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B. RASHYTOV

POWER MOMENTS OF FIRST PASSAGE TIMES FOR SOME OSCILLATING PERTURBED RANDOM WALKS

Let (ξ_1, η_1), $(\xi_2, \eta_2), \ldots$ be a sequence of i.i.d. random vectors taking values in $\mathbb{R}^2$, and let $S_0 := 0$ and $S_n := \xi_1 + \ldots + \ldots \xi_n$ for $n \in \mathbb{N}$. The sequence $(S_{n-1} + \eta_n)_{n\in\mathbb{N}}$ is then called perturbed random walk. For real x, denote by $\tau(x)$ the first time the perturbed random walk exits the interval $(-\infty, x]$. We consider a rather intricate case in which S_n drifts to the left, yet the perturbed random walk oscillates because of occasional big jumps to the right of the perturbating sequence $(\eta_n)_{n\in\mathbb{N}}$. Under these assumptions we provide necessary and sufficient conditions for the finiteness of power moments of $\tau(x)$, thereby solving an open problem posed by Alsmeyer, Iksanov and Meiners in [2].

1. Introduction and main result

Let $(\xi_k, \eta_k)_{k\in\mathbb{N}}$ be a sequence of i.i.d. two-dimensional random vectors with generic copy (ξ, η). No condition is imposed on the dependence structure between ξ and η. Let $(S_n)_{n\in\mathbb{N}_0}$ be the zero-delayed ordinary random walk with increments ξ_n for $n \in \mathbb{N}$, that is, $S_0 := 0$ and $S_n := \xi_1 + \ldots + \xi_n$ for $n \in \mathbb{N}$. Then define its perturbed variant $(T_n)_{n\in\mathbb{N}}$, that we call *perturbed random walk (PRW)*, by

$$T_n := S_{n-1} + \eta_n, \quad n \in \mathbb{N}. \tag{1}$$

Numerous applications of the PRW can be found in [2] and [4].

For $x \in \mathbb{R}$, define the *first passage time* into (x, ∞)

$$\tau(x) := \inf\{n \in \mathbb{N} : T_n > x\}.$$

The purpose of the present paper is to give necessary and sufficient conditions for the finiteness of $\mathbb{E}(\tau(x))^p < \infty$ for $p > 0$ under the assumptions

$$\lim_{n\to\infty} S_n = -\infty \quad \text{a.s. and} \quad \int_{(1,\infty)} \frac{y}{\mathbb{E}(\xi^- \wedge y)} \mathrm{d}\mathbb{P}\{\eta \le y\} = \infty. \tag{2}$$

Although S_n drifts to $-\infty$, the PRW oscillates (see Theorem 2.1 in [2]) because of occasional extremely big jumps of the perturbating sequence $(\eta_k)_{k\in\mathbb{N}}$. Thus, we solve (partially) a problem which remained open in the article [2].

Put $\mathtt{m}^\pm := \mathbb{E}\xi_\pm := \mathbb{E}\max(\pm\xi, 0)$ and, if $\mathtt{m}^+ \wedge \mathtt{m}^- < \infty$, $\mathtt{m} := \mathbb{E}\xi = \mathtt{m}^+ - \mathtt{m}^-$. Note that $\mathtt{m} \in [-\infty, 0)$ by the first condition in (2).

Theorem 1.1. *Let $p > 0$ and $x \in \mathbb{R}$. Suppose that $\lim_{t\to\infty} t\mathbb{P}\{\eta > t\} = s \in [0, \infty]$, $\mathtt{m} \in [-\infty, 0)$ and that at least one parameter s or $\mathtt{m}$ is finite. When $s = 0$, assume also that the second condition in (2) holds. Then*
(a) $\mathbb{E}(\tau(x))^p = \infty$ if $s < -\mathtt{m}p$;
(b) $\mathbb{E}(\tau(x))^p < \infty$ if $s > -\mathtt{m}p$ (so that $\mathtt{m} > -\infty$) and $\mathbb{E}\xi_-^{rp+1} < \infty$ for some $r > s/(s+\mathtt{m}p)$ (where $s/(s + \mathtt{m}p) = 1$ for $s = \infty$).

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The finiteness of the moments of $\tau(x)$ is determined by the distribution tail of η and the behavior in mean of the random walk $(S_n)_{n\in\mathbb{N}_0}$. When $\mathtt{m} > -\infty$, a heuristic argument suggests replacing at the first step S_n with $n\mathtt{m}$ and then use Lemma 2.1 given below. In this way one anticipates a result that may be expected in the general situation treated in Theorem 1.1. At the second step one has to find out what is the price to pay for the replacement of S_n with $n\mathtt{m}$. Our assumption $\mathbb{E}\xi_-^{rp+1} < \infty$ takes care of such a replacement. However, we do not know whether this condition is indeed necessary, nor what happens if $s > -\mathtt{m}p$ and $\mathbb{E}\xi_-^q = \infty$ for $q > 1$ sufficiently close to 1.

After giving several auxiliary results in Section 2 we prove Theorem 1.1 in Section 3.

2. Auxiliary results

The problem of finiteness of $\mathbb{E}(\tau(x))^p < \infty$ which is relatively simple when the distribution of ξ is degenerate at some $-c$ for $c > 0$ is solved in Lemma 2.1. Indeed, it will be shown in the proof of Theorem 1.1 that the convergence of the series $\sum_{n\geq 1} n^{p-1}\mathbb{P}\{\max_{1\leq k\leq n}(-c(k-1)+\eta_k) \leq x\}$ is equivalent to $\mathbb{E}(\tau(x))^p < \infty$, where $\tau(x) = \inf\{n \in \mathbb{N} : -c(n-1)+\eta_n > x\}$ for $x \in \mathbb{R}$. Thus, to prove Theorem 1.1 in full generality we have to be able to cope with complications stemming from the genuine randomness of ξ and a possible dependence of ξ and η.

Lemma 2.1. *Let $p > 0$, $c > 0$ and $x \in \mathbb{R}$. Suppose that $\lim_{t\to\infty} t\mathbb{P}\{\eta > t\} = s \in [0,\infty]$, and that $\mathbb{E}\eta^+ = \infty$ when $s = 0$. Then the series $\sum_{n\geq 1} n^{p-1}\mathbb{P}\{\max_{1\leq k\leq n}(-c(k-1)+\eta_k) \leq x\}$ converges or diverges depending on whether $s > cp$ or $s < cp$.*

For the proof, see p. 34 in [2].

Lemma 2.2. *Let σ be a stopping time for $(\xi_k, \eta_k)_{k\in\mathbb{N}}$ of finite mean. Suppose that $\lim_{t\to\infty} t\mathbb{P}\{\eta > t\} = s \in [0,\infty]$. Then*

$$\lim_{t\to\infty} t\mathbb{P}\{\max_{1\leq i\leq\sigma}(S_{i-1}+\eta_i) > t\} = s\mathbb{E}\sigma, \tag{3}$$

where the right-hand side is equal to 0 if $s = 0$ and ∞ if $s = \infty$.

See the proof of Lemma 5.1 in [1].

3. Proof of Theorem 1.1

Proof of (a). Assuming that $s < -\mathtt{m}p$ we intend to prove that $\mathbb{E}(\tau(x))^p = \infty$ for all $x \in \mathbb{R}$ which is equivalent to

$$\sum_{n\geq 1} n^{p-1}\mathbb{P}\{\max_{1\leq k\leq n}(S_{k-1}+\eta_k) \leq x\} = \infty$$

because

$$\begin{aligned}
\mathbb{E}(\tau(x))^p &= \int_{[0,\infty)} y^p \mathrm{d}\mathbb{P}\{\tau(x)\leq y\} = p\int_0^\infty y^{p-1}\mathbb{P}\{\tau(x)>y\}\mathrm{d}y \\
&\geq p\sum_{n\geq 2}\min((n-1)^{p-1}, n^{p-1})\mathbb{P}\{\tau(x)>n\} \\
&= p\sum_{n\geq 2}\min((n-1)^{p-1}, n^{p-1})\mathbb{P}\{\max_{1\leq k\leq n}(S_{k-1}+\eta_k)\leq x\}
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{E}(\tau(x))^p &= p\int_0^\infty y^{p-1}\mathbb{P}\{\tau(x)>y\}\mathrm{d}y \\
&\leq p\Big(1+\sum_{n\geq 1}\max((n+1)^{p-1}, n^{p-1})\mathbb{P}\{\max_{1\leq k\leq n}(S_{k-1}+\eta_k)\leq x\}\Big).
\end{aligned}$$

For $c > 0$, set

$$\kappa_0(c) := 0, \quad \kappa_n(c) := \inf\{j > \kappa_{n-1}(c) : S_j < S_{\kappa_{n-1}(c)} - c\}, \quad n \in \mathbb{N},$$

and $\kappa(c) := \kappa_1(c)$. Define

$$(\xi_n^*, \eta_n^*) := \big(S_{\kappa_n(c)} - S_{\kappa_{n-1}(c)}, \max_{\kappa_{n-1}(c)+1 \le i \le \kappa_n(c)} (S_{i-1} - S_{\kappa_{n-1}(c)} + \eta_i)\big), \quad n \in \mathbb{N}$$

and observe that the just introduced random vectors are independent copies of

$$(S_{\kappa(c)}, \max_{1 \le i \le \kappa(c)} (S_{i-1} + \eta_i)).$$

Furthermore,

$$\max_{1 \le i \le \kappa_n(c)} (S_{i-1} + \eta_i) = \max_{1 \le j \le n} (S_{j-1}^* + \eta_j^*) \le \max_{1 \le j \le n} (-c(j-1) + \eta_j^*), \tag{4}$$

where $S_0^* := 0$, and $S_n^* := \xi_1^* + \ldots + \xi_n^* = S_{\kappa_n(c)}$ for $n \in \mathbb{N}$.

Hence using the inequalities $\kappa_n(c) \ge n$ a.s. for $n \in \mathbb{N}$ and then (4) we obtain

$$\begin{aligned} \sum_{n \ge 1} n^{p-1} \mathbb{P}\{\max_{1 \le i \le n} (S_{i-1} + \eta_i) \le x\} &\ge \sum_{n \ge 1} n^{p-1} \mathbb{P}\{\max_{1 \le i \le \kappa_n(c)} (S_{i-1} + \eta_i) \le x\} \\ &\ge \sum_{n \ge 1} n^{p-1} \mathbb{P}\{\max_{1 \le j \le n} (-c(j-1) + \eta_j^*) \le x\}. \end{aligned}$$

Since $\lim_{n \to \infty} S_n = -\infty$ a.s., we conclude that $\mathbb{E}\kappa(c) < \infty$. Therefore, an appeal to Lemma 2.2 yields

$$\lim_{t \to \infty} t\mathbb{P}\{\eta_1^* > t\} = \lim_{t \to \infty} t\mathbb{P}\{\max_{1 \le i \le \kappa(c)} (S_{i-1} + \eta_i) > t\} = s\mathbb{E}\kappa(c) < \infty.$$

By Lemma 2.1 the last series diverges if we can find c such that $s\mathbb{E}\kappa(c) < cp$. This holds for any $c > 0$ if $s = 0$. If $s \in (0, \infty)$, then this holds for large enough c because $\lim_{c \to \infty} c^{-1}\mathbb{E}\kappa(c) = (-\mathtt{m})^{-1}$ if $\mathtt{m} > -\infty$ and $= 0$ if $\mathtt{m} = -\infty$ by the elementary renewal theorem. This completes the proof of part (a).

Proof of (b). Suppose now that $s \in (-\mathtt{m}p, \infty)$ (so that necessarily $\mathtt{m}$ is finite) and that $\mathbb{E}\xi_-^{rp+1} < \infty$ for some $r > s/(s + \mathtt{m}p)$. Pick $\delta_1 \in (0, s)$ and $\delta_2 > 0$ such that $r > (s - \delta_1)/(s - \delta_1 + (\mathtt{m} - \delta_2)p)$. To ease the notation we shall write μ for $-\mathtt{m} + \delta_2$ and ν for $s - \delta_1$. In particular, the last inequality reads

$$r > \nu/(\nu - \mu p). \tag{5}$$

Set

$$N := \sup\{n \in \mathbb{N}_0 : S_n + \mu n \le 0\}.$$

Since $\lim_{n \to \infty}(S_n + \mu n) = \lim_{n \to \infty}((S_n - \mathtt{m}n) + \delta_2 n) = +\infty$ a.s. we infer $N < \infty$ a.s. Furthermore, $\mathbb{E}\xi_-^{rp+1} < \infty$ entails $\mathbb{E}N^{rp} < \infty$ by Theorem 1 in [3]. Noting that

$$\max_{0 \le i \le n-1} (S_i + \eta_{i+1}) \ge \max_{N+1 \le i \le n-1} (S_i + \eta_{i+1}) \ge \max_{[n^{1/r}] \le i \le n-1} (-\mu i + \eta_{i+1})$$

on the event $\{N \le [n^{1/r}]-1\}$ (observe that $r>1$) we obtain

$$\begin{aligned}
&\sum_{n\ge1} n^{p-1}\mathbb{P}\{\max_{0\le i\le n-1}(S_i+\eta_{i+1})\le x\}\\
=\;&\sum_{n\ge1} n^{p-1}\mathbb{P}\{\max_{0\le i\le n-1}(S_i+\eta_{i+1})\le x, N\le [n^{1/r}]-1\}\\
+\;&\sum_{n\ge1} n^{p-1}\mathbb{P}\{\max_{0\le i\le n-1}(S_i+\eta_{i+1})\le x, N> [n^{1/r}]-1\}\\
\le\;&\sum_{n\ge1} n^{p-1}\mathbb{P}\{\max_{[n^{1/r}]\le i\le n-1}(-\mu i+\eta_{i+1})\le x\}\\
+\;&\sum_{n\ge1} n^{p-1}\mathbb{P}\{N>[n^{1/r}]-1\}.
\end{aligned}$$

The last series converges in view of

$$\begin{aligned}
\infty > \mathbb{E}N^{rp} = \int_{[0,\,\infty)} y^p \mathrm{d}\mathbb{P}\{N^r\le y\} &= p\int_0^\infty y^{p-1}\mathbb{P}\{N^r>y\}\mathrm{d}y\\
&= p\int_0^\infty y^{p-1}\mathbb{P}\{N>y^{1/r}\}\mathrm{d}y \ge p\sum_{n\ge1} n^{p-1}\mathbb{P}\{N>n^{1/r}\}.
\end{aligned}$$

Denote by $F(x):=\mathbb{P}\{\eta\le x\}$ the distribution function of η. Fix $x\in\mathbb{R}$. The assumption $\lim_{t\to\infty} t(1-F(t)) = s\in(0,\infty)$ implies that

$$\mu i(1-F(x+\mu i))\ge \nu \tag{6}$$

and thereupon

$$F(x+\mu i)\le 1-\nu/(\mu i)\le C(1-i^{-1})^{\nu/\mu}$$

for all i large enough, say $i\ge n_0$, and an appropriate constant $C>1$ when $\nu/\mu\in(0,1)$ and $C=1$ when $\nu/\mu\ge1$. Hence,

$$\begin{aligned}
\sum_{n\ge n_0} n^{p-1}\mathbb{P}\{\max_{[n^{1/r}]\le i\le n-1}(-\mu i+\eta_{i+1})\le x\} &= \sum_{n\ge n_0} n^{p-1}\prod_{i=[n^{1/r}]}^{n-1} F(x+\mu i)\\
&\le C\sum_{n\ge n_0} n^{p-1}\prod_{i=[n^{1/r}]}^{n-1}(1-i^{-1})^{\nu/\mu}\\
&= C\sum_{n\ge n_0} n^{p-1}\Big(\frac{[n^{1/r}]-1}{n-1}\Big)^{\nu/\mu}.
\end{aligned}$$

Since $n^{p-1}\Big(\frac{[n^{1/r}]-1}{n-1}\Big)^{\nu/\mu} \sim n^{-((1-1/r)\nu/\mu)+p-1}$ as $n\to\infty$, and the exponent $-((1-1/r)\nu/\mu)+p-1$ is smaller than -1 in view of (5) the last series converges. The proof of part (b) in the case $s<\infty$ is complete.

If $\lim_{t\to\infty} t(1-F(t))=\infty$ we first set $\mu=-\mathtt{m}+\delta_2$ for any $\delta_2>0$ and then choose ν in (6) large enough to ensure $(1-1/r)\nu/\mu>p$. The proof of Theorem 1.1 is complete.

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Faculty of Computer Science and Cybernetics, Taras Shevchenko National University of Kyiv, 01601, Kyiv, Ukraine

E-mail address: rashitov_b@ukr.net