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MODELLING OF THE QUEUING SYSTEM WITH AN INCREASING DEMAND INTENSITY IN THE EMPTY STATE

The article is dedicated to formation of the served demand flow restoration function and lost demand flow restoration function when the queuing system operates with an increasing demand intensity in the empty state. The paper shows the relation between the input flow and servicing.

1. Introduction

Every queuing system, as a rule, includes the input demand flow, service facility, a queue for service and output flow. There is often no information about the time of the demand arrival and duration of its serving. Therefore, in order to analyze the queuing system, it is assumed [11] that the time of the demand arrival and the service time are random variables which distribution laws are known. The theoretical analysis of the system functioning is based on Markov models, that is, on the description of the system functioning with the use of Markov process with a discrete set of states [4]. The formation of such a model implies determination of [3] the states in which the process may be, and selection of one of the modes [6] of its transient probability characteristics of the transfer from one state to the other. The well-known technique for finding the main characteristics is applied to the formed process, that are interpreted as characteristics of the output system.

Until recently, the mathematical models describing the queuing system functioning were constructed on the assumption that the input flow and servicing are independent. In recent decades, the possibility of modeling the dependence of these two factors has been searched for. We introduced one of the possible options for such a dependency in our work, namely, through the speeding up of the demand arrival and increase of the service intensity.

The results described in this article can be used for calculation of the main characteristics of single-channel systems at the design stage.

The construction of the new queuing mathematical models remains relevant till today, and the researches are conducted both in the areas of the applied modeling [8], and development of its general theory [7].

It should be mentioned that the classical queuing theory, as a rule, postulates a certain type of the input flow and service time distribution law, and the input flow does not depend on servicing. However, the practice needs require construction of the models considering the input flow [10] and service discipline changes [9].

The objective and main task of the article is to establish a relation between the input flow and servicing when constructing a mathematical model for description of the transient service process in the queuing systems in case of the workload increase (demand increased intensity).

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2. Task assignment

Let the demand flow arrive to the service facility, and the time of the first demand is a random variable having an exponent distribution with the parameter $\alpha\lambda$, where $\alpha > 1$. The demand that arrived for serving is serviced during a random time having an exponent distribution with the parameter μ. The time of the next application arrival is a random variable having an exponent distribution with the parameter λ. If the system is empty before arrival of a new demand, then the demand intensity is increased. If a new demand arrives earlier than the previous one is served, then it is lost.

The function of Poisson flow restoration with the parameter $\alpha\lambda$ is equal to

$$H(t) := Ev(t) = \alpha\lambda t,$$

and restoration density (demand arrival intensity)

$$h(t) = H'(t) = \alpha\lambda.$$

Let $v_0(t)$ be a number of the served demands during the time t, and $v_B(t)$—a number of the demands lost during the time t. It is evident, that

$$H_0(t) + H_B(t) = E(v_0(t)) + E(v_B(t)) =$$
$$= E(v_0(t) + v_B(t)) = H(t) = \alpha\lambda t,$$

where $H_0(t)$, $H_B(t)$—flow restoration functions of the served and lost demands.

3. Main material

Theorem 1. *The flow restoration function of the served demands is the following:*

$$H_0(t) = \frac{\alpha\lambda(\lambda+\mu)}{(\alpha+1)\lambda+\mu}t - \frac{\alpha\lambda(\lambda+\mu)}{((\alpha+1)\lambda+\mu)^2}(1 - e^{-((\alpha+1)\lambda+\mu)t}).$$

Proof. Let's have the twocomponents Markov process, which represent the number of demands served during interval $[0;t]$ and the state of the server (empty or not empty).

We denote by e_0 the state of the empty server and by e_1—non-empty server.

ξ—the time of the arrival of the demand. The time of the first demand is a random variable having an exponent distribution with the parameter $\alpha\lambda$ ($\lambda > 1, \lambda > 0$). The time of the next demand arrivals have an exponent distribution with the parameter $\lambda(\lambda > 0)$.

η—the length of time the remand is served. It is a random variable having an exponent distribution with the parameter μ ($\mu > 0$).

Let $P_{n0}(t)$ be the probability that this process at the time t is in the state e_0, and during this time n demands are served, $P_{n1}(t)$—be the probability that the process at the time t is in the state e_1, and during this time n demands are served. Then the system behaviour analysis during the time interval $(t, t+\Delta t)$ and the formula of the total probability leads to the following equalities

$$\overline{P}_{00}(t+\Delta t) = \overline{P}_{00}(t)P(\xi > \Delta t) + o(\Delta t),$$
$$\overline{P}_{01}(t+\Delta t) = \overline{P}_{01}(t)P(min(\xi,\eta) > \Delta t) + \overline{P}_{00}(t)P(\xi < \Delta t) + o(\Delta t),$$
$$\overline{P}_{n0}(t+\Delta t) = \overline{P}_{n0}(t)P(\xi > \Delta t) + \overline{P}_{n-1,1}(t)P(min(\xi,\eta) < \Delta t) + o(\Delta t),$$
$$\overline{P}_{n1}(t+\Delta t) = \overline{P}_{n1}(t)P(min(\xi,\eta) > \Delta t) + \overline{P}_{n0}(t)P(\xi < \Delta t) + o(\Delta t),$$

for $n = 1, 2, \ldots$.

If to consider that

$$P(\xi < \Delta t) = 1 - e^{-\alpha\lambda\Delta t} = \alpha\lambda\Delta t + o(\Delta t),$$
$$P(\xi > \Delta t) = e^{-\alpha\lambda\Delta t} = 1 - \alpha\lambda\Delta t + o(\Delta t),$$
$$P(min(\xi,\eta) < \Delta t) = 1 - e^{-(\lambda+\mu)\Delta t} = (\lambda+\mu)\Delta t + o(\Delta t),$$
$$P(min(\xi,\eta) > \Delta t) = e^{-(\lambda+\mu)\Delta t} = 1 - (\lambda+\mu)\Delta t + o(\Delta t),$$

then the last equalities will be written as

$$\overline{P}_{00}(t+\Delta t)-\overline{P}_{00}(t)=-\alpha\lambda\overline{P}_{00}(t)\Delta t+o(\Delta t),$$
$$\overline{P}_{01}(t+\Delta t)-\overline{P}_{01}(t)=-(\lambda+\mu)\overline{P}_{01}(t)\Delta t+\alpha\lambda\overline{P}_{00}(t)\Delta t+o(\Delta t),$$
$$\overline{P}_{n0}(t+\Delta t)-\overline{P}_{n0}(t)=-\alpha\lambda\overline{P}_{n0}(t)\Delta t+(\lambda+\mu)\overline{P}_{n-1,1}(t)\Delta t+o(\Delta t),$$
$$\overline{P}_{n1}(t+\Delta t)-\overline{P}_{n1}(t)=-(\lambda+\mu)\overline{P}_{n1}(t)\Delta t+\alpha\lambda\overline{P}_{n0}(t)\Delta t+o(\Delta t),$$

for $n=1,2,\ldots$. If to postulate the differentiability of the functions $\overline{P}_{n0}$, $\overline{P}_{n1}$ ($n=0,1,\ldots$) (as it well known from the theory of Markov chains) then, having divided each of the previous equalities by Δt and passing to the limit with $\Delta t\to 0$, we will have an indefinite system of differential equations

$$\overline{P}'_{00}(t)=-\alpha\lambda\overline{P}_{00}(t),$$
$$\overline{P}'_{01}(t)=-(\lambda+\mu)\overline{P}_{01}(t)+\alpha\lambda\overline{P}_{00}(t),$$
$$\overline{P}'_{n0}(t)=-\alpha\lambda\overline{P}_{n0}(t)+(\lambda+\mu)\overline{P}_{n-1,1}(t),$$
$$\overline{P}'_{n1}(t)=-(\lambda+\mu)\overline{P}_{n1}(t)+\alpha\lambda\overline{P}_{n0}(t),$$

for $n=1,2,\ldots$.

Let

$$\Phi_0(t,s):=\sum_{n=0}^{\infty}\overline{P}_{n0}(t)s^n,$$
$$\Phi_1(t,s):=\sum_{n=0}^{\infty}\overline{P}_{n1}(t)s^n$$

be generating functions according to sequences $(\overline{P}_{n0}(t))$,$(\overline{P}_{n1}(t))$. Then multiplying the equation

$$\overline{P}'_{n0}(t)=-\alpha\lambda\overline{P}_{n0}(t)+(\lambda+\mu)\overline{P}_{n-1,1}(t),$$

for $n=1,2,\ldots$ by s^n and adding them to the first one, we get the equation

$$\frac{\partial}{\partial t}\Phi_0(t,s)=-\alpha\lambda\Phi_0(t,s)+s(\lambda+\mu)\Phi_1(t,s).$$

And multiplying the equation

$$\overline{P}'_{n1}(t)=-(\lambda+\mu)\overline{P}_{n1}(t)+\alpha\lambda\overline{P}_{n0}(t)$$

for $n=0,1,2,\ldots$ by s^n and adding them we get the equation

$$\frac{\partial}{\partial t}\Phi_1(t,s)=\alpha\lambda\Phi_0(t,s)-(\lambda+\mu)\Phi_1(t,s).$$

Thus, we get the system of differential equations

$$\left\{\begin{array}{l}\frac{\partial\Phi_0(t,s)}{\partial t}=-\alpha\lambda\Phi_0(t,s)+s(\lambda+\mu)\Phi_1(t,s),\\ \frac{\partial\Phi_1(t,s)}{\partial t}=\alpha\lambda\Phi_0(t,s)-(\lambda+\mu)\Phi_1(t,s).\end{array}\right.$$

If to differentiate the first equation by t again, using the last system, then we get a linear second order equation with constant coefficients

$$\frac{\partial^2\Phi_0(t,s)}{\partial t^2}+((\alpha+1)\lambda+\mu)\frac{\partial\Phi_0(t,s)}{\partial t}+\alpha\lambda(\lambda+\mu)(1-s)\Phi_0(t,s)=0.$$

By analogy, we get one more linear equation from the second equation

$$\frac{\partial^2\Phi_1(t,s)}{\partial t^2}+((\alpha+1)\lambda+\mu)\frac{\partial\Phi_1(t,s)}{\partial t}+\alpha\lambda(\lambda+\mu)(1-s)\Phi_1(t,s)=0.$$

Then the equation

$$\rho^2+((\alpha+1)\lambda+\mu)\rho+\alpha\lambda(\lambda+\mu)-s\alpha\lambda(\lambda+\mu)=0$$

is a characteristic equation of the corresponding equations,

$$\rho_1 = \frac{-(\alpha+1)\lambda - \mu + \sqrt{((\alpha-1)\lambda-\mu)^2 + 4\alpha\lambda(\lambda+\mu)s}}{2},$$

$$\rho_2 = \frac{-(\alpha+1)\lambda - \mu - \sqrt{((\alpha-1)\lambda-\mu)^2 + 4\alpha\lambda(\lambda+\mu)s}}{2}.$$

Considering that $\overline{P}_{00}(0) = 1$, $\overline{P}_{n0}(0) = 0$ $(n = 1, 2, \dots)$,

$$\overline{P}_{n1}(0) = 0 (n = 0, 1, 2, \dots),$$

we have

$$\Phi_0(0, s) = 1,\ \Phi_1(0, s) = 0.$$

Then

$$\frac{\partial \Phi_0(0, s)}{\partial s} = -\alpha\lambda,\ \frac{\partial \Phi_1(0, s)}{\partial s} = \alpha\lambda.$$

Hence we get

$$\Phi_0(t, s) = \frac{\rho_2 + \alpha\lambda}{\rho_2 - \rho_1} e^{\rho_1 t} - \frac{\rho_1 + \alpha\lambda}{\rho_2 - \rho_1} e^{\rho_2 t},$$

$$\Phi_1(t, s) = -\frac{\alpha\lambda}{\rho_2 - \rho_1} e^{\rho_1 t} + \frac{\alpha\lambda}{\rho_2 - \rho_1} e^{\rho_2 t}.$$

Let

$$\overline{P}_n(t) = \overline{P}_{n0}(t) + \overline{P}_{n1}(t)$$

be the probability that n demands $(n = 0, 1, \dots)$ will be served during the t time. Then it is evident that the generating function of the sequence $(\overline{P}_n(t))$ is

$$\Phi(t, s) := \sum_{n=0}^{\infty} \overline{P}_n(t) s^n = \sum_{n=0}^{\infty} (\overline{P}_{n0}(t) + \overline{P}_{n1}) s^n =$$

$$= \Phi_0(t, s) + \Phi_1(t, s) = \frac{\rho_2}{\rho_2 - \rho_1} e^{\rho_1 t} - \frac{\rho_1}{\rho_2 - \rho_1} e^{\rho_2 t}.$$

Finally, as the flow restoration function of the served demands can be presented as

$$H_0(t) = E(v_0(t)) = \sum_{k=1}^{\infty} k\overline{P}_n(t) = \frac{\partial \Phi(t, s)}{\partial s}|_{s=1},$$

then

$$H_0(t) = \Phi'_s(t, 1) = (\frac{\rho_2}{\rho_2 - \rho_1} e^{\rho_1 t} - \frac{\rho_1}{\rho_2 - \rho_1} e^{\rho_2 t})'_s|_{s=1} =$$

$$= (\frac{\rho'_1\rho_2 - \rho_1\rho'_2}{(\rho_2 - \rho_1)^2}(e^{\rho_1 t} - e^{\rho_2 t}) + \frac{t}{\rho_2 - \rho_1}(\rho'_1\rho_2 e^{\rho_1 t} - \rho_1\rho'_2 e^{\rho_2 t}))|_{s=1}.$$

Considering that

$$\rho_1(1) = 0,$$

$$\rho_2(1) = -(\alpha+1)\lambda - \mu,$$

$$\rho'_1(1) = \frac{\alpha\lambda(\lambda+\mu)}{(\alpha+1)\lambda+\mu},$$

$$\rho'_2(1) = -\frac{\alpha\lambda(\lambda+\mu)}{(\alpha+1)\lambda+\mu},$$

we have

$$H_0(t) = \frac{\alpha\lambda(\lambda+\mu)}{(\alpha+1)\lambda+\mu} t - \frac{\alpha\lambda(\lambda+\mu)}{((\alpha+1)\lambda+\mu)^2}(1 - e^{-((\alpha+1)\lambda+\mu)t}).$$

□

4. Test verification

It is evident that when $\alpha = 1$, then we have the simplest model of the queuing one-channel system with losses. Indeed

$$H_0(t) = \frac{\lambda(\lambda+\mu)}{2\lambda+\mu}t - \frac{\lambda(\lambda+\mu)}{(2\lambda+\mu)^2}(1 - e^{-(2\lambda+\mu)t}).$$

Corollary 1. *The lost demand flow restoration function is the following*

$$H_{\mathrm{B}}(t) = \frac{\alpha^2\lambda^2}{(\alpha+1)\lambda+\mu}t + \frac{\alpha\lambda(\lambda+\mu)}{((\alpha+1)\lambda+\mu)^2}(1 - e^{-((\alpha+1)\lambda+\mu)t}).$$

5. Conclusions

In the presented work the functions of the served demand flow restoration and the lost demand flow restoration with an increasing demand intensity of an empty queuing system have been constructed and the relation between the input flow and servicing has been modelled. Of course, such a model can describe certain real situations more effectively than with the use of the independence of the input flow and servicing.

The constructed Markov representation of the distributions made it possible to construct Markov process and to find both stationary and transient characteristics.

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