \in \Gamma$, $X + Y \in \Gamma$, and X, Y are 1-comonotonic, then $H(X + Y) = H(X) + H(Y)$.

Property 3 $H(1) = 1$.

Property 4 $X \in \Gamma \Rightarrow \lim\limits_{d \to +\infty} H[\min(X, d)] = H(X)$.

Remark 2. Properties 1-3 imply that $H(aX + b) = aH(X) + b$, for all $a, b \geq 0$.

4. Layers and distortions

Definition 3. *Let $0 \leq a < b$. A layer at (a, b) of a risk X is defined by*

$$L_{(a,b)} = \begin{cases} 0, & \text{if } 0 \leq X \leq a \\ X - a, & \text{if } a < X < b \\ b - a, & \text{if } X \geq b. \end{cases}$$

Definition 4. *A function $g : \mathcal{F} \to [0,1]$ is called a distortion function if :*

a) $g(\emptyset) = 0$, $g(\Omega) = 1$,

b) $A, B \in \mathcal{F}$, $A \subset B \Rightarrow g(A) \le g(B)$.

Lemma 1. *Let H be a premium principle satisfying Property 1-3, and*

$$g(A) := H(I_A), A \in \mathcal{F}.$$

Then g is a distortion function.

Proof.

$$g(\emptyset) = H(I_\emptyset) = H(0) = 0,$$

see Property 2.

$$g(\Omega) = H(I_\Omega) = H(1) = 1,$$

see Property 3.

$A, B \in \mathcal{F}, A \subset B \Rightarrow I_A \le I_B \Rightarrow g(A) \le g(B)$, see Property 1.

5. Characterization of premium principle

Let $Q(\omega)$ be a certain property of an elementary event $\omega \in \Omega$, which can be satisfied or not. For a distortion function g we write for brevity $g\{Q\} = g\{\omega \in \Omega : Q(\omega)\}$ holds. E.g., for a risk X and $x > 0$ we write

$$g\{X > x\} = g(\{\omega \in \Omega : X(\omega) > x\}).$$

Lemma 2. *Let H be a premium principle with Properties 1-3. Then there exists a unique distortion function g, such that for all discrete risky $X \in \Gamma$ with only finitely many mass points, we have that*

$$H(X) = \int_0^\infty g\{X > x\}\, dx. \tag{1}$$

Proof. Let X be discrete risk with finitely many mass points. Then for certain $n \ge 0$,

$$X = \sum_0^n x_i I_{A_i},$$

where $0 = x_0 < x_1 < ... < x_n, A_i \in \mathcal{F}, \{A_i\}$ form a partition of Ω. Then (we consider the case $n \ge 1$ only)

$$L_{(x_0,x_1)} = (x_1 - x_0) I_{\overline{A_0}},$$

$$L_{(x_1,x_2)} = (x_2 - x_1) I_{\overline{A_0 \cup A_1}},$$

$$\dots$$

$$L_{(x_{n-1},x_n)} = (x_n - x_{n-1}) I_{\overline{A_0 \cup \ldots \cup A_{n-1}}}.$$

We have

$$X = \sum_{i=0}^{n-1} L_{(x_i,x_{i+1})},$$

and the layers $L_{(x_i,x_{i+1})}$, $i=\overline{0,n-1}$, are pairwise mutually 1-comonotonic risks. Then by Property 2

$$H(X)=\sum_{i=0}^{n-1} H\left(L_{(x_i,x_{i+1})}\right).$$

But

$$L_{(x_i,x_{i+1})}=(x_{i+1}-x_i)\, I_{\{X>x_i\}}.$$

Then by Remark 2,

$$H\left(L_{(x_i,x_{i+1})}\right)=(x_{i+1}-x_i)\, g\{X>x_i\},$$

where

$$g(A):=H(I_A),\, A\in\mathcal{F}.$$

Then

$$\begin{aligned} H(X) &= \sum_{i=0}^{n-1}(x_{i+1}-x_i)\, g\{X>x_i\}=\sum_{i=0}^{n-1}\int_{x_i}^{x_{i+1}} g\{X>x\}\, dx \\ &= \int_0^\infty g\{X>x\}\, dx, \end{aligned}$$

since

$$g(\emptyset)=H(0)=0$$

and

$$\{X>x\}=\emptyset$$

for $x\geq x_n$. By Lemma 1 g is a distortion function.

In representation (1) the function $g:\mathcal{F}\to[0,1]$ is unique, since for $A\in\mathcal{F}$ and any g satisfying (1),

$$H(I_A)=\int_0^\infty g(A)\, I_{[0,1)}(x)\, dx=g(A).$$

Theorem 1. *Assume that the premium principle H satisfies Properties 1-4. Then there exists a unique function $g:\mathcal{F}\to[0,1]$, such that for all risks $X\in\Gamma$ we have that*

$$H(X)=\int_0^\infty g\{X>x\}\, dx. \tag{1a}$$

Proof.

1° Let $X\in\Gamma, X(\Omega)\subset[0,b], b>0$. For $n\geq 0$, we use X_n defined in Section 1. We have by Lemma 2

$$H(X_n)=\int_0^b g\{X_n>x\}\, dx, \tag{2}$$

where g is a uniquely defined distortion. Now, $X_n\leq X_{n+1}\leq X$, and $\forall\omega\in\Omega: X_n(\omega)\to X(\omega)$, as $n\to\infty$. Therefore by Property 1,

$$H(X_n)\leq H(X_{n+1})\leq H(X).$$

Moreover

$$X \leq X_n + \frac{b}{2^n} \Rightarrow H(X) \leq H(X_n) + \frac{b}{2^n},$$

we used here Property 1 and Remark 2. Thus

$$\lim_{n\to\infty} H(X_n) = H(X). \tag{3}$$

Next,

$$\{X_n > x\} \uparrow \{X > x\},$$

then

$$g\{X_n > x\} \uparrow g_*(x), \ g_*(x) \leq g\{X > x\}. \tag{4}$$

We used here part b) of definition 4. Next, the function

$$g_*(x) := \lim_{n\to\infty} g\{X_n > x\}, x \geq 0,$$

is non-increasing, therefore it has at most countable number of discontinuous points.

Let x be a point of continuity of the function $g_*(x), x > 0$. Then we will show that $g_*(x) = g\{X > x\}$.

Indeed, for $0 < \varepsilon < x$, we have

$$\{X_n > x\} \subset \{X > x\} \subset \left\{X_n > x - \frac{b}{2^n}\right\} \subset \{X_n > x - \varepsilon\},$$

for $n \geq n_\varepsilon$. Therefore

$$g_*(x) \leq g\{X > x\} \leq g_*(x - \varepsilon).$$

But due to our assumption $g_*(x-\varepsilon) \to g_*(x)$, as $\varepsilon \to 0$, therefore $g_*(x) = g\{X > x\}$, and we proved this equality.

Now, return to (4). We have

$$\lim_{n\to\infty} g\{X_n > x\} = g\{X > x\}$$

for all $x \geq 0$ a.e. with respect to Lebesgue measure.

From this fact, (2) and (3) we obtain finally by the dominated convergence theorem:

$$H(X) = \int_0^b g\{X > x\}\, dx.$$

2° Now let X be an arbitrary risk from Γ. For any $d > 0$ we have by part 1 of the proof that

$$H[\min(X,d)] = \int_0^d g\{X > x\}\, dx.$$

The desired result follows now from Property 4.

6. Connection with Wang's class

Introduce stronger property than Property 1.
Property 1' $X, Y \in \Gamma$, $X \leq_{st} Y \Rightarrow H(X) \leq H(Y)$.
Hereafter $X \leq_{st} Y$ means that X is stochastically dominated by Y.
Denote

$$P(\mathcal{F}) = \{P(A) : A \in \mathcal{F}\}.$$

Corollary 1. *Assume that the premium principle H satisfies Properties 1' and 2 to 4. Then the function $g : \mathcal{F} \to [0,1]$ in Theorem 1 has representation*

$$g(A) = g_0(P(A)), A \in \mathcal{F}, \tag{5}$$

where $g_0 : P(\mathcal{F}) \to [0,1]$ is non-decreasing, with $g_0(0) = 0, g_0(1) = 1$.
Proof. Property 1' implies Property 1, therefore the statement of Theorem 1 holds true for the premium principle H, and the function $g : \mathcal{F} \to [0,1]$ in (1a) exists and unique.

Now let $P(A) = P(B)$. Then $I_A \leq_{st} I_B$ and $I_B \leq_{st} I_A$. By Property 1' we have $H(I_A) = H(I_B)$,therefore $g(A) = g(B)$. Thus representation (5) holds, and the properties of g_0 follow from distortion properties of g.

Consider stronger property than Property 2.
Property 2' If $X, Y \in \Gamma$, $X + Y \in \Gamma$, and X, Y are comonotonic (i.e. $\exists \Omega_0, P(\Omega_0) = 1 : A^0_{XY} := \{(X(\omega), Y(\omega)) : \omega \in \Omega_0\}$ is comonotonic in R^2), then $H(X+Y) = H(X) + H(Y)$.
Corollary 2. *Assume that the premium principle H satisfies Property 1' and 2 to 4. Then it satisfies Property 2' as well, and for all risks $X \in \Gamma$ we have*

$$H(X) = \int_0^\infty g[S_X(x)]\, dx, \tag{6}$$

where $S_X(x) := P\{X > x\}, g : P(\mathcal{F}) \to [0,1]$ is non-decreasing, $g(0) = 0$, $g(1) = 1$. Moreover the function g in representation (6) is unique.

Representation (6) and uniqueness of g follow from Corollary 2. Property 2' then follows from (6), see Wang (1996).

7. Inverse statement

The inverse conclusion of Theorem 1 holds in the next theorem:
Theorem 2. *The premium principle $H : \Gamma \to [0, \infty]$ fulfills the Properties 1-4 if, and only if, there exists a distortion function $g : \mathcal{F} \to [0,1]$, for which:*

$$H(X) = \int_0^\infty g\{X > x\}\, dx, \textit{for any risk } X \in \Gamma. \tag{7}$$

Moreover $g(A) = H(I_A), A \in \mathcal{F}$.
Proof. If H fulfill Properties 1-4 then there exists a distortion function $g : \mathcal{F} \to [0,1]$, for which representation (7) is true, by Theorem 1.

Now let premium principle $H : \Gamma \to [0, \infty]$ is defined by

$$H(X) = \int_0^\infty g\{X > x\}\, dx$$

for some distortion function $g : \mathcal{F} \to [0, 1]$. Then H fulfills the Properties 1-4.

Indeed, the proofs of Properties 1,3 and 4 are straightforward. Now, we prove the key Property 2.

Let X, Y be 1-comonotonic risks from Γ. Then these exists a non-negative r.v. Z and two non-decreasing functions $f, h : [0, \infty) \to [0, \infty)$, such that $X = f(Z), Y = h(Z)$. We have to prove that

$$\int_0^\infty g\{(f+h)(Z) > x\}\, dx = \int_0^\infty g\{f(Z) > x\}\, dx + \int_0^\infty g\{h(Z) > x\}\, dx. \tag{8}$$

We do it in several steps.

1° It is enough to show (8) for bounded r.v. Z only. Indeed, for $d > 0$ define

$$Z_d = \min(d, Z).$$

Then $f(Z_d) = \min(f(d), f(Z)) \in \Gamma$, and $h(Z_d) \in \Gamma$ in a similar way. Suppose that (8) holds for each bounded risk. Then

$$\int_0^\infty g\{(f+h)(Z_d) > x\}\, dx = \int_0^\infty g\{f(Z_d) > x\}\, dx + \int_0^\infty g\{h(Z_d) > x\}\, dx,$$

or

$$\int_0^{(f+h)(d)} g\{(f+h)(Z) > x\}\, dx = \int_0^{f(d)} g\{f(Z) > x\}\, dx + \int_0^{h(d)} g\{h(Z) > x\}\, dx.$$

Tending $d \to \infty$, we immediately obtain (8).

2° Thus we suppose that Z if a bounded non-negative r.v. Assume that f and h are strictly increasing. Then $f + h$ is also strictly increasing. Consider

$$I(f) := \int_0^\infty g\{f(Z) > x\}\, dx. \tag{9}$$

Let $a = f(0), b = f(+\infty)$. Then

$$I(f) = f(0) + \int_a^b g\{f(Z) > x\}\, dx. \tag{10}$$

Let $0 \le t_1 < t_2 < \ldots$ be the set of all points of discontinuity of f (it is a finite or countable set).

Denote $x_i^- = f(t_i-), x_i^+ = f(t_i+), x_i = f(t_i)$.

If $t_1 = 0$ then $x_1^- := x_1 = f(0)$. We have from (10):

$$
\begin{aligned}
I(f) &= f(0) \\
&\quad + \sum_{i\geq 1} \left[\left(x_i - x_i^-\right) g\left\{f(Z) \geq x_i\right\} + \left(x_i^+ - x_i\right) g\left\{f(Z) > x_i\right\} \right] \\
&\quad + \int_{A_f} g\left\{f(Z) > x\right\} dx,
\end{aligned} \tag{11}
$$

where

$$
A_f := (a,b) \diagdown \bigcup_{i\geq 1} \left[x_i^-, x_i^+\right].
$$

Now we define a right-continuous modification of f,

$$
f_{rc}(t) = \begin{cases} f(t), & t \geq 0, t \neq t_i, i \geq 1 \\ f(t+), & t = t_i, i \geq 1. \end{cases}
$$

Denote

$$
I_c(f) = \int_{A_f} g\left\{f(Z) > x\right\} dx.
$$

Then

$$
I_c(f) = \int_{A_f} g\left\{f_{rc}(Z) > x\right\} dx = \int_{A_f} g\left\{Z > f_{rc}^{-1}(x)\right\} dx. \tag{12}
$$

Here f_{rc}^{-1} is inverse mapping for strictly increasing function f_{rc}. Change of variables in Lebesgue integral (12), $t = f_{rc}^{-1}(x)$, leads to the following representation

$$
I_c(f) = \int_{[0,\infty)\diagdown\{t_i, i\geq 1\}} g\left\{Z > t\right\} d\lambda_{f_{rc}}(t), \tag{13}
$$

where $\lambda_{f_{rc}}$ is Lebesgue-Stiltjes measure on Borel σ-field ß$([0,\infty))$,generated by the function f_{rc}. Then

$$
\begin{aligned}
I(f) &= f(0) + \sum_{i\geq 1} \left[\left(x_i - x_i^-\right) g\left\{Z \geq t_i\right\} + \left(x_i^+ - x_i\right) g\left\{Z > t_i\right\} \right] \\
&\quad + \int_{[0,\infty)\diagdown\{t_i, i\geq 1\}} g\left\{Z > t\right\} d\lambda_{f_{rc}}(t).
\end{aligned} \tag{14}
$$

But

$$
I_c(f) = \int_{[0,\infty)} g\left\{Z > t\right\} d\lambda_{f_{rc}}(t) - \sum_{i\geq 1} g\left\{Z > t_i\right\} \left(x_i^+ - x_i^-\right).
$$

We have

$$
\begin{aligned}
& \left(x_i - x_i^-\right) g\left\{Z \geq t_i\right\} + \left(x_i^+ - x_i\right) g\left\{Z > t_i\right\} - \left(x_i^+ - x_i^-\right) g\left\{Z > t_i\right\} \\
= \; & \left(x_i - x_i^-\right) \left(g\left\{Z \geq t_i\right\} - g\left\{Z > t_i\right\}\right).
\end{aligned}
$$

Finally

$$\begin{aligned} I(f) &= f(0) + \sum_{i\geq 1} \left(x_i - x_i^-\right)\left(g\{Z \geq t_i\} - g\{Z > t_i\}\right) \\ &\quad + \int_{[0,\infty)} g\{Z > t\}\, d\lambda_{f_{rc}}(t), \\ x_i &= f(t_i),\, x_i^- = f\left(t_i^-\right). \end{aligned} \tag{15}$$

Now we are able to show (8) for strictly increasing f and h. Let $0 \leq u_1 < u_2 < \ldots$ be the points of discontinuity of $f+h$. Then by (15)

$$\begin{aligned} I(f+h) &= (f+h)(0) + \sum_{i\geq 1} \left[(f+h)(u_i) - (f+h)\left(u_i^-\right)\right] \times \\ & \left(g\{Z \geq u_i\} - g\{Z > u_i\}\right) + \int_{[0,\infty)} g\{Z > t\}\, d\lambda_{f_{rc}+h_{rc}}(t). \end{aligned} \tag{16}$$

But $\lambda_{f_{rc}+h_{rc}} = \lambda_{f_{rc}} + \lambda_{h_{rc}}$,therefore (16) immediately implies

$$I(f+h) = I(f) + I(h),$$

since the set of discontinuity of $f+h$ includes both sets of discontinuity of f and h.

3° Now, Z is a bounded non-negative r.v., and $f, \dot{h}$ are non-decreasing. Let

$$f_n(t) = f(t) + \frac{t}{n},\ h_n(t) = h(t) + \frac{t}{n},\ t \in [0,\infty), n \geq 1.$$

Then f_n, h_n are strictly increasing, and by part 2°,

$$I(f_n + h_n) = I(f_n) + I(h_n).$$

We show first that

$$\lim_{n\to\infty} I(f_n) = I(f).$$

Since Z is bounded, all these integrals equal the corresponding intervals on $[0, b]$, with large enough b. Thus we have to show that

$$\lim_{n\to\infty} \int_0^b g\left\{f(Z) + \frac{Z}{n} > x\right\} dx = \int_0^b g\{f(Z) > x\}\, dx. \tag{17}$$

We have

$$\left\{f(Z) + \frac{Z}{n} > x\right\} \uparrow \{f(Z) > x\}.$$

Let

$$g_*(x) = \lim_{n\to\infty} g\{f_n(Z) > x\}.$$

The function g_* is non-increasing. And similarly to part 1° of Theorem 1 we have the following: if g_* is continuous at point x_0 then

$$g_*(x_0) = g\{f(Z) > x_0\}.$$

But g_* is continuous a.e. with respect to Lebesgue measure, since it is monotone. Then

$$\lim_{n\to\infty} g\left\{f_n(Z) > x\right\} = g\left\{f(Z) > x\right\},$$

$x \in [0.b]$, a.e. with respect to Lebesgue measure. Thus (17) holds from the dominated convergence theorem. By part 1° relation (8) follows now for any non-negative r.v. Z.

Example 1. Let $\{P_\alpha, \alpha \in I\}$ be a family of probability measures on $(\Omega, \mathcal{F})$.

Let $g_\alpha : [0,1] \to [0,1]$ be non-decreasing function, for which $g_\alpha(0) = 0$, and $g_\alpha(1) = 1$. Introduce two premium principles

$$H_1(X) := \int_0^\infty \sup_{\alpha\in I} g_\alpha\left(P_\alpha\left\{X > x\right\}\right) dx,$$

$$H_2(X) := \int_0^\infty \inf_{\alpha\in I} g_\alpha\left(P_\alpha\left\{X > x\right\}\right) dx; x \in \Gamma.$$

Then due to Theorem 2, both principles fulfill the Properties 1-4. We mention that in general these principles do not have the form (6), i.e. they depend not only on the distribution of X under the basic probability measure P, but on the events $\{\omega : X(\omega) > x\}, x > 0$, as well.

Example 2. Another example of this kind could be

$$H_3(X) := \int_I \left[\int_0^\infty g_\alpha\left(P_\alpha\left\{X > x\right\}\right) dx\right] d\mu(\alpha),$$

where μ is a probability measure on $(I, \mathcal{F}_I)$, where $\mathcal{F}_I$ is a σ-field on I.

In this case we have to demand that for any $X \in \Gamma$ the function $h(x,\alpha) := g_\alpha\left(P_\alpha\left\{X > x\right\}\right), \alpha \in I, x > 0$, is measurable with respect to the σ-field $\sigma(S \times \mathcal{F}_I)$, where S is Lebesgue σ-field on $(0, +\infty)$.

For both examples, Property 2' holds as well, if all the probabilities P_α are absolutely continuous w.r.t. P. But Property 1' need not hold for the examples.

8. Greco's Representation Theorem

Given a family Γ' of functions $X : \Omega \to \overline{\mathbb{R}}$ (here $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$) and a functional $H' : \Gamma' \to \overline{\mathbb{R}}$, we list the properties of Γ' and H' which play a role in Greco's Representation Theorem (GRT), stated in Dennenberg (1996).

For Γ' those are

a') $X \geq 0$ for all $X \in \Gamma'$;
b') $X \in \Gamma', d \geq 0 \Rightarrow \min(X,d) \in \Gamma', X - \min(X,d) \in \Gamma'$, and $dX \in \Gamma'$.

For H' the following conditions are relevant:

(i) $X, Y \in \Gamma', X \leq Y \Rightarrow H'(X) \leq H'(Y)$;
(ii) If $X, Y \in \Gamma'$, $X + Y \in \Gamma'$ and X, Y are 1-comonotonic, then $H'(X+Y) = H'(X) + H'(Y)$;
(iii) $X \in \Gamma', X \geq 0 \Rightarrow \lim_{d\searrow 0} H'[X - \min(X,d)] = H'(X)$

(iv) $X \in \Gamma' \Rightarrow \lim_{b\to+\infty} H'[\min(X,b)] = H'(X)$

Theorem 3(GRT). *Given a family Γ' of functions on Ω with properties a') and b') and given a functional $H' : \Gamma' \to \overline{\mathbb{R}}$ with properties (i)-(iv), then there exists a monotone set function $\gamma : 2^\Omega \to \overline{\mathbb{R}}$, such that for all $X \in \Gamma'$ we have that*

$$H'(X) = \int_0^\infty \gamma\{X > x\}dx. \tag{18}$$

Remark 3. In GRT $H'(X) \geq 0$, for all risks $X \in \Gamma'$. This follows from properties (i) and (ii).

Now we demonstrate with two examples, that properties of Γ in Definition 1 do not follow from properties of Γ', listed in GRT, and vice versa.

Example 3. *Let $\Omega = \{\omega\}$, $\mathcal{F} = 2^\Omega$ and Γ' consists of one risk $X = X(\omega) = 0 \in \Gamma'$, then Γ' satisfies properties a'),b') of GRT. But $I_{\{\omega\}} = I_\Omega = 1 \notin \Gamma'$, therefore Γ' does not satisfy properties a)-c), listed in Definition 1.*

Example 4. Let $\Omega = \{\omega_1, \omega_2, \omega_3\}$, $\mathcal{F} = 2^\Omega$. Introduce designation for risk $X = (X(\omega_1), X(\omega_2), X(\omega_3))$ – well-ordered vector of values of risk X in points $\omega_1, \omega_2, \omega_3$. First, $\forall A \in \mathcal{F}, I_A \in \Gamma$ therefore $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, $(0,0,1)$, $(1,1,0)$, $(1,0,1)$, $(0,1,1)$, $(1,1,1) \in \Gamma$. Then we get all risks $X \in \Gamma$, so that Γ fulfills properties a) and b), listed in Definition 1. In that way we get risks with one or two different values, because $\forall d \geq 0$ all risks: $\min(X,d)$, $(X+d)$, dX, X_n (from Definition 1) have less or equal different values then risk X. Later we add to Γ risk X_0 with three values, for example $X_0 = (x_1, x_2, x_3)$, where $0 < x_1 < x_2 < x_3 < \infty$ and those risks, which a necessary by properties a) and b) in Definition 1. It is easy to prove, that for some $d : x_1 < d < x_2$, risk $X_1 = (0, x_2 - d, x_3 - d)$ does not belong to Γ(use that we can get risk X_1 only from risks with three values, and that for getting $X(\omega_1) = 0$ we can use only $\min(X, 0)$ or $0 \cdot X$, which do not lead us to risk X_1). But by property b') from GRT we should have, that $X_1 \in \Gamma'$, because $X_1 = X_0 - \min(X_0, d)$. Thus Γ fulfills properties from Definition 1 and does not fulfill properties from GRT.

9. Premium of risk of any sign

We fix a probability space $(\Omega, \mathcal{F}, P)$, and in this section we call risk to be any real-valued r.v. with finite mean. Thus negative risks are allowed. Definition 1 is still valid.

Let Γ be a set of risks with such properties:

a1) $X \in \Gamma, d \geq 0 \Rightarrow (X+d) \in \Gamma, dX \in \Gamma$

a2) Let $d > 0$. If $X \in \Gamma, X \geq 0$ then $\min(X,d) \in \Gamma$; if $X \in \Gamma, X \leq 0$ then $\max(X, -d) \in \Gamma$.

b1) Coincides with b) from Section 2.

b2) $X \in \Gamma, \Gamma(\Omega) \subset [-b, 0], b > 0 \Rightarrow -(-X)_n \in \Gamma$,where $(-X)_n$ is defined for non-negative r.v. $(-X)$ as in property b) of Γ, see Section 2, where X is replaced to $(-X)$.

c1) $A \in \mathcal{F} \Rightarrow I_A, (-I_A) \in \Gamma$.

d1) $X \in \Gamma \Rightarrow X_+ \in \Gamma, (-X_-) \in \Gamma$.

Hereafter we use common notations $X_+ = \max(X, 0)$, $X_- = -\min(X, 0)$.

Definition 5. *A premium principle is a functional* $H : \Gamma \to (-\infty, +\infty]$.

We have to change Property 4.

Property 4'. If $X \in \Gamma, X \geq 0$ then

$$\lim_{d \to +\infty} H[\min(X, d)] = H(X),$$

and if $X \in \Gamma, X \leq 0$ then

$$\lim_{d \to +\infty} H[\max(X, -d)] = H(X).$$

Introduce condition on a distortion function:

Condition 1. $\forall X \in \Gamma : \quad \int_{-\infty}^{0} (1 - g\{X > x\})dx < \infty.$

Theorem 4. *The premium principle* $H : \Gamma \to (-\infty, +\infty]$ *fulfills the Properties 1 to 3, and 4' if, and only if, there exists a distortion functions* $g : \mathcal{F} \to [0, 1]$ *which satisfies Condition 1 and for all risks* $X \in \Gamma$,

$$H(X) = -\int_{-\infty}^{0} (1 - g\{X > x\})\, dx + \int_{0}^{\infty} g\{X > x\}\, dx. \tag{19}$$

Moreover $g(A) = H(I_A)$, $A \in \mathcal{F}$.

Proof.

1° Let H fulfills the Properties 1 to 3, and 4'. We prove the representation (19).

Let $X \in \Gamma$. Then $X = X_+ + (-X_-)$, and both X_+ and $(-X_-)$ belong to Γ. Moreover

$$X_+ = f_1(X), -X_- = f_2(X),$$

with non-decreasing functions f_1 and f_2. Therefore $X_+, (-X_-)$ are 1-comonotonic. Then by Property 2

$$H(X) = H(X_+) + H(-X_-).$$

Now, $X_+ \in \Gamma_+ := \{Y_+ : Y \in \Gamma\}$. The set of risks Γ_+ satisfies properties a) and c) from Section 2, due to the properties of Γ listed in this section. And H restricted to Γ_+ satisfies Properties 1 to 4. Therefore by Theorem 2 there exists a distortion $g : \mathcal{F} \to [0, 1]$, such that

$$H(X_+) = \int_{0}^{\infty} g\{X_+ > x\}\, dx,$$

But $\forall x > 0 : \quad \{X_+ > x\} = \{X > x\}$, therefore

$$H(X_+) = \int_{0}^{\infty} g\{X > x\}\, dx.$$

Moreover $g(A) = H(I_A), A \in \mathcal{F}$.

2° Next, consider the class of risks

$$\Gamma_- := \{Y_- : Y \in \Gamma\}.$$

Define

$$H_1(Y_-) := -H(-Y_-), Y_- \in \Gamma_-.$$

The class Γ_- satisfies properties a) to c) from Section 2. A premium principle $H_1 : \Gamma_- \to [0, +\infty)$ satisfies the following properties.

-Property 1 follows from Property 1 of H. Indeed, $X_- \le Y_- \Rightarrow H(-X_-) \ge H(-Y_-) \Rightarrow H_1(X_-) \le H_1(Y_-)$.

-Properties 2 and **3** follow from Properties 2 and 3 of H.

-Property 4 follows from the second part of Property 4' for H.

Then by the statement similar to Theorem 2 there exists a distortion h, such that

$$H_1(U) = \int_0^\infty h\{U > x\}\,dx = \int_0^\infty h\{U \ge x\}\,dx,$$

since $h\{U \ge x\} = h\{U > x\}$ a.e. with respect to Lebesgue measure, see the proof of Theorem 1. Moreover $h(A) = H_1(I_A), A \in \mathcal{F}$.

Then define $g_1(A) = 1 - h\left(\overline{A}\right), A \in \mathcal{F}$. It is a distortion as well, and

$$\begin{aligned} g_1(A) &= 1 - H_1(I_{\overline{A}}) = 1 + H(-I_{\overline{A}}) = H(1 - I_{\overline{A}}) \\ &= H(I_A) = g(A). \end{aligned}$$

Therefore $g_1 = g$, and

$$\begin{aligned} H_1(U) &= \int_0^\infty (1 - g\{U < x\})\,dx \\ &= \int_{-\infty}^0 (1 - g\{-U > x\})\,dx. \end{aligned}$$

Finally, for $X \in \Gamma$,

$$\begin{aligned} H(X) &= H(-X_-) + H(X_+) = -H_1(X_-) + H(X_+) \\ &= -\int_{-\infty}^0 (1 - g\{-X_- > x\})\,dx + H(X_+), \end{aligned}$$

$$H(X) = -\int_{-\infty}^0 (1 - g\{X > x\})\,dx + \int_0^\infty g\{X > x\}\,dx,$$

since for $x < 0$, $\{X > x\} = \{-X_- > x\}$.

3° Now, let $H : \Gamma \to (-\infty, +\infty]$ has representation (19), with a distortion function $g : \mathcal{F} \to [0, 1]$, satisfying Condition 1. Then equality $H(I_A) = g(A), A \in \mathcal{F}$, follows immediately, and the properties 1, 3 and 4' are easily verified.

Prove the Property 2 for H. The right hand side of (19) is defined for all risks, there fore we can assume now that H is defined by (19) for all risks, not only for the risks for Γ.

Let X, Y be two 1-comonotonic risks. First consider the case $X \geq -d$, $Y \geq -d$, with fixed positive d. Then since (19) implies $H(Y+c) = H(Y) + c, c \in \mathbb{R}$, for any risk Y, we have

$$H(X+Y) = H((X+d) + (Y+d)) - 2d.$$

But $X+d$ and $Y+d$ are non-negative 1-comonotomic risks. For any non-negative risk Y,

$$H(Y) = \int_0^\infty g\{Y > x\}\, dx =: G(Y).$$

But the functional G fulfills Property 2 for non-negative risks, due to Theorem 2. Then

$$\begin{aligned} H(X+Y) &= H(X+d) + H(Y+d) - 2d \\ &= H(X) + d + H(Y) + d - 2d \\ &= H(X) + H(Y), \end{aligned}$$

and we showed Property 2 for bounded from below risks.

Let now X, Y be arbitrary 1-comonotonic risks. Then $X = f(Z)$, $Y = h(Z)$, where Z is a r.v., and f, h are non-decreasing. We fix $c \in \mathbb{R}$ and for any r.v. U define $U_c = \max(U, c)$. Now, $f(Z_c)$ and $h(Z_c)$ are bounded from below 1-comonotonic risks, then

$$H(f(Z_c) + h(Z_c)) = H(f(Z_c)) + H(h(Z_c)),$$

$$H((f+h)(Z_c)) = H(f(Z_c)) + H(h(Z_c)). \tag{20}$$

Let $\alpha(Z)$ be a risk, with non-decreasing function $\alpha : \mathbb{R} \to \mathbb{R}$. Then $\alpha(Z_\alpha) = (\alpha(Z))_b$, with $b := \alpha(c)$. And due to the integral representation (19),

$$\lim_{c \to -\infty} H(\alpha(Z_c)) = H(\alpha(Z)).$$

Now, in (20) we tend $c \to -\infty$ and obtain

$$H((f+h)(Z)) = H(f(Z)) + H(h(Z)).$$

Now, instead of Property 1 we consider stronger Property 1'. Introduce condition on function $g_0 : P(\mathcal{F}) \to [0,1]$ much as Condition 1 on distortion function.

Condition 2. $\forall X \in \Gamma : \displaystyle\int_{-\infty}^{0} (1 - g_0(S_X(x)))dx < \infty.$

Corollary 3. *The premium principle* $H : \Gamma \to (-\infty, +\infty]$ *fulfills the Properties 1', 2, 3 and 4' if, and only if, these exists a function*

$$g_0 \colon P(\mathcal{F}) \to [0,1],$$

which satisfies Condition 2, $g_0(0) = 0$, $g_0(1) = 1$, g_0 *is non-decreasing, and such that for any risk* $X \in \Gamma$,

$$H(X) = -\int_{-\infty}^{0} (1 - g_0(S_X(x)))\, dx + \int_{0}^{\infty} g_0(S_X(x))\, dx, \tag{21}$$

where $S_X(x) := P\{X > x\}, x \in \mathbb{R}$. *Moreover* $g_0(q) = H(B_q)$, $q \in P(\mathcal{F})$, *where* B_q *is Bernoulli r.v. with parameter* q.

Proof. Let H satisfies the properties listed above. Then weaker Property 1 holds as well, and representation (19) holds, with

$$g(A) = H(I_A).$$

But $I_A \stackrel{d}{=} B_q$, with $q := P(A)$. Thus $g(A)$ depends only on $q = P(A)$, $g(A) = g_0(P(A))$, and (21) follows. The desired properties of g_0 follow from corresponding properties of g.

Next, let representation (21) holds for the principle H. Then (19) holds, with $g(A) := g_0(P(A))$, $A \in \mathcal{F}$. The distortion properties of g follow from the properties of g_0. Then by Theorem 4, H fulfills Properties 1 to 3, and 4'. The Property 1' follows now from Property 1, since $H(X)$ in (21) is determined by the distribution of X.

Remark 4. For the premium principle (19), defined for all risks X, we show the property:

$$H(X - d) = H(X) - d, \ d > 0, \tag{22}$$

which was used in part 3° of the proof of Theorem 4. We have

$$\begin{aligned}
H(X-d) &= -\int_{-\infty}^{0} (1 - g\{X > x + d\})\, dx + \int_{0}^{\infty} g\{X > x + d\}\, dx \\
&= -\int_{-\infty}^{d} (1 - g\{X > t\})\, dt + \int_{d}^{\infty} g\{X > t\}\, dt \\
&= -\int_{-\infty}^{0} (1 - g\{X > t\})\, dt + \int_{0}^{\infty} g\{X > t\}\, dt \\
&\quad -\int_{0}^{d} (1 - g\{X > t\}) - \int_{0}^{d} g\{X > t\}\, dt \\
&= H(X) - d.
\end{aligned}$$

Thus (22) is proven.

10. Corollary

Introduce an important property of a premium principle:

Property 5 $X \in \Gamma \Rightarrow H(X) \geq EX$.

This property means that the net premium is not lower than the pure premium.

Lemma 3. *Let H be a premium principle satisfying Property 1-3,5, and*

$$g(A) := H(I_A), A \in \mathcal{F}.$$

Then g is a distortion function, moreover $g(A) \geq P(A)$, for all $A \in \mathcal{F}$.
Proof. Distortion properties of g follow from Lemma 1.
Let $A \in \mathcal{F}$, then

$$g(A) = H(I_A) \geq EI_A = P(A).$$

Now we have the next corollary of Theorem 2.

Theorem 5. *The premium principle $H : \Gamma \to [0, \infty]$ fulfills the Properties 1-5 if, and only if, there exists a distortion function $g : \mathcal{F} \to [0, 1]$, which satisfies:*

(1) $H(X) = \int_0^\infty g\{X > x\}\, dx$, *for any risk* $X \in \Gamma$,
(2) $g(A) \geq P(A)$, *for any* $A \in \mathcal{F}$.

Moreover $g(A) = H(I_A)$, $A \in \mathcal{F}$.

Now return to Examples 1,2. Introduce property of $\{P_\alpha, \alpha \in I\}$:

$$P_\alpha(A) \geq C_\alpha P(A),$$

for all $\alpha \in \Gamma, A \in \mathcal{F}$.

Here $0 < C_\alpha \leq 1$. We mention that if $C_{\alpha_0} = 1$ then $P_{\alpha_0} = P$.

Let $g_\alpha : [0, 1] \to [0, 1]$ be non-decreasing function, for which

$$g_\alpha(0) = 0, \qquad g_\alpha(1) = 1,$$

and $g_\alpha(q) \geq \min(1, q/C_\alpha)$ for all $q \in [0, 1]$. Then due to Theorem 5, principles H_1, H_2, H_3 fulfill the Properties 1-5.

Now we have the following corollary of Theorem 4.

Theorem 6. *The premium principle $H : \Gamma \to (-\infty, +\infty]$ fulfills the Properties 1, 2, 3', 4', and 5 if, and only if, there exists a distortion functions $g : \mathcal{F} \to [0, 1]$ which satisfies:*

$$\text{(23)} \qquad a)\ H(X) = -\int_{-\infty}^0 (1 - g\{X > x\})\, dx + \int_0^\infty g\{X > x\}\, dx,$$

for any risk $X \in \Gamma$,

$$\text{(24)} \qquad b)\ g(A) \geq P(A),$$

for any $A \in \mathcal{F}$. *Moreover* $g(A) = H(I_A)$, $A \in \mathcal{F}$.

Finally we modify Corollary 3.

Corollary 4. *The premium principle $H : \Gamma \to (-\infty, +\infty]$ fulfills the Properties 1', 2, 3', 4', and 5 if, and only if, these exists a function*

$$g_0 \colon P(\mathcal{F}) \to [0, 1],$$

$g_0(0) = 0$, $g_0(1) = 1$, g_0 *is non-decreasing,* $g_0(q) \geq q$, $q \in P(\mathcal{F})$, *such that for any risk* $X \in \Gamma$,

$$\text{(25)} \qquad H(X) = -\int_{-\infty}^0 (1 - g_0(S_X(x)))\, dx + \int_0^\infty g_0(S_X(x))\, dx,$$

where $S_X(x) := P\{X > x\}, x \in \mathbb{R}$. *Moreover* $g_0(q) = H(B_q)$, $q \in P(\mathcal{F})$, *where* B_q *is Bernoulli r.v. with parameter* q.

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