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ON A PROPERTY OF JOINT TERMINAL DISTRIBUTIONS OF LOCALLY INTEGRABLE INCREASING PROCESSES AND THEIR COMPENSATORS

In this paper we prove that a joint distribution of a locally integrable increasing process X° and its compensator A° at a terminal moment of time can be realized as a joint terminal distribution of another locally integrable increasing process $X^\star$ and its compensator $A^\star$, $A^\star$ being continuous.

1. Introduction

In [2] a class $\mathbb{W}$ of probability measures on the space $\big(\mathbb{R}^2_+,\, \mathcal{B}(\mathbb{R}^2_+)\big)$ is introduced. It includes all measures μ satisfying the following conditions:

1) $\int_{\mathbb{R}^2_+} (x+y)\,\mu(dx,\,dy) < \infty$,
2) $\int_{\mathbb{R}^2_+} x\,\mu(dx,\,dy) = \int_{\mathbb{R}^2_+} y\,\mu(dx,\,dy)$,
3) $\forall \lambda \ge 0\colon \int_{\{y\le\lambda\}} x\,\mu(dx,\,dy) \le \int_{\mathbb{R}^2_+} (y\wedge\lambda)\,\mu(dx,\,dy)$.

It is shown in [2] that the joint distribution of terminal values of an integrable increasing process and its compensator belongs to the class $\mathbb{W}$. Conversely, given $\mu \in \mathbb{W}$ there is constructed an increasing integrable process such that the joint distribution of terminal values of the process and its compensator is μ and, moreover, the compensator is continuous. Thus, if $X^\circ = (X^\circ_t)_{t\in[0;\,\infty)}$ is an integrable increasing process with a compensator $A^\circ = (A^\circ_t)_{t\in[0;\,\infty)}$, one can define on a certain stochastic basis another integrable increasing process $X^\star = (X^\star_t)_{t\in[0;\,\infty)}$ with a compensator $A^\star = (A^\star_t)_{t\in[0;\,\infty)}$, such that

$$\mathrm{Law}\begin{bmatrix} X^\star_\infty \\ A^\star_\infty \end{bmatrix} = \mathrm{Law}\begin{bmatrix} X^\circ_\infty \\ A^\circ_\infty \end{bmatrix}. \tag{1}$$

Moreover, the compensator $A^\star$ is continuous.

The main goal of the article is to extend the last statement to the locally integrable case. Namely, we state the following theorem.

Theorem 1.1. *For any locally integrable increasing process $X^\circ = (X^\circ_t)_{t\in[0;\,\infty)}$ with a compensator $A^\circ = (A^\circ_t)_{t\in[0;\,\infty)}$ on some stochastic basis there exists another locally integrable increasing process $X^\star = (X^\star_t)_{t\in[0;\,\infty)}$ with a compensator $A^\star = (A^\star_t)_{t\in[0;\,\infty)}$, such that relation (1) holds, as well as $A^\star$ is continuous.*

Let us clarify that a complete description of the class of possible distributions of a random vector $(X^\circ_\infty,\, A^\circ_\infty)$ is not known yet in the locally integrable case. Condition 3) is still necessary but its sufficiency is proved under an additional assumption that replaces conditions 1) and 2) (see Proposition 3.6 [2]). However, this additional assumption is not necessary (see [1]). We believe that our Theorem 1.1 sheds extra light on this problem and can simplify its solution.

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As an example of the problem where Theorem 1.1 simplifies the solution, we can mention a well-known result on the sets of convergence of an increasing process and its compensator, see e.g. [5], Theorem 1, § 6, Ch. 2. Namely, the theorem says that for every locally integrable increasing process X with a compensator A one has

$$\big\{A_\infty < \infty\big\} \subseteq \big\{X_\infty < \infty\big\} \quad \mathbb{P}\text{-a.s.} \tag{2}$$

In view of Theorem 1.1, it is enough to consider the case where A is continuous. The reader can look at the proof of relation (2) and make sure that it is much simpler under the assumption of continuity of A.

Before passing to formal constructions from sections 2–4, let us sketch the idea of the proof of Theorem 1.1. We exploit a special construction proposed in [2]. In the proof of Theorem 2.1 [2], given a measure μ, the author constructs on a probability space $(\Omega^{\mathfrak{b}},\, \mathcal{F}^{\mathfrak{b}},\, \mathbb{P}^{\mathfrak{b}})$, where $\mathbb{P}^{\mathfrak{b}}$ depends on μ, an integrable increasing process $X = (X_t)_{t\in[0;\,1]}$ with a continuous compensator $A = (A_t)_{t\in[0;\,1]}$, such that $\mathrm{Law}(X_1,\, A_1) = \mu$. We should mention that the value of $X_t(\omega^{\mathfrak{b}})$ also depends on μ.

Developing the construction mentioned above, let us consider a case, where, instead of measure μ, a Markov kernel Q from a measurable space $(\Omega^{\mathfrak{a}},\, \mathcal{F}^{\mathfrak{a}})$ to $(\mathbb{R}^2_+,\, \mathcal{B}(\mathbb{R}^2_+))$ is given such that, for any fixed $\omega^{\mathfrak{a}} \in \Omega^{\mathfrak{a}}$, the measure $\mathrm{Q}(\omega^{\mathfrak{a}};\,\cdot\,) \in \mathbb{W}$. Taking the measure $\mathrm{Q}(\omega^{\mathfrak{a}};\,\cdot\,)$ in place of μ we apply the mentioned construction, for any $\omega^{\mathfrak{a}} \in \Omega^{\mathfrak{a}}$, in order to build a transition measure $\mathbb{P}^{\mathfrak{a},\mathfrak{b}}(\omega^{\mathfrak{a}};\, d\omega^{\mathfrak{b}})$ and stochastic functions $X_t(\omega^{\mathfrak{a}},\, \omega^{\mathfrak{b}})$ and $A_t(\omega^{\mathfrak{a}},\, \omega^{\mathfrak{b}})$, $t \in [0;\, 1]$. Their measurability as well as other required properties is a subject of separate investigation presented in Lemmas 2.1 and 2.2.

We start the proof of Theorem 1.1 with observing that after some special time-change one can assume that $\mathbb{E}[X^\circ_t] < \infty$, for any $t \geq 0$. The main step of the proof starts with a construction of a process $X^\star_t$, $t \in [0;\, 1]$, with continuous compensator $(A^\star_t)_{t\in[0;\,1]}$, such that $\mathrm{Law}(X^\star_1,\, A^\star_1) = \mathrm{Law}(X^\circ_1,\, A^\circ_1)$, on the space $(\Omega^{\mathfrak{b}},\, \mathcal{F}^{\mathfrak{b}},\, \mathbb{P}^{\mathfrak{b}})$. Further, let us define a Markov kernel Q as a regular conditional distribution of the random vector $(X^\circ_{n+1} - X^\circ_n,\, A^\circ_{n+1} - A^\circ_n)$ under condition $(X^\circ_n,\, A^\circ_n) = \omega^{\mathfrak{a}} \in \Omega^{\mathfrak{a}} = \mathbb{R}^2_+$, $n \geq 1$. The construction from the previous paragraph allows us to construct stochastic processes $X = (X_t)_{t\in[0;\,1]}$ and $A = (A_t)_{t\in[0;\,1]}$, on the space $(\Omega,\, \mathcal{F}) := (\Omega^{\mathfrak{a}} \times \Omega^{\mathfrak{b}},\, \mathcal{F}^{\mathfrak{a}} \otimes \mathcal{F}^{\mathfrak{b}})$. Let us also observe that the conditional distribution of $\omega^{\mathfrak{b}}$ given $\omega^{\mathfrak{a}}$ is $\mathbb{P}^{\mathfrak{a},\mathfrak{b}}(\omega^{\mathfrak{a}};\, d\omega^{\mathfrak{b}})$.

If the process $(X^\star_t)_{t\in[0;\,n]}$ is already set up on some measurable space in such a way that it has a continuous compensator and $\mathrm{Law}(X^\star_n,\, A^\star_n) = \mathrm{Law}(X^\circ_n,\, A^\circ_n)$, then, on a product of this space and the space $(\Omega,\, \mathcal{F})$, we extend the pair of processes $X^\star$ and $A^\star$ on segment $[n;\, n+1]$ by formulae $X^\star_{n+t} = X^\star_n + X_t$ and $A^\star_{n+t} = A^\star_n + A_t$. Then we prove the existence of required stochastic basis and a pair of processes $X^\star = (X^\star_t)_{t\in[0;\,\infty)}$ and $A^\star = (A^\star_t)_{t\in[0;\,\infty)}$ by applying the Ionescu-Tulcea theorem.

We organize the paper in the following way. Section 2 presents the detailed construction considered above, which is required for the proof of Theorem 1.1. Section 3 contains the proof of Theorem 1.1. In the final Section 4 we give proofs of auxiliary statements.

2. Auxiliary construction

In this section we focus on the proof of Lemma 2.2, which serves as the core of the proof of the main theorem. We start with the following auxiliary proposition.

Lemma 2.1. *Consider an arbitrary measurable space* $(\Omega^{\mathfrak{a}},\, \mathcal{F}^{\mathfrak{a}})$ *and a Markov kernel* Q *acting from* $(\Omega^{\mathfrak{a}},\, \mathcal{F}^{\mathfrak{a}})$ *to* $(\mathbb{R}^2_+,\, \mathcal{B}(\mathbb{R}^2_+))$*, and satisfying* $\mathrm{Q}(\omega^{\mathfrak{a}};\,\cdot\,) \in \mathbb{W}$ *for all* $\omega^{\mathfrak{a}} \in \Omega^{\mathfrak{a}}$*. We put*

$$\Omega^{\mathfrak{b}} := [0;\, \infty] \times [0;\, \infty] \times [0;\, 1], \quad \mathcal{F}^{\mathfrak{b}} := \mathcal{B}(\Omega^{\mathfrak{b}}),$$

$$(\Omega,\, \mathcal{F}) := (\Omega^{\mathfrak{a}} \times \Omega^{\mathfrak{b}},\, \mathcal{F}^{\mathfrak{a}} \otimes \mathcal{F}^{\mathfrak{b}}). \tag{3}$$

Further, on the set

$$\Omega = \Big\{(\omega^{\mathfrak{a}}, y^{\mathfrak{b}}, x^{\mathfrak{b}}, u^{\mathfrak{b}})\colon\ \omega^{\mathfrak{a}} \in \Omega^{\mathfrak{a}},\ \ \omega^{\mathfrak{b}} = (y^{\mathfrak{b}}, x^{\mathfrak{b}}, u^{\mathfrak{b}}) \in \Omega^{\mathfrak{b}}\Big\},$$

we define functions

$$K(\omega^{\mathfrak{a}}, y^{\mathfrak{b}}, x^{\mathfrak{b}}, u^{\mathfrak{b}}) = x^{\mathfrak{b}} \quad \text{and} \quad L(\omega^{\mathfrak{a}}, y^{\mathfrak{b}}, x^{\mathfrak{b}}, u^{\mathfrak{b}}) = y^{\mathfrak{b}},$$

which are $\mathcal{F}$-measurable as marginal projections, and a Markov kernel $\mathbb{P}^{\mathfrak{a},\mathfrak{b}}$ from $\Omega^{\mathfrak{a}}$ to $(\Omega^{\mathfrak{b}}, \mathcal{F}^{\mathfrak{b}})$ by

$$\mathbb{P}^{\mathfrak{a},\mathfrak{b}}\big(\omega^{\mathfrak{a}};\, B_1 \times B_2 \times B_3\big) := \mathrm{Q}\big(\omega^{\mathfrak{a}};\, (B_2 \times B_1) \cap \mathbb{R}^2_+\big) \cdot \Lambda(B_3), \tag{4}$$

where $\omega^{\mathfrak{a}} \in \Omega^{\mathfrak{a}}$, $B_1 \times B_2 \times B_3 \in \mathcal{F}^{\mathfrak{b}}$, and Λ is the standard Lebesgue measure.

Then K and L possess the following property:

$$\mathbb{P}^{\mathfrak{a},\mathfrak{b}}\left(\omega^{\mathfrak{a}};\, \left\{\omega^{\mathfrak{b}}\colon\ \left[\begin{smallmatrix} K(\omega^{\mathfrak{a}}, \omega^{\mathfrak{b}}) \\ L(\omega^{\mathfrak{a}}, \omega^{\mathfrak{b}}) \end{smallmatrix}\right] \in C\right\}\right) = \mathrm{Q}(\omega^{\mathfrak{a}};\, C), \quad C \in \mathcal{B}(\mathbb{R}^2_+). \tag{5}$$

In addition, we can find an $\mathcal{F}$-measurable function $Z\colon \Omega \to [0; \infty]$, $Z = Z(\omega^{\mathfrak{a}}, \omega^{\mathfrak{b}})$, $\omega^{\mathfrak{a}} \in \Omega^{\mathfrak{a}}$, $\omega^{\mathfrak{b}} \in \Omega^{\mathfrak{b}}$, which meets, for any $\omega^{\mathfrak{a}} \in \Omega^{\mathfrak{a}}$, the following two requirements:

$$\forall\, \omega^{\mathfrak{b}} \in \Omega^{\mathfrak{b}}\colon\ 0 \le Z(\omega^{\mathfrak{a}}, \omega^{\mathfrak{b}}) \le K(\omega^{\mathfrak{a}}, \omega^{\mathfrak{b}}) \wedge L(\omega^{\mathfrak{a}}, \omega^{\mathfrak{b}}), \tag{6}$$

and, for all $\lambda \ge 0$,

$$\int_{\{\omega^{\mathfrak{b}}\,:\; L(\omega^{\mathfrak{a}}, \omega^{\mathfrak{b}}) - Z(\omega^{\mathfrak{a}}, \omega^{\mathfrak{b}}) \le \lambda\}} \big(K(\omega^{\mathfrak{a}}, \omega^{\mathfrak{b}}) - L(\omega^{\mathfrak{a}}, \omega^{\mathfrak{b}}) + \lambda\big)\, \mathbb{P}^{\mathfrak{a},\mathfrak{b}}\big(\omega^{\mathfrak{a}};\, d\omega^{\mathfrak{b}}\big) = \lambda. \tag{7}$$

Proof. In order to simplify the exposition we will use several constructions from the proof of Proposition 3.6 [2].

For any fixed $\omega^{\mathfrak{a}} \in \Omega^{\mathfrak{a}}$ we define $\mathcal{F}^{\mathfrak{b}}$-measurable function $Z(\omega^{\mathfrak{a}}, \cdot\,)$ as in the proof of Proposition 3.6 [2], taking measure $\mathrm{Q}(\omega^{\mathfrak{a}};\, \varphi^{-1}(B))$, $B \in \mathcal{B}(\mathbb{R}^2_+)$, $\varphi\colon (x, y) \mapsto (y, x)$, in place of measure μ, and $\big(\Omega^{\mathfrak{b}}, \mathcal{F}^{\mathfrak{b}}, \mathbb{P}^{\mathfrak{a},\mathfrak{b}}\big(\omega^{\mathfrak{a}};\, \cdot\,\big)\big)$ instead of measurable space $\big(\Omega, \mathcal{F}, \mathbb{P}\big)$. Referring again to Proposition 3.6, we see that Z satisfies conditions (6) and (7), as well as functions K and L satisfy (5).

Let us show that this function Z is measurable not only as a function of variable $\omega^{\mathfrak{b}}$ for fixed $\omega^{\mathfrak{a}}$, but it is also measurable as a function of two variables with respect to the σ-field $\mathcal{F} = \mathcal{F}^{\mathfrak{a}} \otimes \mathcal{F}^{\mathfrak{b}}$.

Let us notice that Z can be represented as $Z = (L - J) \wedge K$ (see the proof of Proposition 3.6 [2]). So it is enough to check $\mathcal{F}$-measurability of J, whose precise form will be given below in formula (8).

Preliminarily we have to introduce a set S and a function G similarly to the proof of Proposition 3.6 [2]) with slight modifications. Namely, we put

$$S := \Big\{\omega = (\omega^{\mathfrak{a}}, \omega^{\mathfrak{b}})\colon \mathbb{P}^{\mathfrak{a},\mathfrak{b}}(\omega^{\mathfrak{a}};\, [0;\, \omega^{\mathfrak{b}}]) < 1\Big\},$$

where $[0;\, \omega^{\mathfrak{b}}]$ denotes a segment $\big\{\widetilde{\omega}^{\mathfrak{b}} \in \Omega^{\mathfrak{b}}\colon 0 \preceq \widetilde{\omega}^{\mathfrak{b}} \preceq \omega^{\mathfrak{b}}\big\}$ connecting point $0 = (0, 0, 0)$ and point $\omega^{\mathfrak{b}} = (y^{\mathfrak{b}}, x^{\mathfrak{b}}, u^{\mathfrak{b}})$ in space $\Omega^{\mathfrak{b}}$ endowed with a lexicographic order $\preceq$ defined as follows:

$$(\widetilde{y}^{\mathfrak{b}}, \widetilde{x}^{\mathfrak{b}}, \widetilde{u}^{\mathfrak{b}}) \preceq (y^{\mathfrak{b}}, x^{\mathfrak{b}}, u^{\mathfrak{b}}) \ \overset{\text{def}}{\Leftrightarrow}\ \begin{cases} \text{either } \ \widetilde{y}^{\mathfrak{b}} < y^{\mathfrak{b}}, \\ \text{or } \ \widetilde{y}^{\mathfrak{b}} = y^{\mathfrak{b}},\, \widetilde{x}^{\mathfrak{b}} < x^{\mathfrak{b}}, \\ \text{or } \ \widetilde{y}^{\mathfrak{b}} = y^{\mathfrak{b}},\, \widetilde{x}^{\mathfrak{b}} = x^{\mathfrak{b}},\, \widetilde{u}^{\mathfrak{b}} \le u^{\mathfrak{b}}. \end{cases}$$

Let us show that the set S belongs to the σ-field $\mathcal{F} = \mathcal{F}^{\mathfrak{a}} \otimes \mathcal{F}^{\mathfrak{b}}$. For that it suffices to check $\mathcal{F}^{\mathfrak{a}} \otimes \mathcal{F}^{\mathfrak{b}}$-measurability of the function

$$H(\omega^{\mathfrak{a}}, \omega^{\mathfrak{b}}) := \mathbb{P}^{\mathfrak{a},\mathfrak{b}}\big(\omega^{\mathfrak{a}};\, [0;\, \omega^{\mathfrak{b}}]\big).$$

With the notation $\widetilde{\Omega}^{\mathfrak{b}} := \Omega^{\mathfrak{b}}$ and $\widetilde{\mathcal{F}}^{\mathfrak{b}} := \mathcal{F}^{\mathfrak{b}}$, consider a measurable space $(\Omega^{\mathfrak{a}} \times \Omega^{\mathfrak{b}} \times \widetilde{\Omega}^{\mathfrak{b}},\, \mathcal{F}^{\mathfrak{a}} \otimes \mathcal{F}^{\mathfrak{b}} \otimes \widetilde{\mathcal{F}}^{\mathfrak{b}})$ and an indicator function

$$(\omega^{\mathfrak{a}},\, \omega^{\mathfrak{b}},\, \widetilde{\omega}^{\mathfrak{b}}) \mapsto \mathbb{1}\big\{\widetilde{\omega}^{\mathfrak{b}} \preceq \omega^{\mathfrak{b}}\big\},$$

where $\omega^{\mathfrak{a}} \in \Omega^{\mathfrak{a}}$, $\omega^{\mathfrak{b}} = (y^{\mathfrak{b}},\, x^{\mathfrak{b}},\, u^{\mathfrak{b}}) \in \Omega^{\mathfrak{b}}$ and $\widetilde{\omega}^{\mathfrak{b}} = (\widetilde{y}^{\mathfrak{b}},\, \widetilde{x}^{\mathfrak{b}},\, \widetilde{u}^{\mathfrak{b}}) \in \widetilde{\Omega}^{\mathfrak{b}}$. Obviously, this function is $\mathcal{F}^{\mathfrak{a}} \otimes \mathcal{F}^{\mathfrak{b}} \otimes \widetilde{\mathcal{F}}^{\mathfrak{b}}$-measurable due to the representation

$$\mathbb{1}\big\{\widetilde{\omega}^{\mathfrak{b}} \preceq \omega^{\mathfrak{b}}\big\} = \mathbb{1}\big\{\widetilde{y}^{\mathfrak{b}} < y^{\mathfrak{b}}\big\} + \mathbb{1}\big\{\widetilde{y}^{\mathfrak{b}} = y^{\mathfrak{b}}\big\} \cdot \mathbb{1}\big\{\widetilde{x}^{\mathfrak{b}} < x^{\mathfrak{b}}\big\} + \\ + \mathbb{1}\big\{\widetilde{y}^{\mathfrak{b}} = y^{\mathfrak{b}}\big\} \cdot \mathbb{1}\big\{\widetilde{x}^{\mathfrak{b}} = x^{\mathfrak{b}}\big\} \cdot \mathbb{1}\big\{\widetilde{u}^{\mathfrak{b}} \le u^{\mathfrak{b}}\big\}$$

and the fact that all the functions $\widetilde{y}^{\mathfrak{b}}$, $y^{\mathfrak{b}}$, $\widetilde{x}^{\mathfrak{b}}$, $x^{\mathfrak{b}}$, $\widetilde{u}^{\mathfrak{b}}$, $u^{\mathfrak{b}}$, being coordinate projections, are measurable with respect to the σ-field $\mathcal{F}^{\mathfrak{a}} \otimes \mathcal{F}^{\mathfrak{b}} \otimes \widetilde{\mathcal{F}}^{\mathfrak{b}}$.

Let $\overline{\mathbb{P}}^{\mathfrak{a},\mathfrak{b}}\big((\omega^{\mathfrak{a}},\, \omega^{\mathfrak{b}});\, d\widetilde{\omega}^{\mathfrak{b}}\big) := \mathbb{P}^{\mathfrak{a},\mathfrak{b}}(\omega^{\mathfrak{a}};\, d\widetilde{\omega}^{\mathfrak{b}})$. Then $\mathcal{F}^{\mathfrak{a}} \otimes \mathcal{F}^{\mathfrak{b}}$-measurability of function $H(\omega^{\mathfrak{a}},\, \omega^{\mathfrak{b}})$ follows from the formula

$$H(\omega^{\mathfrak{a}},\, \omega^{\mathfrak{b}}) = \int_{\widetilde{\Omega}^{\mathfrak{b}}} \mathbb{1}\big\{\widetilde{\omega}^{\mathfrak{b}} \preceq \omega^{\mathfrak{b}}\big\}\, \overline{\mathbb{P}}^{\mathfrak{a},\mathfrak{b}}\big((\omega^{\mathfrak{a}},\, \omega^{\mathfrak{b}});\, d\widetilde{\omega}^{\mathfrak{b}}\big)$$

and the Fubini theorem for Markov kernels (see Proposition III.2.1 [6] or Lemma 14.20 [4]).

Now, let us define a function G by

$$G(\lambda,\, \omega^{\mathfrak{a}},\, \omega^{\mathfrak{b}}) := \lambda - \int_{[0;\, \omega^{\mathfrak{b}}]} \Big(K(\omega^{\mathfrak{a}},\, \widetilde{\omega}^{\mathfrak{b}}) - L(\omega^{\mathfrak{a}},\, \widetilde{\omega}^{\mathfrak{b}}) + \lambda\Big)^{+}\, \mathbb{P}^{\mathfrak{a},\mathfrak{b}}(\omega^{\mathfrak{a}};\, d\widetilde{\omega}^{\mathfrak{b}}),$$

$\lambda \ge 0$, $\omega^{\mathfrak{a}} \in \Omega^{\mathfrak{a}}$, $\omega^{\mathfrak{b}} \in \Omega^{\mathfrak{b}}$. The proof of measurability of the function G as a function of two variables $\omega^{\mathfrak{a}}$ and $\omega^{\mathfrak{b}}$ with respect to the σ-field $\mathcal{F}^{\mathfrak{a}} \otimes \mathcal{F}^{\mathfrak{b}}$ goes along the same lines as in case of H. Since K, L and $\mathbb{1}\big\{\widetilde{\omega}^{\mathfrak{b}} \preceq \omega^{\mathfrak{b}}\big\}$ are $\mathcal{F}^{\mathfrak{a}} \otimes \mathcal{F}^{\mathfrak{b}} \otimes \widetilde{\mathcal{F}}^{\mathfrak{b}}$-measurable, for any fixed $\lambda \ge 0$ the function

$$g(\omega^{\mathfrak{a}},\, \omega^{\mathfrak{b}},\, \widetilde{\omega}^{\mathfrak{b}}) := \lambda - \mathbb{1}\big\{\widetilde{\omega}^{\mathfrak{b}} \preceq \omega^{\mathfrak{b}}\big\} \cdot \Big(K(\omega^{\mathfrak{a}},\, \widetilde{\omega}^{\mathfrak{b}}) - L(\omega^{\mathfrak{a}},\, \widetilde{\omega}^{\mathfrak{b}}) + \lambda\Big)^{+}$$

is measurable according to the σ-field $\mathcal{F}^{\mathfrak{a}} \otimes \mathcal{F}^{\mathfrak{b}} \otimes \widetilde{\mathcal{F}}^{\mathfrak{b}}$. Hence, $\mathcal{F}^{\mathfrak{a}} \otimes \mathcal{F}^{\mathfrak{b}}$-measurability of function G is obtained from the representation

$$G(\lambda,\, \omega^{\mathfrak{a}},\, \omega^{\mathfrak{b}}) = \int_{\widetilde{\Omega}^{\mathfrak{b}}} g(\omega^{\mathfrak{a}},\, \omega^{\mathfrak{b}},\, \widetilde{\omega}^{\mathfrak{b}})\, \overline{\mathbb{P}}^{\mathfrak{a},\mathfrak{b}}\big((\omega^{\mathfrak{a}},\, \omega^{\mathfrak{b}});\, d\widetilde{\omega}^{\mathfrak{b}}\big)$$

and the Fubini theorem for Markov kernels.

Now we are in a position to recall the definition of the function J:

$$J(\omega^{\mathfrak{a}},\, \omega^{\mathfrak{b}}) := \begin{cases} y^{\mathfrak{b}}, & \text{if } (\omega^{\mathfrak{a}},\, \omega^{\mathfrak{b}}) \notin S, \\ y^{\mathfrak{b}}_{*}, & \text{if } (\omega^{\mathfrak{a}},\, \omega^{\mathfrak{b}}) \in S, \end{cases} \tag{8}$$

where $y^{\mathfrak{b}}_{*}$ is the unique solution of equation $G(y^{\mathfrak{b}}_{*},\, \omega^{\mathfrak{a}},\, \omega^{\mathfrak{b}}) = 0$, $\omega^{\mathfrak{a}} \in \Omega^{\mathfrak{a}}$ and $\omega^{\mathfrak{b}} := (y^{\mathfrak{b}},\, x^{\mathfrak{b}},\, u^{\mathfrak{b}}) \in \Omega^{\mathfrak{b}}$. We have the following representation

$$\Big\{(\omega^{\mathfrak{a}},\, \omega^{\mathfrak{b}})\colon J(\omega^{\mathfrak{a}},\, \omega^{\mathfrak{b}}) \le \lambda\Big\} = \Big\{(\omega^{\mathfrak{a}},\, \omega^{\mathfrak{b}})\colon L(\omega^{\mathfrak{a}},\, \omega^{\mathfrak{b}}) \le \lambda\Big\} \bigcup \\ \bigcup \Big[\Big\{(\omega^{\mathfrak{a}},\, \omega^{\mathfrak{b}})\colon L(\omega^{\mathfrak{a}},\, \omega^{\mathfrak{b}}) > \lambda\Big\} \cap S \cap \Big\{(\omega^{\mathfrak{a}},\, \omega^{\mathfrak{b}})\colon G(\lambda,\, \omega^{\mathfrak{a}},\, \omega^{\mathfrak{b}}) \ge 0\Big\}\Big],$$

(see [2], p. 229). Taking into account $\mathcal{F}^{\mathfrak{a}} \otimes \mathcal{F}^{\mathfrak{b}}$-measurability of L and G, as well as $\mathcal{F}^{\mathfrak{a}} \otimes \mathcal{F}^{\mathfrak{b}}$-measurability of S, we get $\mathcal{F}^{\mathfrak{a}} \otimes \mathcal{F}^{\mathfrak{b}}$-measurability of the function J. $\square$

Lemma 2.2. *Suppose all the conditions of Lemma 2.1 are satisfied. Then one can define a filtration* $(\mathcal{F}_t)_{t \in [0;\, 1]}$ *on the measurable space* $(\Omega,\, \mathcal{F}) = (\Omega^{\mathfrak{a}} \times \Omega^{\mathfrak{b}},\, \mathcal{F}^{\mathfrak{a}} \otimes \mathcal{F}^{\mathfrak{b}})$, *and a pair of increasing processes* $X = (X_t)_{t \in [0;\, 1]}$ *and* $A = (A_t)_{t \in [0;\, 1]}$, $X_0 = 0$, $A_0 = 0$, *such that*

(i) *the processes X and A are adapted, as well as A is continuous, and*

$$\mathbb{P}^{\mathfrak{a},\mathfrak{b}}\Big(\omega^{\mathfrak{a}};\, \Big\{\omega^{\mathfrak{b}}\colon \begin{bmatrix} X_1(\omega^{\mathfrak{a}},\omega^{\mathfrak{b}}) \\ A_1(\omega^{\mathfrak{a}},\omega^{\mathfrak{b}}) \end{bmatrix} \in C\Big\}\Big) = \mathrm{Q}(\omega^{\mathfrak{a}};\, C),\;\; C \in \mathcal{B}(\mathbb{R}^2_+), \tag{9}$$

(ii) *the process $M_t := X_t - A_t$, $t \in [0;\,1]$, satisfies the following condition: for all $0 \le s \le t \le 1$, $\omega^{\mathfrak{a}} \in \Omega^{\mathfrak{a}}$ and $B \in \mathcal{F}_s$,*

$$\int_{\Omega^{\mathfrak{b}}} \big(M_t(\omega^{\mathfrak{a}},\,\omega^{\mathfrak{b}}) - M_s(\omega^{\mathfrak{a}},\,\omega^{\mathfrak{b}})\big)\, \mathbb{1}_B(\omega^{\mathfrak{a}},\,\omega^{\mathfrak{b}})\, \mathbb{P}^{\mathfrak{a},\mathfrak{b}}(\omega^{\mathfrak{a}};\, d\omega^{\mathfrak{b}}) = 0. \tag{10}$$

Proof. Let us denote

$$V(\omega) := K(\omega) - Z(\omega), \quad W(\omega) := L(\omega) - Z(\omega), \quad \omega \in \Omega, \tag{11}$$

with the functions K, L and Z defined in Lemma 2.1. Adjusting the constructions used to establish statement (ii) of Theorem 2.1 [2], we define a filtration on the measurable space $(\Omega,\,\mathcal{F})$ by setting

$$\mathcal{F}_t := \begin{cases} \mathcal{G}_{\frac{t}{1/2-t}}, & t \in [0;\,1/2), \\ \mathcal{F}, & t \in [1/2;\,1], \end{cases} \tag{12}$$

where the σ-field $\mathcal{G}_t$, $t \ge 0$, contains all $\mathcal{F}$-measurable sets B such that

$$B \cap \{W > t\} = \varnothing \qquad \text{or} \qquad B \cap \{W > t\} = \{W > t\}.$$

Further, the processes X and A are determined by the equations

$$\begin{aligned} X_t(\omega) &:= \begin{cases} V(\omega)\mathbb{1}_{\left\{W \le \frac{t}{1/2-t}\right\}}(\omega), & t \in [0;\,1/2), \\ V(\omega) + (2t-1)Z(\omega), & t \in [1/2;\,1], \end{cases} \\ A_t(\omega) &:= \begin{cases} \frac{t}{1/2-t} \wedge W(\omega), & t \in [0;\,1/2), \\ W(\omega) + (2t-1)Z(\omega), & t \in [1/2;\,1]. \end{cases} \end{aligned} \tag{13}$$

Relation (9) follows from formulae (11), (13), and (5), while (10) is a direct application of the proof of statement (ii) of Theorem 2.1 [2]. □

3. Proof of the main theorem

Let a locally integrable increasing process $X^\circ = (X^\circ_t)_{t \in [0;\,\infty)}$ and a localizing sequence of finite stopping times $(T_n)_{n=1}^\infty$ be given. We will show that without loss of generality one can assume that $T_n = n$, $n \in \mathbb{N}$.

Indeed, let us consider an arbitrary continuous strictly increasing function $\psi\colon [0;1] \to [0;\,\infty]$, $\psi(0) = 0$, $\psi(1) = \infty$, and let $T_0 := 0$. Next, define on the measurable space $(\Omega^\circ,\,\mathcal{F}^\circ,\,\mathbb{P}^\circ)$ the flow of σ-fields

$$\tilde{\mathcal{F}}^\circ_t := \mathcal{F}^\circ_{(T_{n-1}+\psi(t-n+1))\wedge T_n} \quad \text{for } \; t \in [n-1;\,n],\;\; n \in \mathbb{N},$$

and the processes

$$\tilde{X}^\circ_t := X^\circ_{(T_{n-1}+\psi(t-n+1))\wedge T_n} \quad \text{for } \; t \in [n-1;\,n],\;\; n \in \mathbb{N},$$

$$\tilde{A}^\circ_t := A^\circ_{(T_{n-1}+\psi(t-n+1))\wedge T_n} \quad \text{for } \; t \in [n-1;\,n],\;\; n \in \mathbb{N}.$$

Let us notice that here we deal with a time-change, since, for all $n \in \mathbb{N}$ and $t \in [0;\,\infty)$, random variables $(T_{n-1} + \psi(t-n+1)) \wedge T_n$ are finite stopping times. Additionally, this time-change is continuous. Thus, the flow of σ-fields $(\tilde{\mathcal{F}}^\circ_t)_{t\in[0;\,\infty)}$ is a filtration, which is right-continuous. It is known (see [3], ch. 10) that a continuous time-change preserves the property of being predictable. Hence, the new pair of processes $\tilde{X}^\circ$ and $\tilde{A}^\circ$ remains a pair of a locally integrable increasing process and its compensator with respect to the stochastic basis $(\Omega^\circ, \mathcal{F}^\circ, \mathbb{P}^\circ, (\tilde{\mathcal{F}}^\circ_t)_{t\in[0;\infty)})$.

Next, we easily see that, $\begin{bmatrix} \tilde{X}^\circ_n \\ \tilde{A}^\circ_n \end{bmatrix} = \begin{bmatrix} X^\circ_{T_n} \\ A^\circ_{T_n} \end{bmatrix}$, for all $n \in \mathbb{N}$, whence, passing to limit $n \to \infty$, it follows that $\begin{bmatrix} \tilde{X}^\circ_\infty \\ \tilde{A}^\circ_\infty \end{bmatrix} = \begin{bmatrix} X^\circ_\infty \\ A^\circ_\infty \end{bmatrix}$. Hence, the random vectors $\begin{bmatrix} \tilde{X}^\circ_\infty \\ \tilde{A}^\circ_\infty \end{bmatrix}$ and $\begin{bmatrix} X^\circ_\infty \\ A^\circ_\infty \end{bmatrix}$ have the same distribution.

Thus, we may assume that $X^\circ = (X^\circ_t)_{t\in[0;\infty)}$ is a locally integrable increasing process with the localizing sequence of finite stopping times $T_n = n$, $n \in \mathbb{N}$, and the process $A^\circ = (A^\circ_t)_{t\in[0;\infty)}$ is the compensator of X°.

We continue with the following auxiliary proposition.

Lemma 3.1. *Let a locally integrable increasing process $X^\circ = (X^\circ_t)_{t\in[0;\infty)}$ such that $\mathbb{E}[X^\circ_n] < \infty$, for any $n \in \mathbb{N}$, be given on a stochastic basis $(\Omega^\circ, \mathcal{F}^\circ, \mathbb{P}^\circ, (\mathcal{F}^\circ_t)_{t\in[0;\infty)})$; $A^\circ = (A^\circ_t)_{t\in[0;\infty)}$ being its compensator. Let also another integrable increasing process $X^{[n]} = (X^{[n]}_t)_{t\in[0;n]}$ on a different stochastic basis $(\Omega^{[n]}, \mathcal{F}^{[n]}, \mathbb{P}^{[n]}, (\mathcal{F}^{[n]}_t)_{t\in[0;n]})$, $n \in \mathbb{N}$, with a compensator $A^{[n]} = (A^{[n]}_t)_{t\in[0;n]}$ be given. Moreover,* $\operatorname{Law}\begin{bmatrix} X^{[n]}_n \\ A^{[n]}_n \end{bmatrix} = \operatorname{Law}\begin{bmatrix} X^\circ_n \\ A^\circ_n \end{bmatrix}$. *Then one can define a pair of processes $X^{[n+1]} = (X^{[n+1]}_t)_{t\in[0;n+1]}$ and $A^{[n+1]} = (A^{[n+1]}_t)_{t\in[0;n+1]}$ on a certain extension $(\Omega^{[n+1]}, \mathcal{F}^{[n+1]}, \mathbb{P}^{[n+1]}, (\mathcal{F}^{[n+1]}_t)_{t\in[0;n+1]})$ of a stochastic basis $(\Omega^{[n]}, \mathcal{F}^{[n]}, \mathbb{P}^{[n]}, (\mathcal{F}^{[n]}_t)_{t\in[0;n]})$, satisfying the following conditions:*

(i) *$X^{[n+1]}$ is an integrable increasing process, and process $A^{[n+1]}$ is its compensator,*
(ii) *the processes $(X^{[n]}_t)_{t\in[0;n]}$ and $(X^{[n+1]}_t)_{t\in[0;n]}$ coincide,*
(iii) *the processes $(A^{[n]}_t)_{t\in[0;n]}$ and $(A^{[n+1]}_t)_{t\in[0;n]}$ coincide,*
(iv) $\operatorname{Law}\begin{bmatrix} X^{[n+1]}_n \\ A^{[n+1]}_n \end{bmatrix} = \operatorname{Law}\begin{bmatrix} X^\circ_n \\ A^\circ_n \end{bmatrix}$ *and* $\operatorname{Law}\begin{bmatrix} X^{[n+1]}_{n+1} \\ A^{[n+1]}_{n+1} \end{bmatrix} = \operatorname{Law}\begin{bmatrix} X^\circ_{n+1} \\ A^\circ_{n+1} \end{bmatrix}$,
(v) *process $(A^{[n+1]}_t)_{t\in[n;n+1]}$ is continuous.*

Proof. Let us set $\Omega^{\mathfrak{a}} := \mathbb{R}^2_+$, $\mathcal{F}^{\mathfrak{a}} := \mathcal{B}(\mathbb{R}^2_+)$ and

$$\Omega^{[n+1]} := \Omega^{[n]} \times \Omega, \quad \mathcal{F}^{[n+1]} := \mathcal{F}^{[n]} \otimes \mathcal{F},$$

where $(\Omega, \mathcal{F})$ is defined by (3).

We define a Markov kernel $\mathrm{Q}(\omega^{\mathfrak{a}}; B)$, $\omega^{\mathfrak{a}} \in \mathbb{R}^2_+$, $B \in \mathcal{B}(\mathbb{R}^2_+)$, as a regular conditional distribution of the random vector $\begin{bmatrix} X^\circ_{n+1} - X^\circ_n \\ A^\circ_{n+1} - A^\circ_n \end{bmatrix}$ under condition $\begin{bmatrix} X^\circ_n \\ A^\circ_n \end{bmatrix} = \omega^{\mathfrak{a}}$. From Proposition 4.1 and Remark 4.1 it follows that, without loss of generality, one can assume that, for any fixed $\omega^{\mathfrak{a}} \in \Omega^{\mathfrak{a}}$, the measure $\mathrm{Q}(\omega^{\mathfrak{a}}, \cdot\,)$ belongs to the class $\mathbb{W}$.

To construct probability measure $\mathbb{P}^{[n+1]}$ let us define three Markov kernels:

- Markov kernel κ_1 from $\Omega^{[n]}$ to $\Omega^{\mathfrak{a}}$:

$$\kappa_1(\omega^{[n]}; B^{\mathfrak{a}}) := \mathbb{1}\left\{ \begin{bmatrix} X^{[n]}_n(\omega^{[n]}) \\ A^{[n]}_n(\omega^{[n]}) \end{bmatrix} \in B^{\mathfrak{a}} \right\}, \quad \omega^{[n]} \in \Omega^{[n]}, \quad B^{\mathfrak{a}} \in \mathcal{F}^{\mathfrak{a}};$$

- Markov kernel κ_2 from $\Omega^{[n]} \times \Omega^{\mathfrak{a}}$ to $\Omega^{\mathfrak{b}}$:

$$\kappa_2\big((\omega^{[n]}, \omega^{\mathfrak{a}}); B^{\mathfrak{b}}\big) := \mathbb{P}^{\mathfrak{a},\mathfrak{b}}(\omega^{\mathfrak{a}}; B^{\mathfrak{b}}), \quad \omega^{[n]} \in \Omega^{[n]}, \quad \omega^{\mathfrak{a}} \in \Omega^{\mathfrak{a}}, \quad B^{\mathfrak{b}} \in \mathcal{F}^{\mathfrak{b}};$$

- Markov kernel κ from $\Omega^{[n]}$ to Ω:

$$\kappa(\omega^{[n]}; B) := \int_{\Omega^{\mathfrak{a}}} \left[\int_{\Omega^{\mathfrak{b}}} \mathbb{1}_B(\omega^{\mathfrak{a}}, \omega^{\mathfrak{b}})\, \kappa_2\big((\omega^{[n]}, \omega^{\mathfrak{a}}); d\omega^{\mathfrak{b}}\big) \right] \kappa_1(\omega^{[n]}; d\omega^{\mathfrak{a}}),$$

$\omega^{[n]} \in \Omega^{[n]}$, $B \in \mathcal{F}$, which is the product $\kappa_1 \otimes \kappa_2$ of Markov kernels κ_1 and κ_2 (the details about the product of Markov kernels can be found, for example, in [4]).

We define a probability measure $\mathbb{P}^{[n+1]}$ on the measurable space $(\Omega^{[n+1]}, \mathcal{F}^{[n+1]})$ by

$$\mathbb{P}^{[n+1]}(B^{[n]} \times B) := \int_{B^{[n]}} \kappa\big(\omega^{[n]}; B\big)\, \mathbb{P}^{[n]}(d\omega^{[n]}), \quad B^{[n]} \in \mathcal{F}^{[n]},\ B \in \mathcal{F}. \tag{14}$$

Let us set a filtration on probability space $(\Omega^{[n+1]}, \mathcal{F}^{[n+1]}, \mathbb{P}^{[n+1]})$ by

$$\mathcal{F}_t^{[n+1]} := \begin{cases} \mathcal{F}_t^{[n]} \otimes \{\emptyset, \Omega\}, & t \in [0; n], \\ \mathcal{F}_n^{[n]} \otimes \mathcal{F}_{t-n}, & t \in (n; n+1], \end{cases}$$

with the σ-field $\mathcal{F}_t$, $t \in [0; 1]$, given by (12).

Using Lemma 2.2, let us define a pair of stochastic processes $X = (X_t)_{t\in[0;1]}$ and $A = (A_t)_{t\in[0;1]}$ on the measurable space $(\Omega, \mathcal{F})$. Now, we are in a position to construct required processes:

$$\begin{aligned} X_t^{[n+1]}(\omega^{[n+1]}) &:= \begin{cases} X_t^{[n]}(\omega^{[n]}), & t \in [0; n], \\ X_n^{[n]}(\omega^{[n]}) + X_{t-n}(\omega), & t \in (n; n+1], \end{cases} \\ A_t^{[n+1]}(\omega^{[n+1]}) &:= \begin{cases} A_t^{[n]}(\omega^{[n]}), & t \in [0; n], \\ A_n^{[n]}(\omega^{[n]}) + A_{t-n}(\omega), & t \in (n; n+1], \end{cases} \\ M_t^{[n+1]}(\omega^{[n+1]}) &:= X_t^{[n+1]}(\omega^{[n+1]}) - A_t^{[n+1]}(\omega^{[n+1]}),\ t \in [0; n+1], \end{aligned} \tag{15}$$

where $\omega^{[n+1]} = (\omega^{[n]}, \omega) \in \Omega^{[n]} \times \Omega = \Omega^{[n+1]}$. By Propositions 4.2 and 4.3, $X^{[n+1]}$ and $A^{[n+1]}$ is a required pair of processes. $\square$

Now we are ready to continue the proof of Theorem 1.1. We start with the following recursive procedure.

Step 1. Applying Theorem 2.1 (i) [2] to the integrable increasing process $(X_t^\circ)_{t\in[0;1]}$, as well as its compensator $(A_t^\circ)_{t\in[0;1]}$ and a stopping time $T = 1$, we get $\mathrm{Law}\begin{bmatrix} X_1^\circ \\ A_1^\circ \end{bmatrix} \in \mathbb{W}$. Then by Theorem 2.1 (ii) [2], there exists a stochastic basis $\big(\Omega^{[1]}, \mathcal{F}^{[1]}, \mathbb{P}^{[1]}, (\mathcal{F}_t^{[1]})_{t\in[0;1]}\big)$, and an integrable process $(X_t^{[1]})_{t\in[0;1]}$ on it with a continuous compensator $(A_t^{[1]})_{t\in[0;1]}$, such that $\mathrm{Law}\begin{bmatrix} X_1^{[1]} \\ A_1^{[1]} \end{bmatrix} = \mathrm{Law}\begin{bmatrix} X_1^\circ \\ A_1^\circ \end{bmatrix}$.

All the steps starting from the second are performed similarly.

Step $n+1$, $n \ge 1$. Remark that the pair of processes $(X_t^\circ)_{t\in[0;\infty)}$ and $(A_t^\circ)_{t\in[0;\infty)}$ and the pair of processes $(X_t^{[n]})_{t\in[0;n]}$ and $(A_t^{[n]})_{t\in[0;n]}$ fit the requirements of Lemma 3.1. So, applying this lemma, we build a stochastic basis

$$\big(\Omega^{[n+1]}, \mathcal{F}^{[n+1]}, \mathbb{P}^{[n+1]}, (\mathcal{F}_t^{[n+1]})_{t\in[0;n+1]}\big),$$

and an integrable increasing process $(X_t^{[n+1]})_{t\in[0;n+1]}$ with a continuous compensator $(A_t^{[n+1]})_{t\in[0;n+1]}$, satisfying the condition

$$\mathrm{Law}\begin{bmatrix} X_{n+1}^{[n+1]} \\ A_{n+1}^{[n+1]} \end{bmatrix} = \mathrm{Law}\begin{bmatrix} X_{n+1}^\circ \\ A_{n+1}^\circ \end{bmatrix}.$$

Now, we are ready to define the required stochastic basis

$$\big(\Omega^\star,\, \mathcal{F}^\star,\, \mathbb{P}^\star,\, (\mathcal{F}^\star_t)_{t\in[0;\,\infty)}\big)$$

and a locally integrable increasing process $X^\star = (X^\star_t)_{t\in[0;\,\infty)}$ on it with a continuous compensator $A^\star = (A^\star_t)_{t\in[0;\,\infty)}$. Put:

$$\Omega^\star := \Omega^{[1]} \times (\Omega)^\infty, \quad \mathcal{F}^\star := \mathcal{F}^{[1]} \otimes \bigotimes_{i=2}^{\infty} \mathcal{F},$$

$$\mathcal{F}^\star_t := \begin{cases} \mathcal{F}^{[1]}_t \otimes \{\emptyset,\, \Omega\}^\infty, & t \in [0;\, 1], \\ \mathcal{F}^{[1]}_1 \otimes \mathcal{F}_{t-1} \otimes \{\emptyset,\, \Omega\}^\infty, & t \in (1;\, 2], \\ \mathcal{F}^{[1]}_1 \otimes \Big(\bigotimes_{i=2}^{n-1} \mathcal{F}_1\Big) \otimes \mathcal{F}_{t-n+1} \otimes \{\emptyset,\, \Omega\}^\infty, & t \in (n-1;\, n],\ n \geq 3. \end{cases}$$

Next, in view of the Ionescu-Tulcea theorem (see e.g. [7]) on the measurable space $(\Omega^\star,\, \mathcal{F}^\star)$ there exists a unique probability measure $\mathbb{P}^\star$, such that

$$\forall n \in \mathbb{N}\ \ \forall B^{[n]} \in \mathcal{F}^{[n]}: \ \ \mathbb{P}^\star\big(B^{[n]} \times (\Omega)^\infty\big) = \mathbb{P}^{[n]}\big(B^{[n]}\big).$$

Further, let $\omega^\star = \big(\omega^{[1]},\, \omega_2,\, \ldots,\, \omega_n,\, \ldots\big) \in \Omega^\star$. Set

$$X^\star_t\,(\omega^\star) := \begin{cases} X^{[1]}_t\,\big(\omega^{[1]}\big), & t \in [0;\, 1], \\ X^{[n]}_t\,\big(\omega^{[1]},\, \omega_2,\, \ldots,\, \omega_n\big), & t \in (n-1;\, n],\ \ n \geq 2, \end{cases}$$

$$A^\star_t\,(\omega^\star) := \begin{cases} A^{[1]}_t\,\big(\omega^{[1]}\big), & t \in [0;\, 1], \\ A^{[n]}_t\,\big(\omega^{[1]},\, \omega_2,\, \ldots,\, \omega_n\big), & t \in (n-1;\, n],\ \ n \geq 2, \end{cases}$$

$$M^\star_t\,(\omega^\star) := X^\star_t\,(\omega^\star) - A^\star_t\,(\omega^\star), \quad t \geq 0.$$

We will show that $M^\star = (M^\star_t)_{t\in[0;\,\infty)}$ is a martingale on $\big(\Omega^\star,\, \mathcal{F}^\star,\, \mathbb{P}^\star,\, (\mathcal{F}^\star_t)_{t\in[0;\,\infty)}\big)$. For this purpose we will prove that for any $0 \leq s < t$ and a set $B^\star_s \in \mathcal{F}^\star_s$ we have

$$\int_{B^\star_s} M^\star_t\,(\omega^\star)\ \mathbb{P}^\star(d\omega^\star) = \int_{B^\star_s} M^\star_s\,(\omega^\star)\ \mathbb{P}^\star(d\omega^\star). \tag{16}$$

Indeed, there is an integer $n \geq t$. Then the condition $B^\star_s \in \mathcal{F}^\star_s$ implies $B^\star_s = B^{[n]}_s \times (\Omega)^\infty$, where $B^{[n]}_s \in \mathcal{F}^{[n]}_s$ or $B^\star_s = \varnothing$. So, in what follows we consider the case $B^\star_s = B^{[n]}_s \times (\Omega)^\infty$, because (16) trivially holds, if $B^\star_s = \varnothing$. Now, let $\omega^\star = \big(\omega^{[1]},\, \omega_2,\, \ldots,\, \omega_n,\, \ldots\big) \in \Omega^\star$ and $\omega^{[n]} := \big(\omega^{[1]},\, \omega_2,\, \ldots,\, \omega_n\big) \in \Omega^{[n]}$. Then for $t \in [0;\, n]$ one has $M^\star_t(\omega^\star) = M^{[n]}_t(\omega^{[n]})$. Taking into account that the process $M^{[n]} = (M^{[n]}_t)_{t\in[0;\,n]}$ is a martingale, we come to the required relation (16):

$$\int_{B^\star_s} M^\star_t\,(\omega^\star)\ \mathbb{P}^\star(d\omega^\star) = \int_{B^{[n]}_s \times (\Omega)^\infty} M^{[n]}_t(\omega^{[n]})\,\mathbb{P}^\star(d\omega^\star) =$$

$$= \int_{B^{[n]}_s} M^{[n]}_t(\omega^{[n]})\,\mathbb{P}^{[n]}(d\omega^{[n]}) = \int_{B^{[n]}_s} M^{[n]}_s(\omega^{[n]})\,\mathbb{P}^{[n]}(d\omega^{[n]}) =$$

$$= \int_{B^\star_s} M^\star_s\,(\omega^\star)\ \mathbb{P}^\star(d\omega^\star).$$

The process $A^\star = (A^\star_t)_{t\in[0;\,\infty)}$ is a predictable (by continuity) increasing process.

Finally, formula (1) is obtained from the relations

$$\lim_{n\to\infty} \begin{bmatrix} X^\star_n \\ A^\star_n \end{bmatrix} = \begin{bmatrix} X^\star_\infty \\ A^\star_\infty \end{bmatrix}, \quad \lim_{n\to\infty} \begin{bmatrix} X^\circ_n \\ A^\circ_n \end{bmatrix} = \begin{bmatrix} X^\circ_\infty \\ A^\circ_\infty \end{bmatrix},$$

$$\operatorname{Law} \begin{bmatrix} X^\star_n \\ A^\star_n \end{bmatrix} = \operatorname{Law} \begin{bmatrix} X^\circ_n \\ A^\circ_n \end{bmatrix}, \quad n \in \mathbb{N},$$

and the fact that almost sure convergence implies weak convergence. □

Remark 3.1. We have already proved that the process $M^\star = (M^\star_t)_{t\in[0;\,\infty)}$ is a martingale with respect to filtration $(\mathcal{F}^\star_t)_{t\in[0;\,\infty)}$. The filtration $(\mathcal{F}^\star_t)_{t\in[0;\,\infty)}$ is not right-continuous in general. However, every right-continuous process $M = (M_t)_{t\in[0;\,\infty)}$ which is a martingale with respect to some filtration $(\mathcal{F}_t)_{t\in[0;\,\infty)}$ is also a martingale with respect to the right-continuous filtration generated by $(\mathcal{F}_t)_{t\in[0;\infty)}$:

$$\mathcal{H}_t := \bigcap_{\varepsilon>0} \mathcal{F}_{t+\varepsilon}, \quad t \geq 0,$$

as it follows form Corollary 8.9 [8]. This shows that a filtration in Theorem 1.1 can be taken right-continuous.

4. Auxiliary propositions

Proposition 4.1. *Let $(\Omega^\circ,\, \mathcal{F}^\circ,\, \mathbb{P}^\circ,\, (\mathcal{F}^\circ_t)_{t\in[0;\,\infty)})$ be a stochastic basis. Consider a locally integrable increasing process $X^\circ = (X^\circ_t)_{t\in[0;\,\infty)}$ on it with a compensator $A^\circ = (A^\circ_t)_{t\in[0;\,\infty)}$, such that $\mathbb{E}[X^\circ_n] < \infty$ for all $n \in \mathbb{N}$. Let us fix an arbitrary $n \in \mathbb{N}$. Denote $\mathrm{Q}(\omega^{\mathfrak{a}};\, B)$, $\omega^{\mathfrak{a}} \in \Omega^{\mathfrak{a}} = \mathbb{R}^2_+$, $B \in \mathcal{B}(\mathbb{R}^2_+)$, a regular conditional distribution of the random vector $\begin{bmatrix} X^\circ_{n+1}-X^\circ_n \\ A^\circ_{n+1}-A^\circ_n \end{bmatrix}$ under the condition $\begin{bmatrix} X^\circ_n \\ A^\circ_n \end{bmatrix} = \omega^{\mathfrak{a}}$, and $\mu := \mathrm{Law}\begin{bmatrix} X^\circ_n \\ A^\circ_n \end{bmatrix}$. Then for μ-a.e. $\omega^{\mathfrak{a}} \in \Omega^{\mathfrak{a}}$ we have*

$$\mathrm{Q}(\omega^{\mathfrak{a}};\, \cdot\,) \in \mathbb{W}. \tag{17}$$

Remark 4.1. Let us notice that a Dirac measure $\delta_{\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]}$ at a point $\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]$ belongs to class $\mathbb{W}$. Then without loss of generality one can assume that (17) holds not only for μ-a.e. $\omega^{\mathfrak{a}} \in \Omega^{\mathfrak{a}}$, but for all $\omega^{\mathfrak{a}} \in \Omega^{\mathfrak{a}}$. Indeed, the distribution $\mathrm{Q}(\omega^{\mathfrak{a}};\, \cdot\,)$ can be redefined as the Dirac measure $\delta_{\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]}$ for $\omega^{\mathfrak{a}} \in \Omega^{\mathfrak{a}}$ such that $\mathrm{Q}(\omega^{\mathfrak{a}};\, \cdot\,) \notin \mathbb{W}$.

Proof. We have to show that for μ-a.e. $\omega^{\mathfrak{a}} \in \Omega^{\mathfrak{a}}$ the following conditions hold:

1) $\int_{\mathbb{R}^2_+} (x+y)\, \mathrm{Q}(\omega^{\mathfrak{a}};\, d(x,\, y)) < \infty$,
2) $\int_{\mathbb{R}^2_+} x\, \mathrm{Q}(\omega^{\mathfrak{a}};\, d(x,\, y)) = \int_{\mathbb{R}^2_+} y\, \mathrm{Q}(\omega^{\mathfrak{a}};\, d(x,\, y))$,
3) $\forall \lambda \geq 0\colon\ \int_{\{y\leq\lambda\}} x\, \mathrm{Q}(\omega^{\mathfrak{a}};\, d(x,\, y)) \leq \int_{\mathbb{R}^2_+} (y \wedge \lambda)\, \mathrm{Q}(\omega^{\mathfrak{a}};\, d(x,\, y))$.

First, let us verify condition 2). Since $M^\circ_t := X^\circ_t - A^\circ_t$, $t \in [0;\, n+1]$, is a martingale we have

$$\mathbb{E}^\circ\big[X^\circ_{n+1} - A^\circ_{n+1}\big|\mathcal{F}^\circ_n\big] = X^\circ_n - A^\circ_n \quad \mathbb{P}^\circ\text{-a.s.},$$

whence,

$$\mathbb{E}^\circ\big[X^\circ_{n+1} - X^\circ_n\big|\sigma(X^\circ_n,\, A^\circ_n)\big] = \mathbb{E}^\circ[A^\circ_{n+1} - A^\circ_n|\sigma(X^\circ_n,\, A^\circ_n)] \quad \mathbb{P}^\circ\text{-a.s.}$$

Hence, for μ-a.e. $\omega^{\mathfrak{a}}$ the following identity holds

$$\mathbb{E}^\circ\big[X^\circ_{n+1} - X^\circ_n\big|(X^\circ_n,\, A^\circ_n) = \omega^{\mathfrak{a}}\big] = \mathbb{E}^\circ\big[A^\circ_{n+1} - A^\circ_n\big|(X^\circ_n,\, A^\circ_n) = \omega^{\mathfrak{a}}\big]. \tag{18}$$

Wherefrom, taking into account that the left-hand and right-hand sides of equation (18) are equal to $\int_{\mathbb{R}^2_+} x\, \mathrm{Q}\big(\omega^{\mathfrak{a}};\, d(x,\, y)\big)$ and $\int_{\mathbb{R}^2_+} y\, \mathrm{Q}\big(\omega^{\mathfrak{a}};\, d(x,\, y)\big)$ respectively, we arrive at statement 2).

Further, from integrability of $\mathbb{E}^\circ\big[X^\circ_{n+1}-X^\circ_n\big|\sigma(X^\circ_n,\, A^\circ_n)\big]$ and $\mathbb{E}^\circ[A^\circ_{n+1}-A^\circ_n|\sigma(X^\circ_n,\, A^\circ_n)]$, one can deduce finiteness of integrals $\int_{\mathbb{R}^2_+} x\, \mathrm{Q}\big(\omega^{\mathfrak{a}};\, d(x,\, y)\big)$ and $\int_{\mathbb{R}^2_+} y\, \mathrm{Q}\big(\omega^{\mathfrak{a}};\, d(x,\, y)\big)$ for μ-a.e. $\omega^{\mathfrak{a}}$. Thus, statement 1) is established.

Finally, let us proof condition 3). Let us fix an arbitrary $B^\circ \in \mathcal{F}^\circ_n$ and consider an integrable increasing process

$$Z^\circ_t(\omega^\circ) := \big(X^\circ_t(\omega^\circ) - X^\circ_n(\omega^\circ)\big)\, \mathbb{1}\{t \geq n\}\, \mathbb{1}\{\omega^\circ \in B^\circ\},$$

$\omega^\circ \in \Omega^\circ$, $t \in [0;\, n+1]$. Clearly, it has a compensator

$$C^\circ_t(\omega^\circ) := \big(A^\circ_t(\omega^\circ) - A^\circ_n(\omega^\circ)\big)\, \mathbb{1}\{t \ge n\}\, \mathbb{1}\{\omega^\circ \in B^\circ\},$$

$\omega^\circ \in \Omega^\circ$, $t \in [0;\, n+1]$. Then, by Theorem 2.1(i) [2], the joint distribution γ of the random variables Z°_{n+1} and C°_{n+1} belongs to the class $\mathbb{W}$. In particular, it means that, for any $\lambda \ge 0$, we have the inequality

$$\int_{\{y \le \lambda\}} x\, \gamma(d(x,\, y)) \le \int_{\mathbb{R}^2_+} (y \wedge \lambda)\, \gamma(d(x,\, y)),$$

which can be represented in the following form:

$$\mathbb{E}^\circ\big[Z^\circ_{n+1}\, \mathbb{1}\{C^\circ_{n+1} \le \lambda\}\big] \le \mathbb{E}^\circ\big[C^\circ_{n+1} \wedge \lambda\big]. \tag{19}$$

In can by easily seen that the condition (19) is equivalent to

$$\begin{aligned} &\mathbb{E}^\circ\Big[(X^\circ_{n+1} - X^\circ_n)\, \mathbb{1}\big\{A^\circ_{n+1} - A^\circ_n \le \lambda\big\}\, \mathbb{1}_{B^\circ}\Big] \le \\ &\le \mathbb{E}^\circ\Big[\Big((A^\circ_{n+1} - A^\circ_n) \wedge \lambda\Big)\, \mathbb{1}_{B^\circ}\Big]. \end{aligned} \tag{20}$$

Next, since the condition (20) holds for any $B^\circ \in \mathcal{F}^\circ_n$, it follows that

$$\begin{aligned} &\mathbb{E}^\circ\Big[(X^\circ_{n+1} - X^\circ_n)\, \mathbb{1}\{A^\circ_{n+1} - A^\circ_n \le \lambda\}\Big|\mathcal{F}^\circ_n\Big] \le \\ &\le \mathbb{E}^\circ\Big[(A^\circ_{n+1} - A^\circ_n) \wedge \lambda\,\Big|\mathcal{F}^\circ_n\Big] \quad \mathbb{P}^\circ\text{-a.s.}, \end{aligned}$$

whence, for μ-a.e. points $\omega^{\mathfrak{a}}$ one has

$$\begin{aligned} &\mathbb{E}^\circ\Big[(X^\circ_{n+1} - X^\circ_n)\, \mathbb{1}\{A^\circ_{n+1} - A^\circ_n \le \lambda\}\Big|(X^\circ_n,\, A^\circ_n) = \omega^{\mathfrak{a}}\Big] \le \\ &\le \mathbb{E}^\circ\Big[(A^\circ_{n+1} - A^\circ_n) \wedge \lambda\,\Big|(X^\circ_n,\, A^\circ_n) = \omega^{\mathfrak{a}}\Big]. \end{aligned} \tag{21}$$

Thus, taking into account that the left-hand side and the right-hand side of (21) are equal to

$$\int_{\mathbb{R}^2_+} x\, \mathbb{1}\{y \le \lambda\}\ \mathrm{Q}\big(\omega^{\mathfrak{a}};\, d(x,\, y)\big) \quad \text{and} \quad \int_{\mathbb{R}^2_+} (y \wedge \lambda)\ \mathrm{Q}\big(\omega^{\mathfrak{a}};\, d(x,\, y)\big),$$

respectively, we arrive at condition 3). □

Proposition 4.2. *The processes X°, A°, $X^{[n+1]}$ and $A^{[n+1]}$ introduced in Lemma 3.1 have the following property*

$$\mathrm{Law}\left(\begin{bmatrix} X^{[n+1]}_n \\ A^{[n+1]}_n \end{bmatrix}, \begin{bmatrix} X^{[n+1]}_{n+1} \\ A^{[n+1]}_{n+1} \end{bmatrix}\right) = \mathrm{Law}\left(\begin{bmatrix} X^\circ_n \\ A^\circ_n \end{bmatrix}, \begin{bmatrix} X^\circ_{n+1} \\ A^\circ_{n+1} \end{bmatrix}\right). \tag{22}$$

Proof. Let us observe that to prove formula (22) it is enough to check that, for any sets C_1, $C_2 \in \mathcal{B}(\mathbb{R}^2)$, one has

$$\begin{aligned} &\mathbb{P}^{[n+1]}\left\{\left(\begin{bmatrix} X^{[n+1]}_n \\ A^{[n+1]}_n \end{bmatrix}, \begin{bmatrix} X^{[n+1]}_{n+1} - X^{[n+1]}_n \\ A^{[n+1]}_{n+1} - A^{[n+1]}_n \end{bmatrix}\right) \in C_1 \times C_2\right\} = \\ &= \mathbb{P}^\circ\left\{\left(\begin{bmatrix} X^\circ_n \\ A^\circ_n \end{bmatrix}, \begin{bmatrix} X^\circ_{n+1} - X^\circ_n \\ A^\circ_{n+1} - A^\circ_n \end{bmatrix}\right) \in C_1 \times C_2\right\}. \end{aligned} \tag{23}$$

Let $\omega^{[n+1]} = (\omega^{[n]},\, \omega) \in \Omega^{[n+1]}$. Then in view of formulae (15), (13), (11), and (14), the left part of relation (23) can be represented in the form

$$\begin{aligned}
&\mathbb{P}^{[n+1]}\left\{\left(\begin{bmatrix} X_n^{[n+1]} \\ A_n^{[n+1]} \end{bmatrix},\, \begin{bmatrix} X_{n+1}^{[n+1]} - X_n^{[n+1]} \\ A_{n+1}^{[n+1]} - A_n^{[n+1]} \end{bmatrix}\right) \in C_1 \times C_2\right\} = \\
&= \mathbb{P}^{[n+1]}\left(\left\{\omega^{[n]}\colon\ \begin{bmatrix} X_n^{[n]}(\omega^{[n]}) \\ A_n^{[n]}(\omega^{[n]}) \end{bmatrix} \in C_1\right\} \times \left\{\omega\colon\ \begin{bmatrix} K(\omega) \\ L(\omega) \end{bmatrix} \in C_2\right\}\right) \overset{(14)}{=} \\
&= \int_{\left\{\omega^{[n]}\colon\ \left[\begin{smallmatrix} X_n^{[n]}(\omega^{[n]}) \\ A_n^{[n]}(\omega^{[n]}) \end{smallmatrix}\right] \in C_1\right\}} \kappa\left(\omega^{[n]};\ \left\{\omega\colon\ \left[\begin{smallmatrix} K(\omega) \\ L(\omega) \end{smallmatrix}\right] \in C_2\right\}\right) \mathbb{P}^{[n]}(d\omega^{[n]}).
\end{aligned} \tag{24}$$

Since $\omega = (\omega^{\mathfrak{a}},\, y^{\mathfrak{b}},\, x^{\mathfrak{b}},\, u^{\mathfrak{b}})$, $K(\omega) = x^{\mathfrak{b}}$, and $L(\omega) = y^{\mathfrak{b}}$, the set $\left\{\omega\colon\ \left[\begin{smallmatrix} K(\omega) \\ L(\omega) \end{smallmatrix}\right] \in C_2\right\}$ in the right-hand side of (24) can be represented in the form

$$\left\{\omega\colon\ \begin{bmatrix} K(\omega) \\ L(\omega) \end{bmatrix} \in C_2\right\} = \Omega^{\mathfrak{a}} \times \varphi^{-1}(C_2) \times [0;\, 1], \tag{25}$$

with $\varphi\colon (x,\, y) \mapsto (y,\, x)$.

Now, we are going to use a formula proved later:

$$\kappa(\omega^{[n]};\ B) = \mathbb{P}^{\mathfrak{a},\mathfrak{b}}\left(\eta^{[n]}(\omega^{[n]});\ B\left(\eta^{[n]}(\omega^{[n]})\right)\right), \tag{26}$$

where $\omega^{[n]}$ is an arbitrary element of $\Omega^{[n]}$, set $B \in \mathcal{F}$, vector $\eta^{[n]}(\omega^{[n]}) := \left[\begin{smallmatrix} X_n^{[n]}(\omega^{[n]}) \\ A_n^{[n]}(\omega^{[n]}) \end{smallmatrix}\right]$, and, finally, $B\left(\eta^{[n]}(\omega^{[n]})\right) \in \mathcal{F}^{\mathfrak{b}}$ denotes the section of set B at point $\eta^{[n]}(\omega^{[n]}) \in \mathcal{F}^{\mathfrak{a}}$.

Using formulae (25), (26), and (4), we get

$$\begin{aligned}
&\kappa\left(\omega^{[n]};\ \left\{\omega\colon\ \left[\begin{smallmatrix} K(\omega) \\ L(\omega) \end{smallmatrix}\right] \in C_2\right\}\right) = \\
&= \mathbb{P}^{\mathfrak{a},\mathfrak{b}}\left(\eta^{[n]}(\omega^{[n]});\ \varphi^{-1}(C_2) \times [0;\, 1]\right) = \\
&= \mathrm{Q}\left(\eta^{[n]}(\omega^{[n]});\, C_2\right).
\end{aligned} \tag{27}$$

Now, substituting (27) into (24), then applying a change of variable $\omega^{\mathfrak{a}} = \eta^{[n]}(\omega^{[n]})$ in the Lebesgue integral, and using the identity $\mathrm{Law}(\eta^{[n]}) = \mathrm{Law}\left[\begin{smallmatrix} X_n^{[n]} \\ A_n^{[n]} \end{smallmatrix}\right] = \mathrm{Law}\left[\begin{smallmatrix} X_n^{\circ} \\ A_n^{\circ} \end{smallmatrix}\right] = \mu$, we get the required formula (23):

$$\begin{aligned}
&\mathbb{P}^{[n+1]}\left\{\left(\begin{bmatrix} X_n^{[n+1]} \\ A_n^{[n+1]} \end{bmatrix},\, \begin{bmatrix} X_{n+1}^{[n+1]} - X_n^{[n+1]} \\ A_{n+1}^{[n+1]} - A_n^{[n+1]} \end{bmatrix}\right) \in C_1 \times C_2\right\} = \\
&= \int_{\left\{\omega^{[n]}\colon\ \eta^{[n]}(\omega^{[n]}) \in C_1\right\}} \mathrm{Q}\left(\eta^{[n]}(\omega^{[n]});\, C_2\right) \mathbb{P}^{[n]}(d\omega^{[n]}) = \\
&= \int_{C_1} \mathrm{Q}\left(\omega^{\mathfrak{a}};\ C_2\right) \mu(d\omega^{\mathfrak{a}}) = \\
&= \mathbb{P}^{\circ}\left(\left\{\begin{bmatrix} X_n^{\circ} \\ A_n^{\circ} \end{bmatrix} \in C_1\right\} \bigcap \left\{\begin{bmatrix} X_{n+1}^{\circ} - X_n^{\circ} \\ A_{n+1}^{\circ} - A_n^{\circ} \end{bmatrix} \in C_2\right\}\right) = \\
&= \mathbb{P}^{\circ}\left\{\left(\begin{bmatrix} X_n^{\circ} \\ A_n^{\circ} \end{bmatrix},\, \begin{bmatrix} X_{n+1}^{\circ} - X_n^{\circ} \\ A_{n+1}^{\circ} - A_n^{\circ} \end{bmatrix}\right) \in C_1 \times C_2\right\}.
\end{aligned}$$

□

Proposition 4.3. *The process $M^{[n+1]} = \left(M_t^{[n+1]}\right)_{t \in [0;\, n+1]}$ defined in Lemma 3.1 is a martingale with respect to filtration $(\mathcal{F}_t^{[n+1]})_{t \in [0;\, n+1]}$.*

Proof. Let us show that, for all $0 \le s < t \le n+1$ and any $B^{[n+1]} \in \mathcal{F}_s^{[n+1]}$, one has

$$\int_{B^{[n+1]}} \Big(M_t^{[n+1]}(\omega^{[n+1]}) - M_s^{[n+1]}(\omega^{[n+1]})\Big)\, \mathbb{P}^{[n+1]}(d\omega^{[n+1]}) = 0. \tag{28}$$

1) To prove (28) in the case $0 \le s < t \le n$ it is enough to restrict ourselves to the sets $B^{[n+1]} = B^{[n]} \times \Omega$, where $B^{[n]} \in \mathcal{F}_s^{[n]}$. Using the fact that $M_t^{[n]}(\omega^{[n]})$, $t \in [0;\, n]$, is a martingale with respect to $(\mathcal{F}_t^{[n]})_{t\in[0;\,n]}$ we get the required identity

$$\begin{aligned}
&\int_{B^{[n+1]}} M_t^{[n+1]}(\omega^{[n+1]})\, \mathbb{P}^{[n+1]}(d\omega^{[n+1]}) = \int_{B^{[n]}\times\Omega} M_t^{[n]}(\omega^{[n]})\, \mathbb{P}^{[n+1]}(d\omega^{[n+1]}) = \\
&= \int_{B^{[n]}} M_t^{[n]}(\omega^{[n]})\, \mathbb{P}^{[n]}(d\omega^{[n]}) = \int_{B^{[n]}} M_s^{[n]}(\omega^{[n]})\, \mathbb{P}^{[n]}(d\omega^{[n]}) = \\
&= \int_{B^{[n+1]}} M_s^{[n+1]}(\omega^{[n+1]})\, \mathbb{P}^{[n+1]}(d\omega^{[n+1]}).
\end{aligned}$$

2) Let us prove (28) in the case $n < s < t \le n+1$. In this case

$$M_t^{[n+1]}(\omega^{[n+1]}) - M_s^{[n+1]}(\omega^{[n+1]}) = M_{t-n}(\omega) - M_{s-n}(\omega).$$

Hence, using properties of the Lebesgue integral, in order to prove (28) it is enough to show that for all sets $B^{[n+1]} = B^{[n]} \times B \in \mathcal{F}_n^{[n]} \times \mathcal{F}_{s-n}$ one has

$$\int_{B^{[n+1]}} \Big(M_{t-n}(\omega) - M_{s-n}(\omega)\Big)\, \mathbb{P}^{[n+1]}(d\omega^{[n+1]}) = 0.$$

In view of the formula

$$\begin{aligned}
J(\omega^{[n]}) &:= \int_B M_{t-n}(\omega)\, \kappa\big(\omega^{[n]};\, d\omega\big) = \\
&= \int_{\Omega^{\mathfrak{a}}} \left[\int_{B(\omega^{\mathfrak{a}})} M_{t-n}(\omega)\, \kappa_2\big((\omega^{[n]},\, \omega^{\mathfrak{a}});\, d\omega^{\mathfrak{b}}\big)\right] \kappa_1\big(\omega^{[n]};\, d\omega^{\mathfrak{a}}\big) = \\
&= \int_{\Omega^{\mathfrak{a}}} \left[\int_{B(\omega^{\mathfrak{a}})} M_{s-n}(\omega)\, \kappa_2\big((\omega^{[n]},\, \omega^{\mathfrak{a}});\, d\omega^{\mathfrak{b}}\big)\right] \kappa_1\big(\omega^{[n]};\, d\omega^{\mathfrak{a}}\big),
\end{aligned} \tag{29}$$

established later, we get the required equation

$$\begin{aligned}
&\int_{B^{[n+1]}} M_{t-n}(\omega)\, \mathbb{P}^{[n+1]}(d\omega^{[n+1]}) = \int_{B^{[n]}} J(\omega^{[n]})\, \mathbb{P}^{[n]}(d\omega^{[n]}) = \\
&= \int_{B^{[n+1]}} M_{s-n}(\omega)\, \mathbb{P}^{[n+1]}(d\omega^{[n+1]}).
\end{aligned}$$

3) It remains to verify (28) in the case: $n = s < t \le n+1$. Taking the identity $M_t^{[n+1]}(\omega^{[n+1]}) - M_n^{[n+1]}(\omega^{[n+1]}) = M_{t-n}(\omega)$, in order to prove the relation (28) it is sufficient to show the equality

$$\int_{B^{[n+1]}} M_{t-n}(\omega)\, \mathbb{P}^{[n+1]}(d\omega^{[n+1]}) = 0,$$

for any set $B^{[n+1]} = B^{[n]} \times \Omega$, with $B^{[n]} \in \mathcal{F}_n^{[n]}$. Then using a formula

$$\int_{\Omega} M_{t-n}(\omega)\, \kappa(\omega^{[n]};\, d\omega) = 0, \tag{30}$$

proved later, we get the required property

$$\int_{B^{[n+1]}} M_{t-n}(\omega)\,\mathbb{P}^{[n+1]}(d\omega^{[n+1]}) = \int_{B^{[n]}\times\Omega} M_{t-n}(\omega)\,\mathbb{P}^{[n+1]}(d\omega^{[n+1]}) =$$
$$= \int_{B^{[n]}} \left[\int_{\Omega} M_{t-n}(\omega)\,\kappa(\omega^{[n]};\, d\omega)\right] \mathbb{P}^{[n]}(d\omega^{[n]}) = 0.$$

□

Proof of Formula (26). Let $\omega^{[n]} \in \Omega^{[n]}$, the notation $B\left(\eta^{[n]}(\omega^{[n]})\right)$ stands for the section of a set $B \in \mathcal{F}$ at a point $\eta^{[n]}(\omega^{[n]}) := \left[\begin{smallmatrix} X_n^{[n]}(\omega^{[n]}) \\ A_n^{[n]}(\omega^{[n]}) \end{smallmatrix}\right]$, further, $\delta_{\eta^{[n]}(\omega^{[n]})}(B^{\mathfrak{a}})$, $B^{\mathfrak{a}} \in \mathcal{F}^{\mathfrak{a}}$, denotes a Dirac measure at point $\eta^{[n]}(\omega^{[n]})$. Then the required representation (26) follows from

$$\kappa(\omega^{[n]};\, B) = \int_{\Omega^{\mathfrak{a}}} \left[\int_{\Omega^{\mathfrak{b}}} \mathbb{1}_B(\omega^{\mathfrak{a}},\, \omega^{\mathfrak{b}})\,\mathbb{P}^{\mathfrak{a},\mathfrak{b}}(\omega^{\mathfrak{a}};\, d\omega^{\mathfrak{b}})\right] \delta_{\eta^{[n]}(\omega^{[n]})}(d\omega^{\mathfrak{a}}) =$$
$$= \int_{\Omega^{\mathfrak{b}}} \mathbb{1}_B\left(\eta^{[n]}(\omega^{[n]}),\, \omega^{\mathfrak{b}}\right) \mathbb{P}^{\mathfrak{a},\mathfrak{b}}\left(\eta^{[n]}(\omega^{[n]});\, d\omega^{\mathfrak{b}}\right) =$$
$$= \mathbb{P}^{\mathfrak{a},\mathfrak{b}}\left(\eta^{[n]}(\omega^{[n]});\; B\left(\eta^{[n]}(\omega^{[n]})\right)\right). \ \square$$

Let us recall the notation:

$$\eta^{[n]}(\omega^{[n]}) := \left[\begin{smallmatrix} X_n^{[n]}(\omega^{[n]}) \\ A_n^{[n]}(\omega^{[n]}) \end{smallmatrix}\right].$$

Proof of Formula (29). Let $B \in \mathcal{F}_{s-n}$. Then by (10), we have

$$\int_{\Omega^{\mathfrak{a}}} \left[\int_{B(\omega^{\mathfrak{a}})} M_{t-n}(\omega)\,\kappa_2\big((\omega^{[n]},\, \omega^{\mathfrak{a}});\, d\omega^{\mathfrak{b}}\big)\right] \kappa_1\big(\omega^{[n]};\, d\omega^{\mathfrak{a}}\big) =$$

$$= \int_{\Omega^{\mathfrak{a}}} \left[\int_{B(\omega^{\mathfrak{a}})} M_{t-n}(\omega^{\mathfrak{a}},\, \omega^{\mathfrak{b}})\,\mathbb{P}^{\mathfrak{a},\mathfrak{b}}\big(\omega^{\mathfrak{a}};\, d\omega^{\mathfrak{b}}\big)\right] \delta_{\eta^{[n]}(\omega^{[n]})}(d\omega^{\mathfrak{a}}) =$$

$$= \int_{B\left(\eta^{[n]}(\omega^{[n]})\right)} M_{t-n}\left(\eta^{[n]}(\omega^{[n]}),\, \omega^{\mathfrak{b}}\right) \mathbb{P}^{\mathfrak{a},\mathfrak{b}}\left(\eta^{[n]}(\omega^{[n]});\, d\omega^{\mathfrak{b}}\right) \overset{(10)}{=}$$

$$= \int_{B\left(\eta^{[n]}(\omega^{[n]})\right)} M_{s-n}\left(\eta^{[n]}(\omega^{[n]}),\, \omega^{\mathfrak{b}}\right) \mathbb{P}^{\mathfrak{a},\mathfrak{b}}\left(\eta^{[n]}(\omega^{[n]});\, d\omega^{\mathfrak{b}}\right) =$$

$$= \int_{\Omega^{\mathfrak{a}}} \left[\int_{B(\omega^{\mathfrak{a}})} M_{s-n}(\omega)\,\kappa_2\big((\omega^{[n]},\, \omega^{\mathfrak{a}});\, d\omega^{\mathfrak{b}}\big)\right] \kappa_1\big(\omega^{[n]};\, d\omega^{\mathfrak{a}}\big). \ \square$$

Proof of Formula (30). Let us, first, remark that it follows from relations (7) and (11) with $\lambda = 0$, as well as non-negativity of W, that

$$\int_{\{\omega^{\mathfrak{b}}\,:\, W(\omega^{\mathfrak{a}},\, \omega^{\mathfrak{b}}) \le 0\}} V(\omega^{\mathfrak{a}},\, \omega^{\mathfrak{b}})\,\mathbb{P}^{\mathfrak{a},\mathfrak{b}}(\omega^{\mathfrak{a}};\, d\omega^{\mathfrak{b}}) = 0. \tag{31}$$

Then using (10), (13) and (31), we get (30):

$$\begin{aligned}
&\int_{\Omega} M_{t-n}(\omega)\, \kappa(\omega^{[n]};\, d\omega) = \\
&= \int_{\Omega^{\mathfrak{a}}} \left[\int_{\Omega^{\mathfrak{b}}} M_{t-n}(\omega^{\mathfrak{a}},\, \omega^{\mathfrak{b}})\, \mathbb{P}^{\mathfrak{a},\mathfrak{b}}(\omega^{\mathfrak{a}};\, d\omega^{\mathfrak{b}}) \right] \delta_{\eta^{[n]}(\omega^{[n]})}(d\omega^{\mathfrak{a}}) = \\
&= \int_{\Omega^{\mathfrak{b}}} M_{t-n}\Big(\eta^{[n]}(\omega^{[n]}),\, \omega^{\mathfrak{b}}\Big)\, \mathbb{P}^{\mathfrak{a},\mathfrak{b}}\Big(\eta^{[n]}(\omega^{[n]});\, d\omega^{\mathfrak{b}}\Big) \overset{(10)}{=} \\
&= \int_{\Omega^{\mathfrak{b}}} M_0\Big(\eta^{[n]}(\omega^{[n]}),\, \omega^{\mathfrak{b}}\Big)\, \mathbb{P}^{\mathfrak{a},\mathfrak{b}}\Big(\eta^{[n]}(\omega^{[n]});\, d\omega^{\mathfrak{b}}\Big) = \\
&= \int_{\Omega} M_0(\omega)\, \kappa(\omega^{[n]};\, d\omega) \overset{(13)}{=} \\
&= \int_{\Omega} V(\omega)\, \mathbb{1}_{\big\{\omega:\, W(\omega)\le 0\big\}}(\omega)\, \kappa(\omega^{[n]};\, d\omega) =
\end{aligned}$$

$$= \int_{\Omega^{\mathfrak{a}}} \Bigg[\underbrace{\int_{\Omega^{\mathfrak{b}}} V(\omega^{\mathfrak{a}},\, \omega^{\mathfrak{b}})\, \mathbb{1}_{\{W\le 0\}}(\omega^{\mathfrak{a}},\, \omega^{\mathfrak{b}})\, \mathbb{P}^{\mathfrak{a},\mathfrak{b}}(\omega^{\mathfrak{a}};\, d\omega^{\mathfrak{b}})}_{=0,\ \text{see (31)}} \Bigg] \delta_{\eta^{[n]}(\omega^{[n]})}(d\omega^{\mathfrak{a}}) = 0. \ \square$$

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