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DMITRY B. BUKIN AND ELENA P. KRUGOVA

TRANSPORTATION COSTS FOR OPTIMAL AND TRIANGULAR TRANSFORMATIONS OF GAUSSIAN MEASURES

We study connections between transportation costs (with the quadratic Kantorovich distance) for Monge optimal mappings and increasing triangular mappings between Gaussian measures. We show that the second cost cannot be estimated by the first cost with a dimension-free coefficient, but under certain restrictions a comparison is possible.

Many results of the theory of extremal problems, measure theory and nonlinear analysis are closely connected with the problem of transforming one given measure into another by means of mappings from some special classes. Widely used classes of this type consist of monotone mappings (gradients of convex functions) and triangular mappings. Gradients of convex functions arise as optimal transformations in the Monge transportation problem. The goal of this paper is to compare transportation costs for optimal and triangular transformations of Gaussian measures.

A mapping $T = (T_1, \dots, T_n)\colon \mathbb{R}^n \to \mathbb{R}^n$ is called triangular if T_1 is a function of x_1, T_2 is a function of (x_1, x_2), T_3 is a function of (x_1, x_2, x_3) and so on: T_i is a function of $(x_1, x_2, \dots, x_i)$. A triangular mapping T is called increasing if every component T_i is increasing with respect to the variable x_i; we do not require monotonicity with respect to other variables. A measurable increasing triangular mapping is sometimes called canonical; the existence and uniqueness theorems are known for such (finite-dimensional) mappings under broad assumptions (see [3], [4], [8], [9]). The term "triangular mapping" is explained by the fact that for such differentiable mappings the Jacobi matrix has a triangular form. An obvious advantage of triangular mappings is their constructiveness, the possibility to obtain explicit (although cumbersome) formulas. Another advantage of the class of triangular mappings is that it is closed under compositions, unlike the class of optimal mappings.

Another important class of mappings is provided by the Monge–Kantorovich optimal transportation problem (see [1], [7], [14], [22], [28], [29]). Let X be a measurable space with two measures γ and μ defined on it, and let $c(x, y)$ be a nonnegative measurable function on $X \times X$ (it is called a cost function). The Kantorovich problem for the cost function c is to minimize the transportation cost

$$\mathcal{K}(\nu) = \int_{X\times X} c(x_1, x_2)\, \nu(dx_1 dx_2)$$

over all measures ν on $X \times X$ with marginals γ and μ. If X is a metric space with a metric d, the quadratic Kantorovich–Rubinstein distance (see [3], [5]) between γ and μ is defined using the functional $\mathcal{K}(\nu)$ with the cost function $c(x, y) = d(x, y)^2$:

$$W_2(\gamma, \mu) = \inf \left[\int_{X\times X} d(x_1, x_2)^2 \nu(dx_1 dx_2) \right]^{1/2},$$

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where the infimum is taken over all measures ν on $X \times X$ with marginals γ and μ on the first and second factors, respectively.

In many particular cases there exists a mapping $T\colon X \to X$, called an optimal mapping, such that $\mu = \gamma \circ T^{-1}$ and

$$W_2(\gamma, \mu)^2 = \int_X d(x, T(x))^2 \, \gamma(dx).$$

This mapping is said to be a solution of the Monge problem of minimization of the integral

$$\int_X d(x, T(x))^2 \, \gamma(dx)$$

(called the transportation cost for T) in the class of Borel mappings taking γ to μ (actually, the classical Monge problem, as well as the Kantorovich problem [17], deals with the distance cost function $d(x, y)$, not with its square, but the squared distance leads to much more regular solutions). Under broad assumptions such mappings are unique. In particular, for any two absolutely continuous probability measures $\rho_1 dx$ and $\rho_2 dx$ on the space $X = \mathbb{R}^n$ there is an optimal transportation T of $\rho_1 dx$ into $\rho_2 dx$ which is $\rho_1 dx$-unique and has the form $T = \nabla\Psi$, where Ψ is a convex function satisfying the Monge–Ampére equation

$$\rho_2(\nabla\Psi) \det D^2\Psi = \rho_1.$$

Suppose that the measure μ is absolutely continuous with respect to the measure γ. The entropy $\mathrm{Ent}_\gamma(\mu)$ is defined by

$$\mathrm{Ent}_\gamma(\mu) = \mathrm{Ent}_\gamma(\rho) = \int_X \rho \log \rho \, d\gamma,$$

where $\rho = d\mu/d\gamma$ is the Radon–Nikodym density of μ with respect to γ.

It was shown by Talagrand [27] that for the standard Gaussian measure γ on $\mathbb{R}^n$ and any probability measure $\mu \ll \gamma$ with finite entropy $\mathrm{Ent}_\gamma(\mu)$ there exists an increasing triangular mapping T such that $\mu = \gamma \circ T^{-1}$ and

$$\int_{\mathbb{R}^n} |T(x) - x|^2 \, \gamma(dx) \le 2\, \mathrm{Ent}_\gamma(\mu).$$

The same inequality holds for the optimal mapping (see [6], [19], [20]), so it is natural to compare the values of the Monge transportation costs for these two classes of mappings. We do this for Gaussian measures μ. Let us mention the papers [25], [26] concerned with the quadratic Kantorovich distance in the class of Gaussian measures (see also [21], [23], [24] on the Riemmannian geometry of Gaussian distributions).

We investigate how great can the difference be between the quantity

$$\int_{\mathbb{R}^n} |T(x) - x|^2 \, \gamma(dx)$$

for the canonical triangular mapping T and the quantity

$$\int_{\mathbb{R}^n} |T_0(x) - x|^2 \, \gamma(dx)$$

for the optimal mapping T_0.

It turns out that these quantities are comparable only if the corresponding measures and mappings satisfy quite narrow restrictions. In particular, even for linear images of the standard Gaussian measure the best possible constant K in the estimate

$$\int_{\mathbb{R}^n} |T(x) - x|^2 \, \gamma(dx) \le K \int_{\mathbb{R}^n} |T_0(x) - x|^2 \, \gamma(dx) \tag{1}$$

is increasing to infinity when the dimension n increases. However, a dimension-free estimate holds if the eigenvalues of the covariance matrix of the image measure belong to $[1/2, 3/2]$.

We employ the following fact about centered Gaussian measures.

Lemma 1. *Let γ and μ be two centered nondegenerate Gaussian measures on $\mathbb{R}^n$. Then the optimal mapping T_0 that transforms γ into μ is linear and the canonical triangular mapping T of γ into μ is also linear.*

Proof. First we prove that T_0 is linear. It is known (see [28]) that it is unique γ-almost everywhere and has the form $T_0 = \nabla\Psi$ for some convex function $\Psi\colon \mathbb{R}^n \to \mathbb{R}$. Hence it suffices to construct a linear operator with a symmetric matrix which transforms γ into μ (this operator will be the gradient of a quadratic form).

The measures γ and μ are defined by their Fourier transforms φ_γ and φ_μ, respectively (see [2]), given by

$$\varphi_\gamma(x) = \exp\biggl(-\frac{(Mx,x)}{2}\biggr), \quad \varphi_\mu(x) = \exp\biggl(-\frac{(Nx,x)}{2}\biggr),$$

where the symmetric and positive definite matrices M and N are the covariance matrices of the measures γ and μ, respectively. Given two symmetric positive definite matrices M and N, one can find a positive symmetric matrix A such that $N = AMA$ (which is called the Riccati matrix equation): the solution is

$$A = M^{-1/2}\Bigl(M^{1/2}NM^{1/2}\Bigr)^{1/2}M^{-1/2}.$$

Indeed, substituting A into the required equality we obtain

$$\begin{aligned}AMA = \biggl(M^{-1/2}\Bigl(M^{1/2}NM^{1/2}\Bigr)^{1/2}M^{-1/2}\biggr)\times \\ \times\Bigl(M^{1/2}M^{1/2}\Bigr)\biggl(M^{-1/2}\Bigl(M^{1/2}NM^{1/2}\Bigr)^{1/2}M^{-1/2}\biggr) \\ = M^{-1/2}\Bigl(M^{1/2}NM^{1/2}\Bigr)^{1/2}\Bigl(M^{1/2}NM^{1/2}\Bigr)^{1/2}M^{-1/2} \\ = M^{-1/2}M^{1/2}NM^{1/2}M^{-1/2} = N.\end{aligned}$$

Due to the symmetry of the matrix A the corresponding linear mapping is the gradient of the quadratic form $2^{-1}(Ax,x)$. On the other hand, γ is transformed into μ by the linear operator with the matrix A. It follows from the uniqueness of the gradient defining the optimal map that this linear operator is optimal.

Let us now construct a linear triangular mapping T. There exists a lower triangular matrix D such that $N = DMD^T$. Indeed, we can apply the Cholesky decomposition for the positive symmetric matrices M and N (see [16]):

$$M = BB^T \quad \text{and} \quad N = CC^T,$$

where B and C are lower triangular matrices with positive elements on the diagonal. The required matrix is $D = CB^{-1}$. The class of positive lower triangular matrices is a group, so the matrix D is lower triangular. It is shown in the following equality that the linear operator T defined by the matrix D transforms the measure γ into the measure μ:

$$DMD^T = \bigl(CB^{-1}\bigr)\bigl(BB^T\bigr)\bigl(CB^{-1}\bigr)^T = CB^{-1}BB^T\bigl(B^{-1}\bigr)^TC^T = CC^T = N.$$

The mapping T is triangular, hence it is the unique canonical triangular mapping transforming γ into μ. □

We now obtain some preliminary estimate for the coefficient K in (1).

Theorem 1. *Let γ be the standard Gaussian measure on $\mathbb{R}^n$ and let M be a positive definite matrix with eigenvalues μ_i such that its determinant is equal to 1. Then there is an orthonormal basis in $\mathbb{R}^n$ such that the centered Gaussian measure μ with the covariance M in this new basis has the following property: for the optimal linear mapping T_0 that transforms γ into μ (which is $\sqrt{M}$ in this basis) and for the canonical triangular transformation T of γ to μ (with respect to the new basis) one has*

$$K \geq 1 + 2\frac{\sum_{i=1}^{n}(\lambda_i - 1)}{\sum_{i=1}^{n}(\lambda_i - 1)^2}, \tag{2}$$

where the numbers λ_i are all eigenvalues of the matrix A of the mapping T_0 in the standard basis, i.e., $\lambda_i = \sqrt{\mu_i}$.

Proof. The optimal mapping T_0 is linear by Lemma 1. The corresponding matrix is $A = \sqrt{M}$. Of course, it is diagonal in some basis, but it can fail to be triangular in the standard basis. It follows that

$$\sum_{i,j=1}^{n} a_{ij}^2 = \operatorname{tr} M. \tag{3}$$

Let us consider the matrix M. We shall construct a sequence of orthogonal operators transforming it to a form where all principal minors are equal to 1. There exists an orthogonal matrix U which reduces the matrix M to the diagonal form

$$U^{-1}\begin{pmatrix} \mu_1 & 0 & \dots & 0 \\ 0 & \mu_2 & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & \mu_n \end{pmatrix} U = M, \quad U^{-1} = U^T.$$

Without loss of generality we can assume that all diagonal elements μ_i are different from 1 for $i = 1, \dots, n$ and the product of any subset of these elements is different from 1 except for the product $\mu_1\mu_2\dots\mu_n$. Otherwise we apply the orthogonal transformations described below to the corresponding subspaces.

Inductively we consider the rotations of the pairs of coordinate vectors reducing the diagonal matrix to the form with unit principal minors. Without loss of generality we assume that $\mu_1 > 1 > \mu_n$ (which can always be obtained by the permutation of coordinate vectors taking for μ_1 and μ_n the maximal and the minimal values among all μ_i, respectively). The rotation in the (x_1, x_n)-plane by the angle φ leads to

$$\begin{pmatrix} \cos\varphi & 0 & \dots & 0 & \sin\varphi \\ 0 & 1 & & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & & 1 & 0 \\ -\sin\varphi & 0 & \dots & 0 & \cos\varphi \end{pmatrix}\begin{pmatrix} \mu_1 & 0 & \dots & 0 & 0 \\ 0 & \mu_2 & & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & & \mu_{n-1} & 0 \\ 0 & 0 & \dots & 0 & \mu_n \end{pmatrix} \times$$

$$\times \begin{pmatrix} \cos\varphi & 0 & \dots & 0 & -\sin\varphi \\ 0 & 1 & & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & & 1 & 0 \\ \sin\varphi & 0 & \dots & 0 & \cos\varphi \end{pmatrix}.$$

After multiplication we have

$$\begin{pmatrix} \mu_1\cos^2\varphi+\mu_n\sin^2\varphi & 0 & \dots & 0 & (\mu_n-\mu_1)\sin\varphi\cos\varphi \\ 0 & \mu_2 & & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & & \mu_{n-1} & 0 \\ (\mu_n-\mu_1)\sin\varphi\cos\varphi & 0 & \dots & 0 & \mu_1\sin^2\varphi+\mu_n\cos^2\varphi \end{pmatrix}.$$

For the equality $\mu_1\cos^2\varphi+\mu_n\sin^2\varphi=1$ we see that it is sufficient to take

$$\operatorname{tg}\varphi=\sqrt{\frac{\mu_1-1}{1-\mu_n}}.$$

Let now the first $i-1$ principal minors of the covariance matrix be equal to 1. By a suitable permutation of coordinate vectors we obtain $\mu_i>1>\mu_1\dots\mu_{i-1}\mu_n$ or $\mu_i<1<\mu_1\dots\mu_{i-1}\mu_n$. Rotating in the (x_i,x_n) plain we have ($\widetilde{\mu}$ is the element in the lower right angle of the matrix calculated after all previous rotations)

$$\left(\begin{array}{ccc|cc} 1 & & 0 & 0 & 0 \\ & \ddots & & \vdots & \vdots \\ 0 & & 1 & 0 & 0 \\ \hline 0 & \dots & 0 & \cos\varphi & \sin\varphi \\ 0 & \dots & 0 & -\sin\varphi & \cos\varphi \end{array}\right)\left(\begin{array}{ccc|cc} m_{11} & \dots & m_{1,i-1} & 0 & m_{1n} \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ m_{1,i-1} & \dots & m_{i-1,i-1} & 0 & m_{i-1,n} \\ \hline 0 & \dots & 0 & \mu_i & 0 \\ m_{1n} & \dots & m_{i-1,n} & 0 & \widetilde{\mu} \end{array}\right)\times$$

$$\times\left(\begin{array}{ccc|cc} 1 & & 0 & 0 & 0 \\ & \ddots & & \vdots & \vdots \\ 0 & & 1 & 0 & 0 \\ \hline 0 & \dots & 0 & \cos\varphi & -\sin\varphi \\ 0 & \dots & 0 & \sin\varphi & \cos\varphi \end{array}\right),$$

which can be written as

$$\left(\begin{array}{ccc|cc} m_{11} & \dots & m_{1,i-1} & m_{1n}\sin\varphi & m_{1n}\cos\varphi \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ m_{1,i-1} & \dots & m_{i-1,i-1} & m_{i-1,n}\sin\varphi & m_{i-1,n}\cos\varphi \\ \hline m_{1n}\sin\varphi & \dots & m_{i-1,n}\sin\varphi & \mu_i\cos^2\varphi+\widetilde{\mu}\sin^2\varphi & (\widetilde{\mu}-\mu_i)\sin\varphi\cos\varphi \\ m_{1n}\cos\varphi & \dots & m_{i-1,n}\cos\varphi & (\widetilde{\mu}-\mu_i)\sin\varphi\cos\varphi & \mu_i\sin^2\varphi+\widetilde{\mu}\cos^2\varphi \end{array}\right).$$

The variables $x_{i+1},\dots,x_{n-1}$ that are not employed in rotations are omitted in this formula.

We need φ such that the following determinant is equal to 1:

$$D=\begin{vmatrix} m_{11} & \dots & m_{1,i-1} & m_{1n}\sin\varphi \\ \vdots & \ddots & \vdots & \vdots \\ m_{1,i-1} & \dots & m_{i-1,i-1} & m_{i-1,n}\sin\varphi \\ m_{1n}\sin\varphi & \dots & m_{i-1,n}\sin\varphi & \mu_i\cos^2\varphi+\widetilde{\mu}\sin^2\varphi \end{vmatrix}.$$

Under the inductive hypothesis, the principal minor M_{i-1} of order $(i-1)\times(i-1)$ is equal to 1. The minor corresponding to the subspace $(x_1,\dots,x_{i-1},x_n)$ is invariant under all considered rotations, therefore, the following equality holds:

$$\begin{vmatrix} m_{11} & \dots & m_{1,i-1} & m_{1n} \\ \vdots & \ddots & \vdots & \vdots \\ m_{1,i-1} & \dots & m_{i-1,i-1} & m_{i-1,n} \\ m_{1n} & \dots & m_{i-1,n} & \widetilde{\mu} \end{vmatrix}=\mu_1\cdots\mu_{i-1}\mu_n.$$

We expand the obtained determinant along the last row:

$$\mu_1 \cdots \mu_{i-1}\mu_n = \widetilde{\mu} M_{i-1} + \sum_{k=1}^{i-1} m_{kn} M_{i-1,k} = \widetilde{\mu} + \sum_{k=1}^{i-1} m_{kn} M_{i-1,k},$$

where $M_{i-1,k}$ are the adjugate minors of order $(i-1)\times(i-1)$ with the corresponding signs. We expand the determinant D along the last row as well:

$$D = (\mu_i \cos^2\varphi + \widetilde{\mu}\sin^2\varphi) M_{i-1} + \sum_{k=1}^{i-1} (m_{kn}\sin\varphi)(M_{i-1,k}\sin\varphi)$$

$$= \mu_i \cos^2\varphi + \Big(\widetilde{\mu} + \sum_{k=1}^{i-1} m_{kn} M_{i-1,k}\Big)\sin^2\varphi = \mu_i\cos^2\varphi + \mu_1\cdots\mu_{i-1}\mu_n \sin^2\varphi.$$

Similarly to the very first rotation we take

$$\operatorname{tg}\varphi = \sqrt{\frac{\mu_i - 1}{1 - \mu_1\cdots\mu_{i-1}\mu_n}}.$$

Eventually we obtain the matrix M with unit principal minors and an orthogonal basis in which this matrix has this form.

By Lemma 1 the canonical triangular transformation of the standard Gaussian measure γ into the measure μ (with respect to the obtained orthogonal basis) is a linear operator T defined by a lower triangular matrix B in the obtained orthogonal basis, B has positive elements on the diagonal and

$$M = BB^T \quad \text{and} \quad \operatorname{tr} M = \sum_{i,j=1}^{n} b_{ij}^2. \tag{4}$$

We prove that all diagonal elements of B are equal to 1. Let us express them through the principal minors of M. The matrix B has the form

$$B = \begin{pmatrix} b_{11} & 0 & \dots & 0 \\ b_{12} & b_{22} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ b_{1n} & b_{2n} & \dots & b_{nn} \end{pmatrix}.$$

We denote its principal $(i\times i)$-minor by B_i. According to (4), the principle minor M_i of the matrix M is equal to the determinant of the following matrix product:

$$\begin{pmatrix} b_{11} & 0 & \dots & 0 \\ b_{12} & b_{22} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ b_{1i} & b_{2i} & \dots & b_{ii} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1i} \\ 0 & b_{22} & \dots & b_{2i} \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & b_{ii} \end{pmatrix},$$

hence $M_i = B_i^2$. Since B is lower triangular, B_i is equal to the product of its diagonal elements, so the following equalities hold:

$$M_1 = b_{11}^2, \quad M_2 = b_{11}^2 b_{22}^2, \ \dots\ , \quad M_n = |M| = b_{11}^2\cdots b_{nn}^2.$$

Hence the diagonal elements of the matrix B are represented as follows:

$$b_{11} = \sqrt{M_1}, \quad b_{22} = \frac{\sqrt{M_2}}{\sqrt{M_1}}, \ \dots\ , \quad b_{nn} = \frac{\sqrt{M_n}}{\sqrt{M_{n-1}}}.$$

All M_i are equal to 1, therefore,

$$b_{11} = b_{22} = \cdots = b_{nn} = 1.$$

Thus, we have a basis in which both linear operators T_0 and T are represented by the matrices A and B, respectively, and B has unit diagonal elements. Estimate (1) takes the form

$$\int_{\mathbb{R}^n} |Bx - x|^2 \, \gamma(dx) \le K \int_{\mathbb{R}^n} |Ax - x|^2 \, \gamma(dx). \tag{5}$$

Let us calculate the value of the transportation cost for an arbitrary linear operator represented by a matrix C in some orthogonal basis of the space $\mathbb{R}^n$:

$$\int_{\mathbb{R}^n} |Cx - x|^2 \, \gamma(dx) = \operatorname{tr}(C^T C) - 2 \operatorname{tr} C + n = \sum_{i,j=1}^{n} c_{ij}^2 - 2 \sum_{i=1}^{n} c_{ii} + n.$$

Given (3) and (4), let us calculate the values of the transportation cost for the mappings T and T_0:

$$\int_{\mathbb{R}^n} |Ax - x|^2 \, \gamma(dx) = \operatorname{tr} M - 2 \operatorname{tr} A + n,$$

$$\int_{\mathbb{R}^n} |Bx - x|^2 \, \gamma(dx) = \operatorname{tr} M - 2 \operatorname{tr} B + n = \operatorname{tr} M - n.$$

It is worth noting that the first formula with $A = \sqrt{M}$ is a particular case of a more general formula for the quadratic Kantorovich–Rubinstein distance between two arbitrary Gaussian measures on $\mathbb{R}^n$ (see [15] and [12]). It will be important below that this formula and the first equality in the second formula hold for arbitrary linear operators taking γ to μ, but the optimal mapping has the maximal trace among such operators.

Substituting the obtained values into (5), we arrive at the following estimate for the constant K:

$$K \ge \frac{\operatorname{tr} M - 2 \operatorname{tr} B + n}{\operatorname{tr} M - 2 \operatorname{tr} A + n} = \frac{\operatorname{tr} M - n}{\operatorname{tr} M - 2 \operatorname{tr} A + n} = 1 + 2 \frac{\operatorname{tr} A - n}{\operatorname{tr} M - 2 \operatorname{tr} A + n}. \tag{6}$$

Since the trace of a matrix is invariant under orthogonal changes of variables, we calculate the traces in the orthogonal basis in which A has a diagonal form with eigenvalues $\lambda_1, \ldots, \lambda_n$ on the diagonal. Then

$$\operatorname{tr} A = \sum_{i=1}^{n} \lambda_i, \quad \operatorname{tr} M = \operatorname{tr}(A^2) = \sum_{i=1}^{n} \lambda_i^2.$$

Substituting into (6) we have

$$K \ge 1 + 2 \frac{\sum_{i=1}^n \lambda_i - n}{\sum_{i=1}^n \lambda_i^2 - 2 \sum_{i=1}^n \lambda_i + n} = 1 + 2 \frac{\sum_{i=1}^n (\lambda_i - 1)}{\sum_{i=1}^n (\lambda_i - 1)^2},$$

as announced. □

Let now μ be the centered Gaussian measure with a nondegenerate covariance matrix M. One can obtain this measure from the centered Gaussian measure μ_1 with a covariance matrix M_1 such that $\det M_1 = 1$. It is sufficient to multiply all the elements of M by $|M|^{-1/n}$ and take the resulting matrix for M_1. Then μ_1 is transformed into μ by the homothety with the ratio $\alpha = (\det M)^{1/(2n)}$. We now use Theorem 1 and the orthogonal basis constructed there. The linear optimal mapping transforming γ into μ_1 is given by the symmetric matrix A_1 with unit determinant defined by $A_1^2 = M_1$. The matrix A obtained by multiplying all the elements of A_1 by α is also symmetric and satisfies the equality $A^2 = M$, so A is the matrix of the optimal mapping of γ into μ (the proof of this statement replicates the corresponding part of Lemma 1). The canonical triangular mapping of γ into μ_1 is given by a lower triangular matrix B_1 in the constructed orthogonal basis. Then we multiply all the elements of B_1 by α and obtain the matrix of the canonical triangular mapping that transforms γ into μ. The trace of this matrix is equal to $n\alpha$. The traces of the matrices A and M are represented by the eigenvalues $\lambda_1, \dots, \lambda_n$ of the matrix A_1 as follows:

$$\operatorname{tr} A = \alpha \sum_{i=1}^n \lambda_i, \quad \operatorname{tr} M = \alpha^2 \sum_{i=1}^n \lambda_i^2.$$

We substitute these expressions into (6) for the coefficient K:

$$K \geq 1 + 2\frac{\operatorname{tr} A - \operatorname{tr} B}{\operatorname{tr} M - 2\operatorname{tr} A + n} = 1 + 2\frac{\sum_{i=1}^n \alpha\lambda_i - n\alpha}{\sum_{i=1}^n (\alpha\lambda_i)^2 - 2\sum_{i=1}^n \alpha\lambda_i + n} = 1 + 2\frac{\alpha\sum_{i=1}^n (\lambda_i - 1)}{\sum_{i=1}^n (\alpha\lambda_i - 1)^2}. \tag{7}$$

In the next theorem we investigate the possible values of the expression in (7) and find a lower bound for the coefficient K. We also show that K is increasing to infinity when n increases. Thus, this theorem implies that in the general case the triangular mapping and the Monge optimal mapping are not comparable for the considered cost function. Before proving this result, let us observe that if $\|M\| \geq 4$, then for an arbitrary linear operator L transforming the standard Gaussian measure γ on $\mathbb{R}^n$ to the measure with covariance M we have

$$\int_{\mathbb{R}^n} |Lx - x|^2 \, \gamma(dx) \leq (5n+4) \int_{\mathbb{R}^n} |T_0 x - x|^2 \, \gamma(dx),$$

where $T_0 = \sqrt{M}$ is the optimal mapping. Indeed, we know that the left-hand side equals $\operatorname{tr} M - 2\operatorname{tr} L + n$ and $L^T L = M$, so $\|L\|^2 = \|M\|$. Then $2|\operatorname{tr} L| \leq 2n\|M\|^{1/2}$ and the left-hand side does not exceed $\operatorname{tr} M + 2n\|M\|^{1/2} + n$, which is estimated by the right-hand side, since $M^{1/2} - I \geq 2^{-1}M^{1/2}$ and $\|M\|^{1/2} \leq \operatorname{tr} M/2$.

On the other hand, let γ be the same standard Gaussian measure on $\mathbb{R}^n$ and let μ be the centered Gaussian measure whose covariance matrix M has eigenvalues $\mu_1, \dots \mu_n$. Then the entropy equals

$$\int \rho \log \rho \, d\gamma = \int \log \rho \, d\mu = 2^{-1} \sum_{i=1}^n (\mu_i - 1 - \log \mu_i),$$

where $\rho = d\mu/d\gamma$. Indeed, we can assume that M is diagonal, then

$$\log \rho = \sum_{i=1}^n (-2^{-1} \log \mu_i + 2^{-1} x_i^2 (1 - 1/\mu_i)),$$

where the integral of x_i^2 with respect to μ equals μ_i. If $|\mu_i - 1| \leq 1/2$, then

$$\int \rho \log \rho \, d\gamma \leq 2 \sum_{i=1}^n (\mu_i - 1)^2 \leq \frac{25}{2} \sum_{i=1}^n (\mu_i^{1/2} - 1)^2,$$

since $(\mu_i - 1)^2 \leq 25(\mu_i^{1/2} - 1)^2/4$. On the right we have the cost of the optimal transportation $T_0 = M^{1/2}$. Hence by the Talagrand inequality we obtain

$$\int |Tx - x|^2 \, \gamma(dx) \leq 25 \int |T_0 x - x|^2 \, \gamma(dx) \quad \text{if } \|M - I\| \leq 1/2. \tag{8}$$

As we shall now see the bound on the norm of M is important.

Theorem 2. *Let γ be the standard Gaussian measure on $\mathbb{R}^n$. Suppose that K is a constant such that the inequality*

$$\int_{\mathbb{R}^n} |T(x) - x|^2 \, \gamma(dx) \leq K \int_{\mathbb{R}^n} |T_0(x) - x|^2 \, \gamma(dx)$$

holds for every nondegenerate centered Gaussian measure μ on $\mathbb{R}^n$ for which T is the canonical triangular mapping and T_0 is the optimal mapping transforming γ into μ. Then

$$K \geq n + \sqrt{n^2 - n}.$$

Moreover, even if we consider only measures μ with the determninant of the covariance matrix equal to 1, the coefficient K cannot be smaller than $\sqrt{n}$.

Proof. The last assertion follows directly from Theorem 1 by taking the matrix M such that $\lambda_i = \sqrt{\mu_i} = 1 + n^{-1/2}$ for $i = 1, \ldots, n-1$ and $\lambda_n = \sqrt{\mu_n} = (1 + n^{-1/2})^{1-n}$. Note that $(1 + n^{-1/2})^n = \Bigl((1 + n^{-1/2})^{n^{1/2}}\Bigr)^{n^{1/2}} = (e + o(1))^{n^{1/2}}$. Hence

$$\sum_{i=1}^{n} (\lambda_i - 1) = (n-1)n^{-1/2} - 1 + o(1), \quad \sum_{i=1}^{n} (\lambda_i - 1)^2 = (n-1)n^{-1} + 1 + o(1),$$

so the ratio is $n^{1/2} + o(1)$.

Let us show that in the general case the ratio can be even larger. We already know that the mappings T_0 and T are linear and their matrices A (which is positive definite) and B satisfy the conditions

$$A^2 = BB^T = M,$$

where M is the covariance matrix of the measure μ. Let A_1 be the matrix with unit determinant obtained from the matrix A by dividing all its elements by the number $\alpha = (\det M)^{1/(2n)}$. Then A_1 is symmetric. Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of the matrix A_1. We maximize the quantity

$$F_{\lambda_1,\ldots,\lambda_n}(\alpha) = \frac{\alpha \sum_{i=1}^{n} (\lambda_i - 1)}{\sum_{i=1}^{n} (\alpha\lambda_i - 1)^2} \tag{9}$$

as the function of α with fixed λ_k, $k = 1, \ldots, n$.

First we note that the case where all λ_i are equal is not meaningful, since if the product of all numbers λ_i is equal to 1 (it is the determinant of A_1), then $\lambda_i = 1$ for all $i = 1, \ldots, n$. Then the function (8) is undefined when $\alpha = 1$, which is explained by the fact that the measure γ is transformed into itself by the linear mapping with the matrix A, so the optimal mapping is identical and the transportation cost equals 0. This case is trivial.

Let us assume that there are at least two different λ_i, so the function $F_{\lambda_1,\ldots,\lambda_n}(\alpha)$ is defined and differentiable for all $\alpha > 0$. Note that its limits as $\alpha \to 0$ and $\alpha \to \infty$ are equal to 0, hence it is bounded on $(0, +\infty)$ and has an extremum on this interval. The derivative has the form

$$F'_{\lambda_1,\ldots,\lambda_n}(\alpha) = \frac{\Bigl(\sum_{i=1}^{n} (\alpha\lambda_i - 1)^2 - \alpha \sum_{i=1}^{n} 2\lambda_i(\alpha\lambda_i - 1)\Bigr) \sum_{i=1}^{n} (\lambda_i - 1)}{\Bigl(\sum_{i=1}^{n} (\alpha\lambda_i - 1)^2\Bigr)^2} = 0. \tag{10}$$

Not all numbers λ_i are equal, hence

$$\sum_{i=1}^{n} (\lambda_i - 1) = \sum_{i=1}^{n} \lambda_i - n > 0,$$

since we have

$$\frac{1}{n} \sum_{i=1}^{n} \lambda_i \geq \Bigl(\prod_{i=1}^{n} \lambda_i\Bigr)^{1/n} = 1,$$

where the equality is only possible when all numbers λ_i are equal.

Let us return to equation (9). After simplifications we obtain

$$\begin{aligned}\sum_{i=1}^{n}(\alpha\lambda_i-1)^2-\alpha\sum_{i=1}^{n}2\lambda_i(\alpha\lambda_i-1)&=\sum_{i=1}^{n}(\alpha^2\lambda_i^2-2\alpha\lambda_i+1-2\alpha^2\lambda_i^2+2\alpha\lambda_i)\\&=\sum_{i=1}^{n}(1-\alpha^2\lambda_i^2)=n-\alpha^2\sum_{i=1}^{n}\lambda_i^2=0,\end{aligned}$$

hence the expected extremum point is

$$\alpha_0=\frac{\sqrt{n}}{\sqrt{\sum_{i=1}^n\lambda_i^2}}.$$

The derivative $F'_{\lambda_1,\dots,\lambda_n}(\alpha)$ is positive if $0<\alpha<\alpha_0$ and is negative if $\alpha>\alpha_0$, therefore, the maximum of $F_{\lambda_1,\dots,\lambda_n}(\alpha)$ over $\alpha>0$ with fixed $\lambda_1,\dots,\lambda_n$ is achieved when $\alpha=\alpha_0$:

$$\begin{aligned}F_{\lambda_1,\dots,\lambda_n}(\alpha_0)&=\frac{\sum_{i=1}^n(\lambda_i-1)}{\alpha_0\sum_{i=1}^n\lambda_i^2-2\sum_{i=1}^n\lambda_i+n/\alpha_0}\\&=\frac{\sum_{i=1}^n(\lambda_i-1)}{\sqrt{n\sum_{i=1}^n\lambda_i^2}-2\sum_{i=1}^n\lambda_i+\sqrt{n\sum_{i=1}^n\lambda_i^2}}=\frac{\sum_{i=1}^n(\lambda_i-1)}{2\Big(\sqrt{n\sum_{i=1}^n\lambda_i^2}-\sum_{i=1}^n\lambda_i\Big)}.\end{aligned}$$

Then estimate (7) takes the form

$$K\ge 1+2\cdot\frac{\sum_{i=1}^n\lambda_i-n}{2\Big(\sqrt{n\sum_{i=1}^n\lambda_i^2}-\sum_{i=1}^n\lambda_i\Big)}=\frac{\sqrt{n\sum_{i=1}^n\lambda_i^2}-n}{\sqrt{n\sum_{i=1}^n\lambda_i^2}-\sum_{i=1}^n\lambda_i}.\tag{11}$$

We now consider the following special case. Let $\lambda_1=\dots=\lambda_{n-1}=\lambda,\ \lambda_n=\lambda^{1-n}$. Then the right-hand side in (10) takes has the form

$$\begin{aligned}\frac{\sqrt{n\sum_{i=1}^n\lambda_i^2}-n}{\sqrt{n\sum_{i=1}^n\lambda_i^2}-\sum_{i=1}^n\lambda_i}&=\frac{\sqrt{n(n-1)\lambda^2+n\lambda^{2-2n}}-n}{\sqrt{n(n-1)\lambda^2+n\lambda^{2-2n}}-(n-1)\lambda-\lambda^{1-n}}\\&=\frac{\sqrt{n(n-1)+n\lambda^{-2n}}-n\lambda^{-1}}{\sqrt{n(n-1)+n\lambda^{-2n}}-(n-1)-\lambda^{-n}}.\end{aligned}$$

Letting $\lambda\to\infty$, we obtain the limit

$$\frac{\sqrt{n(n-1)}}{\sqrt{n(n-1)}-(n-1)}=\frac{\sqrt{n}}{\sqrt{n}-\sqrt{n-1}}=n+\sqrt{n^2-n}.$$

Inequality (10) holds for all λ, hence

$$K\ge n+\sqrt{n^2-n},$$

which completes the proof. □

Thus, for the linear images of the measure γ the coefficient K in the inequality

$$\int_{\mathbb{R}^n}|T(x)-x|^2\,\gamma(dx)\le K\int_{\mathbb{R}^n}|T_0(x)-x|^2\,\gamma(dx)$$

has a lower bound $n+\sqrt{n^2-n}$, so it increases to infinity when n tends to infinity. In addition, in the class of image measures with unit determinant of covariance matrices the coefficient cannot be made smaller than $n^{1/2}$.

This means that in the Monge problem for the linear images of Gaussian measures on infinite-dimensional spaces with the cost function defined by the Euclidean distance the value of the transportation cost for the triangular mapping cannot be estimated by the minimal value with any constant (about infinite-dimensional Monge problems, see [7], [10], [11], [13], [19]). However, some positive result holds. Let γ be the measure on the space $\mathbb{R}^\infty$ of all sequences that is the countable power of the standard Gaussian measure on the real line. Its Cameron–Martin space is the classical Hilbert space $H = l^2$. For Gaussian measures μ equivalent to γ it is natural to consider the Monge and Kantorovich problems corresponding to the cost function $|x-y|_H^2 = \sum_{i=1}^\infty (x_i - y_i)^2$. It is known (see [7]) that for μ equivalent to γ the cost is finite and there is a measurable linear optimal operator T_0 taling γ to μ and minimizing the value

$$\int |T_0 x - x|_H^2 \,\gamma(dx).$$

A typical example of μ equivalent to γ is this: $\mu = \gamma \circ A^{-1}$, where $A = I + B$ and B is the measurable linear operator with values H generated by a Hilbert–Schmidt operator $B_0\colon H \to H$ with the Hilbert–Schmidt norm less than 1. The operator B is defined by $Bx = \sum_{i=1}^\infty x_i B_0 e_i$, where the series converges γ-almost everywhere (this converges follows from the assumption that B_0 is Hilbert–Schmidt). From the considered finite-dimensional case and the known method of constructing triangular transformations in infinite dimensions (see [9]) we obtain the following assertion.

Corollary 1. *Suppose that the Hilbert–Schmidt norm of B_0 does not exceed $1/2$. Then the measure $\mu = \gamma \circ (I+B)^{-1}$ is equivalent to γ and has finite entropy, there is a measurable linear triangular mapping T taking γ to μ and for the optimal operator T_0 there holds estimate (8) along with Talagrand's estimates of the integrals of $|T_0 x - x|_H^2$ and $|Tx - x|_H^2$ by $2\mathrm{Ent}_\gamma(\mu)$.*

Note that if $|\mu_i - 1| < q$, where $q < 1$, then (8) holds with another constant depending on q.

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Department of Mechanics and Mathematics, Moscow State University, Moscow 119991, Russia
E-mail address: d.b.bukin@gmail.com

Russian Institute for Scientific and Technical Information, Usievicha 20, Moscow 125190, Russia
E-mail address: ekrugo@mail.ru