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ON UNIFORM CONVERGENCE OF WAVELET EXPANSIONS OF SOME RANDOM PROCESSES

In the paper there are found conditions for uniform convergence with probability one of wavelet expansion of \( g \)-sub-Gaussian random processes under additional condition for norm of such process

1. Introduction

It this paper I proceed with research presented in [1] and derive conditions for uniform convergence of wavelet expansions of \( g \)-sub-Gaussian random processes on the finite interval in case when norm \( \tau_g \) of such process

\[ X = \{ X(t), t \in R \} \]

increases for positive \( t \).

2. Main results

Definition 1. [2] Let \( g = \{ g(x), x \in R \} \) be a continuous even convex function; \( g \) is called an \( N \)-function if \( g(0) = 0, g(x) > 0 \) as \( x \neq 0 \) and

\[ \lim_{x \to 0} \frac{g(x)}{x} = 0, \lim_{x \to \infty} \frac{g(x)}{x} = \infty. \]

Condition Q. [3] An \( N \)-function \( g \) satisfies condition Q if \( \lim \inf_{x \to 0} \frac{g(x)}{x^2} = C > 0 \). It may happen that \( C = \infty \).

Definition 2. [2, 3] Let \( g \) be an \( N \)-function, which satisfies condition Q. Let \( \{ \Omega, L, P \} \) be a standard probability space. A random variable \( \xi = \{ \xi(\omega), \omega \in \Omega \} \) belongs to the space \( \text{Sub}_g(\Omega) \) (is \( g \)-sub-Gaussian) if \( E \xi = 0, E \exp \{ \lambda \xi \} \) exists for all \( \lambda \in R \) and there exists a constant \( a > 0 \) such that the following inequality holds for all \( \lambda \in R : E \exp \{ \lambda \xi \} \leq \exp \{ g(a\lambda) \} \).

The space \( \text{Sub}_g(\Omega) \) is a Banach space with respect to the norm

\[ \tau_g(\xi) = \sup_{\lambda \neq 0} g^{-1}(\ln E \exp \{ \lambda \xi \}) \]

\[ \lambda \]

Definition 3. [2] A random process \( \{ X(t), t \in T \} \) belongs to the space \( \text{Sub}_g(\Omega) \) (is \( g \)-sub-Gaussian) if the random variable \( X(t) \in \text{Sub}_g(\Omega) \) for all \( t \in T \).

Let \( \varphi = \{ \varphi(x), x \in R \} \) be an \( f \)-wavelet and \( \psi = \{ \psi(x), x \in R \} \) be the \( m \)-wavelet, which corresponds to \( \varphi \).

Define a family of functions \( \{ \varphi_{jk}, j \in Z, k \in Z \} \) in the following way:

\[ \varphi_{jk}(x) = 2^j/2 \cdot \varphi(2^j x - k), \psi_{jk}(x) = 2^j/2 \cdot \psi(2^j x - k). \]

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It is known that the family of functions \( \{ \varphi_{0k}, \psi_{jk}, j = 0, 1, \ldots k \in \mathbb{Z} \} \) is an orthonormal basis in \( L_2 (R) \).

**Definition 4.** [1] Let \( \varphi \) be an \( f \)-wavelet (\( \psi \) be an \( m \)-wavelet). The assumption \( S \) holds for \( \varphi \) (or \( \psi \)) if there exists a function \( \Phi = \{ \Phi (x), x \geq 0 \} \) such that \( \Phi (x) \) decreases, \( |\varphi (x)| \leq \Phi (|x|) \) (or \( \psi (x) \leq \Phi (|x|) \)) almost everywhere and \( \int R \Phi (|x|) \ dx < \infty \).

The following theorem is a particular case of the theorem 4.1 from the paper [1].

**Theorem 1.** Let \( X = \{ X (t), t \in R \} \) be a separable \( g \)-sub-Gaussian random process, \( B_l = [a_l, a_{l+1}], a_{l+1} - a_l = e, l \in \mathbb{Z}, a_l \to +\infty \) as \( l \to +\infty \), \( a_l \to -\infty \) as \( l \to -\infty \). Assume that there exists an increasing continuous function \( \sigma = \{ \sigma (h), h > 0 \} \) such that \( \sup_{|t-s| \leq h} \tau_g (X (t) - X (s)) \leq \sigma (h) \).

Let \( c = \{ c (t) , t \in R \} \) be a continuous even positive function such that for sufficiently large \( x \) we have: \( c (ax) \leq c (x) A (a), A (a) \in (0; \infty) \). Denote \( \delta_l = \sup_{t \in B_l} (c (t))^{-1}, \chi_l = \sup_{l \in B_l} \tau_g (X (t) - X (a_l+1)), Z_l = \tau_g (X (a_l+1)), l \in \mathbb{Z} \).

Assume that for any \( \varepsilon > 0 \):

\[
\int_0^\varepsilon a_g \left( \ln \left( \left( 2 \sigma^{-1} (u) \right)^{-1} + 1 \right) \right) \ du < \infty, \tag{1}
\]

and

\[
\sum_{l \in \mathbb{Z}} \delta_l Z_l < \infty, \tag{2}
\]

\[
\sup_{l \in \mathbb{Z}} \frac{\chi_l}{Z_l} \leq \beta < \infty, \tag{3}
\]

\[
\sum_{l \in \mathbb{Z}} \delta_l \int_0^{\chi_l} a_g \left( \ln \left( \frac{a_{l+1} - a_l}{2 \sigma^{-1} (u)} + 1 \right) \right) \ du < \infty, \tag{4}
\]

where \( a_g (x) = \frac{x}{g^{(-1)} (x)} \). Let \( \varphi \) be an \( f \)-wavelet and \( \psi \) be the \( m \)-wavelet, which corresponds to \( \varphi \), and suppose that the assumption \( S \) holds for \( \varphi \) and \( \psi \) with respect to a function \( \Phi \) and

\[
\int R c (x) \Phi (|x|) \ dx < \infty. \tag{5}
\]

Then with probability one there exist

\[
a_{0k} = \int R X (t) \varphi_{0k} (t) \ dt \quad \text{and} \quad b_{jk} = \int R X (t) \psi_{jk} (t) \ dt, \quad k \in \mathbb{Z}, j = 0, +\infty
\]
and wavelet expansion \( X_m(t) = \sum_{k \in \mathbb{Z}} \alpha_{0k} \varphi_{0k}(x) + \sum_{j=0}^{m-1} \sum_{k \in \mathbb{Z}} \beta_{jk} \psi_{jk}(x) \) converges to \( X(t) \) as \( m \to \infty \) uniformly on each interval \([a, b] \) with probability one \((-\infty < a < b < +\infty)\).

**Theorem 2.** Let the assumptions (1) and (2) of the Theorem 1 hold and assume that

\[
\sum_{l \in \mathbb{Z}} \delta_l \chi_l a_g \left( \ln \left( 1 + (a_{l+1} - a_l) \right) \right) < \infty, \tag{6}
\]

\[
\sum_{l \in \mathbb{Z}} \delta_l \int_0^{\chi_l} a_g \left( \ln \left( \frac{a_{l+1} - a_l}{2\sigma^{(-1)}(u)} + 1 \right) \right) du < \infty. \tag{7}
\]

Also suppose that \( \tau_g(X(t)) = \tau_g(X(-t)) > 0, t \neq 0, \) and norm \( \tau_g(X(t)) \) increases as \( t > 0 \).

Then the assertion of the Theorem 1 follows.

**Proof.** It follows from Lemma 2.2.3 of the book [2] that the function \( a_g(x) = \frac{x+y}{g^{(-1)}(x+y)} \) increases as \( x > 0 \). If \( x > 0 \) and \( y > 0 \) then

\[
a_g(x+y) = \frac{x+y}{g^{(-1)}(x+y)} = \frac{x}{g^{(-1)}(x+y)} + \frac{y}{g^{(-1)}(x+y)} \leq \frac{x}{g^{(-1)}(x)} + \frac{y}{g^{(-1)}(y)} = a_g(x) + a_g(y). \]

Therefore

\[
\int_0^{\chi_l} a_g \left( \ln \left( \frac{a_{l+1} - a_l}{2\sigma^{(-1)}(u)} + 1 \right) \right) du \leq \int_0^{\chi_l} a_g \left( \ln \left( 1 + \left( \frac{a_{l+1} - a_l}{2\sigma^{(-1)}(u)} \right)^{-1} \right) \right) du \leq \chi_l a_g \left( \ln \left( 1 + \left( \frac{a_{l+1} - a_l}{2\sigma^{(-1)}(u)} \right)^{-1} \right) \right) du
\]

and the assumption (4) follows from (6) and (7).

Since \( \sup_{l \in \mathbb{Z}} \frac{\chi_l}{\delta_l} = \sup_{l>0} \frac{\chi_l}{\delta_l} \), then

\[
\tau_g(X(t)) - X(a_{l+1}) \leq \tau_g(X(t)) + \tau_g(X(a_{l+1})) \leq 2\tau_g(a_{l+1})
\]

for any \( t \in B_l, l > 0. \) Therefore \( \frac{\chi_l}{\delta_l} \leq 2 \) and assumption (3) holds true.

**Example 1.** Let the assumptions of the Theorem 2 hold true for the function \( \sigma(u) = \frac{c}{(\ln(1+\frac{1}{2u}))}, \) where \( c > 0, \gamma > 0. \) Then \( \sigma^{(-1)}(u) = \)
\[
\frac{1}{2} \left( \exp \left( \frac{1}{\tau^{1/\gamma}} \right) \right) - 1 \\
\int_0^{\chi_l} a_g \left( \ln \left( 1 + (2\sigma^{(-1)}(u))^{-1} \right) \right) \, du = \int_0^{\chi_l} a_g \left( \frac{c}{u} \right)^{1/\gamma} \, du. 
\] (8)

Since \( a_g \left( \frac{x}{u} \right)^{1/\gamma} = \frac{(\frac{x}{u})^{1/\gamma}}{g^{(-1)} \left( (\frac{x}{u})^{1/\gamma} \right)} \leq \frac{(\frac{x}{u})^{1/\gamma}}{g^{(-1)} \left( \frac{c}{\chi_l} \right)^{1/\gamma}} \), as \( u < \chi_l \) then

\[
\int_0^{\chi_l} a_g \left( \ln \left( 1 + (2\sigma^{(-1)}(u))^{-1} \right) \right) \, du \leq \frac{c^{1/\gamma}}{g^{(-1)} \left( \frac{c}{\chi_l} \right)^{1/\gamma}} \cdot \frac{\chi_l^{1-\frac{1}{\gamma}}}{1 - \frac{1}{\gamma}}
\]
and assumption (7) holds true if

\[
\sum_{l \in Z} \delta_l \chi_l^{1-\frac{1}{\gamma}} \left( g^{(-1)} \left( \frac{c}{\chi_l} \right)^{1/\gamma} \right)^{-1} < \infty. 
\] (9)

If \( g(x) = |x|^\alpha \), \( 1 < \alpha \leq 2 \), then \( a_g \left( \frac{x}{u} \right)^{1/\gamma} = \left( \frac{x}{u} \right)^{\frac{1}{\gamma} - \frac{1}{\alpha}} \) and if \( \gamma > 1 - \frac{1}{\alpha} \) then

\[
\int_0^{\chi_l} a_g \left( \ln \left( (2\sigma^{(-1)}(u))^{-1} + 1 \right) \right) \, du = \frac{c^{1-\frac{1}{\gamma}}}{\gamma \chi_l^{1-\frac{1}{\gamma} + \frac{1}{\gamma\alpha}}} 
\]

Thus assumption (7) holds true if

\[
\sum_{l \in Z} \delta_l \chi_l^{1-\frac{1}{\gamma} + \frac{1}{\gamma\alpha}} < \infty. 
\] (10)

**Theorem 3.** Let \( X = \{ X(t), t \in R \} \) be a separable \( g \)-sub-Gaussian random process, where \( g(x) = |x|^\alpha \), \( 1 < \alpha \leq 2 \); \( X(t) = X(-t) \) with probability one; \( B_l = [a_l, a_{l+1}] \), \( l = 0, 1, 2, ..., a_0 = 0, a_{l+1} - a_l > \epsilon, a_l \to \infty, l \to \infty \), and

\[
\sup_{|t-s| \leq \epsilon} \tau_g \left( X(t) - X(s) \right) \leq \frac{c}{\left( \ln \left( 1 + \frac{1}{2\alpha} \right) \right)^\gamma}, c > 0, \gamma > 1 - \frac{1}{\alpha}.
\]

Let \( \tau_g(X(t)) \) increase as \( t > 0 \) and

\[
\sum_{l=0}^{\infty} \delta_l Z_l < \infty, 
\] (11)

\[
\sum_{l=0}^{\infty} \delta_l \chi_l \left( \ln \left( 1 + (a_{l+1} - a_l) \right) \right)^{1-\alpha} < \infty, 
\] (12)

\[
\sum_{l=0}^{\infty} \delta_l \chi_l^{1-\frac{1}{\gamma} + \frac{1}{\gamma\alpha}} < \infty. 
\] (13)

Then with probability one \( X_m(t) \to X(t) \) as \( m \to \infty \) uniformly on each bounded interval \([a, b]\).

Theorem 3 follows from Example 1 and Theorem 2.
Remark 1. Since $\chi_l \leq 2Z_l$ then from the assumption
\[
\sum_{l=0}^{\infty} \delta_l Z_l (\ln (1 + (a_{l+1} - a_l)))^{1-\alpha} < \infty
\]
the assumptions (11)–(13) follow, if $\chi_l > c > 0$.

If $a_l = e^l$ and $\tau_g (X(t)) = t$ then $c(t) = t \cdot (\ln t)^\beta$, $t > 1$, and

\[
\Phi(|t|) = \frac{1}{p(|t| (\ln t)^v)}, \quad v > 1, \ |t| > 1.
\]

Conclusions. In the paper there are found conditions for uniform convergence with probability one of the wavelet expansion of $g$-sub-Gaussian random process such that $\tau_g (X(t))$ increases for $t > 0$.

I plan to obtain similar results for random processes from $\text{Sub}_g (\Omega)$ such that

\[
\tau_g (X(t) - X(s)) \leq c \cdot |t - s|^{\alpha}.
\]

References


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