Theory of Stochastic Processes
Vol. 12 (28), no. 3–4, 2006, pp. 12–19

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ON EXTENSION OF THE LIMIT THEOREM OF RENEWAL THEORY AND ITS APPLICATION

An extension of the limit theorem of renewal theory on the case of semi-Markov process with an atom of the stationary distribution of the embedded Markov chain is obtained in this paper. The obtained result is used for finding an analytic solution of a reliability problem for a system with protection.

1. Introduction

An extension of the limit theorem of renewal theory on the case of semi-Markov process with an atom of the stationary distribution of the embedded Markov chain is obtained in this paper. The result is illustrated by application to the solution of reliability problem for system with protection.

Consider the renewal process $\xi(t)$ which satisfies conditions:

B1. *Distribution function of time between renewals is absolutely continuous:* $Q(t) = \int\limits_0^t q(s)ds$;

B2. $m_l = \int\limits_0^\infty t^l q(t)dt < \infty$, $l = \overline{1, k+2}$, $k \geq 1$.

Let $h(t)$ be the renewal density of the renewal process $\xi(t)$.
In the paper by C. J. Stone (1965) and B. A. Sevastianov (1968) it is proved that under conditions B1, B2

$$\int\limits_0^\infty t^n |h_*(t)| dt < \infty, \quad n = \overline{0, k}, \quad \text{where } h_*(t) = h(t) - \frac{1}{m_1}.$$

In the present paper we extend this result on the case of strongly regular semi-Markov process $\xi(t)$ that satisfies conditions C1 – C3, stated below, and the condition that the stationary distribution of embedded in $\xi(t)$ Markov chain has an atom of distribution.

Note that conditions B1, B2 are a particular case of conditions C1-C3 for the renewal process.

2. Extension of the limit theorem of renewal theory

2000 *Mathematics Subject Classification.* Primary 60-XX, 60J35.

Key words and phrases. Strongly regular semi-Markov process, embedded Markov chain, uniformly recurring, atom of stationary distribution.

Let semi-Markov process $\xi(t)$ be given by the Markov renewal process $\{\xi_n, \tau_n;\ n \geqslant 0\}$, where ξ_n is an embedded Markov chain, τ_n are moments of renewal. Let the following conditions holds:

C1. The Markov chain $\xi_n, n \geq 0$, embedded in the $\xi(t)$ is uniformly recurring;

C2. $\sup\limits_{x\in X} M_l(x,B) < \infty, \quad B \in \mathcal{B},\ l = \overline{1, k+2},\ k \geq 1$, where $M_l(x,B) = \int\limits_0^\infty t^l Q(dt,x,B)$;

C3. The semi-Markov kernel of the process $\xi(t)$ is absolutely continuous in t:

$$Q(t,x,B) = \int\limits_0^t q(s,x,B)ds, \quad t \geq 0,\ x \in X,\ B\in\mathcal{B}.$$

Suppose that there exists a point $x_0 \in X$ such that $\rho(\{x_0\}) > 0$, in other words, there exists at least one point from the set of states in which stationary distribution ρ of the embedded Markov chain has an atom of distribution.

Denote by ${}_{x_0}F(t,x,B)$ the distribution function of the time of the first attainment of the set of states B by process $\xi(t)$ from the initial state x and with forbidenness to get to x_0:

$${}_{x_0}F(t,x,B) = \mathcal{P}\{\tau_n \leqslant t, \xi_n \in B, n \geqslant 1, \xi_\nu \notin B, \xi_\nu \neq x_0, \nu = \overline{1, n-1}/\xi_0 = x\}$$

Denote by $h(t,x,B)$ and $h_c(B)$ the density of the Markov renewal function of the process $\xi(t)$ and the stationary density of the Markov renewal function correspondingly. Let ${}_{x_0}M_n(x,B) \overset{\text{def}}{=} \int\limits_0^\infty t^n {}_{x_0}F(dt,x,B)$.

Theorem 1. *Let a strongly regular semi-Markov process $\xi(t)$ satisfies conditions C1 – C3 and let the stationary distribution of the embedded in $\xi(t)$ Markov chain has an atom of distribution at the point x_0. Then there exist*

$$H_n(x,B) = \int\limits_0^\infty t^n h_*(t,x,B)dt, \quad n = \overline{0,k}, \quad x \in X,\ B\in\mathcal{B},$$

where $h_*(t,x,B) = h(t,x,B) - h_c(B)$, and the following relations hold true:

$$\begin{aligned} H_n(x,x_0) &= {}_{x_0}M_n(x,x_0) - \frac{h_c(x_0)}{n+1} {}_{x_0}M_{n+1}(x,x_0) + \\ &+ \sum_{r=0}^n C_n^r\ {}_{x_0}M_{n-r}(x,x_0) H_r(x_0,x_0), \end{aligned} \tag{1}$$

$$\begin{aligned} H_n(x,B) &= {}_{x_0}M_n(x,B) - \frac{h_c(x_0)}{n+1} {}_{x_0}M_{n+1}(x_0,B) + \\ &+ \sum_{r=0}^n C_n^r\ H_{n-r}(x,x_0) {}_{x_0}M_r(x_0,B). \end{aligned} \tag{2}$$

Proof. Let $\xi^{x_0}(t)$ be the sparse semi-Markov process with the state set $\{x, x_0\}$, where $x = \xi(0)$ (see, for example, paper by V. S. Korolyuk, A. A. Tomusyak and A. F. Turbin (1979)). It is a general renewal process with the initial state x and moments of renewal which coincides with moments of hit of the process $\xi(t)$ the state x_0. Then from definition of ${}_{x_0}F(t, y, x_0)$ it follows that for $y = x$ it is the distribution function of time until the first renewal of $\xi^{x_0}(t)$, and for $y = x_0$ it is the distribution function of time between any other renewals of $\xi^{x_0}(t)$.

Denote by $h^{x_0}(t, y, x_0)$, where $y = x$ or x_0, the renewal density of the process $\xi^{x_0}(t)$, which, by definition of this process, coincides with $h(t, y, x_0)$, the density of the Markov renewal function of the process $\xi(t)$, were $y = x$ or x_0, $\{x_0\} \in \mathcal{B}$.

From the book by A. N. Korlat, V. N. Kuznetsov, M. M. Novikov and A. F. Turbin (1991) it follows, that under conditions of our theorem we get

$$ {}_{x_0}M_l(y, x_0) < \infty,\ l = \overline{1, k+2},\ y = x, x_0, $$

and the distribution function of $\xi^{x_0}(t)$ is absolutely continuous:

$$ {}_{x_0}F(t, y, x_0) = \int_0^t {}_{x_0}f(s, y, x_0)ds, \quad y = x, x_0. $$

So the renewal process $\xi^{x_0}(t)$ satisfies conditions B1, B2. Therefore, as it was proved by C. J. Stone (1965) and B. A. Sevastianov (1968)

$$ \int_0^\infty t^n |h_*(t, y, x_0)| dt < \infty, \quad n = \overline{0, k}, \quad y = x, x_0. \tag{3} $$

Thus by Lebesgue theorem we may pass to the limit under the integral sign in the Laplace transform:

$$ H_n(y, x_0) = \int_0^\infty t^n h_*(t, y, x_0)dt = \lim_{p \to 0} \int_0^\infty e^{-pt} h_*(t, y, x_0)dt,\ y = x, x_0 \tag{4} $$

By applying the formula of total probability and taking into consideration the first jump of the process $\xi^{x_0}(t)$ we will have the equation

$$ h(t, x, x_0) = {}_{x_0}f(t, x, x_0) + \int_0^t {}_{x_0}f(s, x, x_0)h(t - s, x_0, x_0)ds. $$

Note that $t^n = (t - s + s)^n = \sum_{r=0}^{n} C_n^r s^{n-r}(t - s)^r$. So if multiply the last equation by t^n, $n = \overline{1, k}$, after not complicated algebraic transformation we will get

$$ t^n h_*(t, x, x_0) = t^n {}_{x_0}f(t, x, x_0) - h_c(x_0)t^n(1 - {}_{x_0}F(t, x, x_0)) + $$

$$+\sum_{r=0}^{n} C_n^r \int_0^t s^{n-r}{}_{x_0}f(s,x,x_0)(t-s)^r h_*(t-s,x_0,x_0)ds.$$

Then applying the Laplace transform to the last equation taking into consideration (4)and passing to the limit as the parameter of the Laplace transform p tends to 0 we will have (1).
By applying the formula of total probability and taking into consideration the last until time t jump of the process $\xi(t)$ in state x_0 we will get

$$h(t,x,B) = {}_{x_0}f(t,x,B) + \int_0^t h(s,x,x_0)\ {}_{x_0}f(t-s,x_0,B)ds. \tag{5}$$

It follows from the results by A. N. Korlat, V. N. Kuznetsov, M. M. Novikov and A. F. Turbin (1991) that

$$h_c(B) = \int_0^\infty h_c(x_0)\ {}_{x_0}f(t,x_0,B)dt. \tag{6}$$

Since $t^n = (t-s+s)^n = \sum_{r=0}^{n} C_n^r s^{n-r}(t-s)^r$, from (5), (6) we get

$$t^n h_*(t,x,B) = t^n\ {}_{x_0}f(t,x,B) - h_c(x_0)t^n({}_{x_0}F(\infty,x_0,B) - {}_{x_0}F(t,x_0,B))+$$

$$+\sum_{r=0}^{n} C_n^r \int_0^t s^{n-r}h_*(s,x,x_0)(t-s)^r\ {}_{x_0}f(t-s,x_0,B)ds. \tag{7}$$

Applying the Laplace transform to the last equation taking into consideration (4) and passing to the limit as p tends to 0 we will have

$$\lim_{p\to 0}\int_0^\infty e^{-pt}t^n h_*(t,x,B)dt = {}_{x_0}M_n(x,B) - \frac{h_c(x_0)}{n+1}{}_{x_0}M_{n+1}(x_0,B)+$$

$$+\sum_{r=0}^{n} C_n^r\ H_{n-r}(x,x_0)\ {}_{x_0}M_r(x_0,B).$$

So, to prove (2) we should to prove the existence of $\int_0^\infty t^n h_*(t,x,B)dt$, $n = \overline{0,k}$. From (7) it follows that

$$|t^n h_*(t,x,B)| \le t^n\ {}_{x_0}f(t,x,B) + h_c(x_0)t^n({}_{x_0}F(\infty,x_0,B) - {}_{x_0}F(t,x_0,B))+$$

$$+\sum_{r=0}^{n} C_n^r \int_0^t |s^{n-r}h_*(s,x,x_0)|(t-s)^r\ {}_{x_0}f(t-s,x_0,B)ds.$$

Applying the Laplace transform to the right side of the last unequality and then taking the parameter of the Laplace transform p equal to 0 we will get

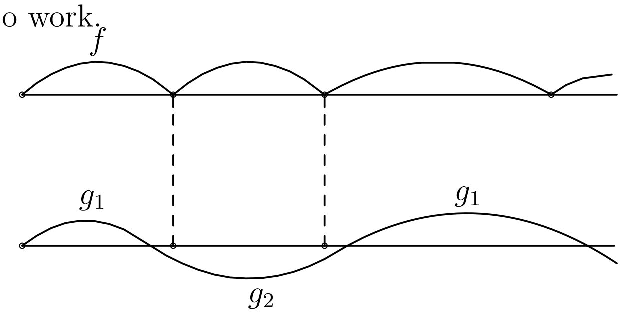
$$\int_0^\infty |t^n h_*(t,x,B)|dt \le {}_{x_0}M_n(x,B) - \frac{h_c(x_0)}{n+1}{}_{x_0}M_{n+1}(x_0,B) + + \sum_{r=0}^{n} C_n^r \int_0^\infty t^n |h_*(t,x,x_0)|dt \; {}_{x_0}M_r(x_0,B).$$

From the last unequality and (3) we have that $\int_0^\infty t^n|h_*(t,x,B)|dt < \infty$. □

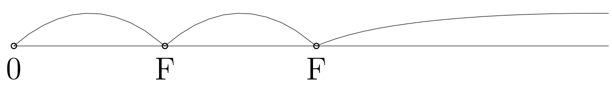
2. Reliability problem for system with protection

Consider a system with protection which consist of two independent elements. Functioning of the first element is described by a renewal process with the density of distribution function of the time of renewal $f(t)$. Functioning of the second element (system of protection) is described by an alternating renewal process. This process models the alternating sequence of periods of work (state 1) and periods of repair (state 2) of system of protection, with densities of distribution function $g_1(t)$ and $g_2(t)$ correspondingly. Let $f(t)$ be the density of the uniform distribution in interval $[0,1]$, let $g_1(t) = \dfrac{\mu_1^n}{\Gamma(n)} t^{n-1} e^{-\mu_1 t}$, $t \ge 0$, $n \in N$, $\mu_1 > 0$ (Erlang distribution), and let $g_2(t) = \mu_2 e^{-\mu_2 t}$, $t \ge 0$, $\mu_2 > 0$ (exponential distribution). If the moment of renewal of the first element occurs in period of repair of the second element, then the system faults.

Our problem is to find the average of distribution of time until the first system fault (M), under condition that at starting time $t = 0$ both elements start to work.



We will suppose that after a fault the system keep on functioning. Let us describe functioning of this system by a semi-Markov process $\eta(t)$ (general renewal process) with two states 0 and F. Let at starting time $t = 0$ the process be in state 0 and stay there until the first system fault. In the moment of system fault the process $\eta(t)$ transfers to state F and stay there until the next fault.



Let $h(t,x,F)$ be the renewal density of the process $\eta(t)$, let h_c be the stationary renewal density of the process $\eta(t)$. The process $\eta(t)$ satisfies conditions of Theorem 1. For this reason from equation (1) for $n=0$ we have:

$$\int_0^\infty (h(t,0,F)-h_c)dt = 1+\int_0^\infty (h(t,F,F)-h_c)dt - h_cM \tag{8}$$

Let's calculate integrals that appear in equation (8). Let $h_1(t)$ be the density of renewal of the first element, let $h_{1c} \stackrel{\text{def}}{=} \lim\limits_{t\to\infty} h_1(t)$, let $\Pi_i(t)$ be the probability of being of the second element in state of repair at time t, under condition that at starting time $t=0$ it was in state i, $i=\overline{1,2}$, and let $\Pi_c \stackrel{\text{def}}{=} \lim\limits_{t\to\infty} \Pi_1(t)$. Then because of independence of elements we find
$h(t,0,F)=h_1(t)\,\Pi_1(t), \quad h(t,F,F)=h_1(t)\,\Pi_2(t), \quad h_c=h_{1c}\,\Pi_c.$
As it is well known (see, for example, D. Koks and V. Smeet (1967))
$\tilde{\Pi}_1(p) = \dfrac{\tilde{g}_1(1-\tilde{g}_2(p))}{p(1-\tilde{g}_1(p)\tilde{g}_2(p))}, \ \tilde{\Pi}_2(p) = \dfrac{1-\tilde{g}_2(p)}{p(1-\tilde{g}_1(p)\tilde{g}_2(p))}, \ \tilde{h}_1(p) = \dfrac{\tilde{f}(p)}{1-\tilde{f}(p)},$
where symbol $\sim$ denotes the Laplace transform.
As $\tilde{g}_1(p) = \dfrac{\mu_1^n}{(\mu_1+p)^n}, \ \tilde{g}_2(p) = \dfrac{\mu_2}{(\mu_2+p)}, \ \tilde{f}(p) = \dfrac{1-e^{-p}}{p}$, then

$$\tilde{\Pi}_1(p) = \frac{(\mu_1)^n}{(\mu_1+p)^n(p+\mu_2)-\mu_1^n\mu_2}, \quad \Pi_c = \frac{\mu_1}{\mu_1+n\mu_2}, \tag{9}$$

$$\tilde{\Pi}_2(p) = \frac{(\mu_1+p)^n}{(\mu_1+p)^n(p+\mu_2)-\mu_1^n\mu_2}, \ \tilde{h}_1(p) = \frac{1-e^{-p}}{(p-1+e^{-p})}, \ h_{1c}=2. \tag{10}$$

From Theorem 1 follows existence of $\int_0^\infty (h_1(t)-h_{1c})dt$ and $\int_0^\infty (\Pi_i(t)-\Pi_c)dt$, $i=\overline{1,2}$ in our case. Thus from (9), (10) it follows that there exists $\delta>0$ such that $\{p\in\mathbb{C} : Rep \geq -\delta\}$ is singularity-free domain for the functions $\tilde{h}_1(p)-\dfrac{h_{1c}}{p}$ and $\tilde{\Pi}_i(p)-\dfrac{\Pi_c}{p}$, $i=\overline{1,2}$. Thus we may apply theorem about composition of original functions and residue theorem (see, for example V. Martynenko (1965)), according to which

$$\int_0^\infty (h_1(t)-h_{1c})(\Pi_i(t)-\Pi_c)dt = \sum Res(\widetilde{\Pi_i}(p)-\frac{1}{p}\Pi_c)(\widetilde{h_1}(-p)+\frac{1}{p}h_{1c}), i=\overline{1,2}$$

where residues are calculated at singular points of function $(\widetilde{\Pi_i}(p) - \frac{1}{p}\Pi_c)$. Consequently from (8) it follows that

$$M = \frac{1}{h_{1c}\Pi_c} + \frac{1}{\Pi_c}\int\limits_0^\infty (\Pi_2(t) - \Pi_1(t))dt + \tag{11}$$

$$+\frac{1}{h_{1c}\Pi_c}\sum Res(\widetilde{\Pi_2}(p) - \frac{1}{p}\Pi_c)(\widetilde{h_1}(-p) + \frac{1}{p}h_{1c}) -$$

$$-\frac{1}{h_{1c}\Pi_c}\sum Res(\widetilde{\Pi_1}(p) - \frac{1}{p}\Pi_c)(\widetilde{h_1}(-p) + \frac{1}{p}h_{1c}),$$

where residues are calculated at singular points of functions $(\widetilde{\Pi_2}(p) - \frac{1}{p}\Pi_c)$ and $(\widetilde{\Pi_1}(p) - \frac{1}{p}\Pi_c)$ correspondingly. From (9), (10) it follows that functions $(\widetilde{\Pi_2}(p) - \frac{1}{p}\Pi_c)$ and $(\widetilde{\Pi_1}(p) - \frac{1}{p}\Pi_c)$ have the same singular points, which are nonzero roots of the equation $(\mu_1 + p)^n(p + \mu_2) - \mu_1^n\mu_2 = 0$. Noting that 0 is the root of this equation, but it is not the singular point for this functions. Thus considering (11), (9), (10), we get

$$M = \frac{(\mu_1 + n\mu_2)}{2\mu_1} + \frac{n}{\mu_1} + \frac{(\mu_1 + n\mu_2)}{2\mu_1}\sum Res\frac{(2 - 2e^p + p + pe^p)}{(p + 1 - e^p)p} \times$$

$$\times\frac{(\mu_1 + p)^n - (\mu_1)^n}{(\mu_1 + p)^n(p + \mu_2) - \mu_1^n\mu_2}$$

where residues are calculated at points $p_i,\ i = \overline{1,n}$, which are nonzero roots of the equation $(\mu_1 + p)^n(p + \mu_2) - \mu_1^n\mu_2 = 0$.
In particular case, where $\mu_1 = \mu_2$, we have

$$M = \frac{(n+1)}{2} + \frac{n}{\mu_1} + \frac{(n+1)}{2}\sum_{k=1}^{n}\frac{(2 - 2e^{p_k} + p_k + p_k e^{p_k})[(\mu_1 + p_k)^n - (\mu_1)^n]}{(n+1)(p_k + 1 - e^{p_k})p_k(\mu_1 + p_k)^n},$$

$$p_k = \mu_1 cos\frac{2k\pi}{(n+1)} + isin\frac{2k\pi}{(n+1)},\ k = \overline{1,n}.$$

Conclusion

Due to Theorem 1, proved in this paper, we succeeded in finding an analytic solution of reliability problem for system with protection. Other methods result in more complicated system of equations for which analytic solution are not known.

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