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BERRY-ESSEEN TYPE BOUND FOR FRACTIONAL ORNSTEIN-UHLENBECK TYPE PROCESS DRIVEN BY SUB-FRACTIONAL BROWNIAN MOTION

We obtain a Berry-Esseen type bound for the distribution of the maximum likelihood estimator of the drift parameter for fractional Ornstein-Uhlenbeck type process driven by sub-fractional Brownian motion.

1. Introduction

Statistical inference for fractional diffusion processes satisfying stochastic differential equations driven by a fractional Brownian motion (fBm) has been studied earlier and a comprehensive survey of various methods is given in Prakasa Rao [17]. There has been a recent interest to study similar problems for stochastic processes driven by a sub-fractional Brownian motion. Bojdecki et al. [2] introduced a centered Gaussian process $\zeta^H = \{\zeta^H(t), t \geq 0\}$ called *sub-fractional Brownian motion* (sub-fBm) with the covariance function

$$C_H(s,t) = s^{2H} + t^{2H} - \frac{1}{2}[(s+t)^{2H} + |s-t|^{2H}]$$

where $0 < H < 1$. The increments of this process are not stationary and are more weakly correlated on non-overlapping intervals than those of a fBm. Tudor [25] introduced a Wiener integral with respect to a sub-fBm. Tudor [22, 23, 24, 25] discussed some properties related to sub-fBm and its corresponding stochastic calculus. By using a fundamental martingale associated to sub-fBm, a Girsanov type theorem is obtained in Tudor[25]. Diedhiou et al. [3] investigated parametric estimation for a stochastic differential equation (SDE) driven by a sub-fBm. Mendy [13] studied parameter estimation for the sub-fractional Ornstein-Uhlenbeck process defined by the stochastic differential equation

$$dX_t = \theta X_t dt + d\zeta^H(t), t \geq 0$$

where $H > \frac{1}{2}$. This is an analogue of the Ornstein-Uhlenbeck process, that is, a continuous time first order autoregressive process $X = \{X_t, t \geq 0\}$ which is the solution of a one-dimensional homogeneous linear stochastic differential equation driven by a sub-fBm $\zeta^H = \{\zeta^H_t, t \geq 0\}$ with Hurst parameter H. Mendy [13] proved that the least squares estimator estimator $\tilde{\theta}_T$ is strongly consistent as $T \to \infty$. Kuang and Xie [10] studied properties of maximum likelihood estimator for sub-fBm through approximation by a random walk. Kuang and Liu [9] discussed about the L^2-consistency and strong consistency of the maximum likelihood estimators for the sub-fBm with drift based on discrete observations. Yan et al. [26] obtained the Ito's formula for sub-fractional Brownian motion with Hurst index $H > \frac{1}{2}$. Shen and Yan [21] studied estimation for the drift of sub-fractional Brownian motion and constructed a class of biased estimators of James-Stein type which dominate the maximum likelihood estimator under the quadratic risk. El Machkouri et al. [5] investigated the asymptotic properties of the least squares

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estimator for non-ergodic Ornstein-Uhlenbeck process driven by Gaussian processes, in particular, sub-fractional Brownian motion. In a recent paper, we have investigated optimal estimation of a signal perturbed by a sub-fractional Brownian motion in Prakasa Rao [19]. Some maximal and integral inequalities for a sub-fBm were derived in Prakasa Rao [18]. Parametric estimation for linear stochastic differential equations driven by a sub-fractional Brownian motion is studied in Prakasa Rao [20]. We now obtain a Berry-Esseen type bound for the distribution of the maximum likelihood estimator for the drift parameter of a fractional Ornstein-Uhlenbeck type process driven by a sub-fractional Brownian motion.

2. Preliminaries

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ be a stochastic basis satisfying the usual conditions and the processes discussed in the following are $(\mathcal{F}_t)$-adapted. Further the natural filtration of a process is understood as the P-completion of the filtration generated by this process.

Let $\zeta^H = \{\zeta_t^H, t \geq 0\}$ be a normalized *sub-fractional Brownian motion* (sub-fBm) with Hurst parameter $H \in (0,1)$, that is, a Gaussian process with continuous sample paths such that $\zeta_0^H = 0, E(\zeta_t^H) = 0$ and

$$E(\zeta_s^H \zeta_t^H) = t^{2H} + s^{2H} - \frac{1}{2}[(s+t)^{2H} + |s-t|^{2H}], t \geq 0, s \geq 0. \tag{2.1}$$

Bojdecki et al. [2] noted that the process

$$\frac{1}{\sqrt{2}}[W^H(t) + W^H(-t)], t \geq 0,$$

where $\{W^H(t), -\infty < t < \infty\}$ is a fBm, is a centered Gaussian process with the same covariance function as that of a sub-fBm. This proves the existence of a sub-fBm. Let $D_H(s,t)$ denote the covariance function of a standard fractional Brownian motion with Hurst index H. Note that

$$D_H(s,t) = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H}).$$

Bojdecki et al. [2] proved the following result concerning properties of a sub-fBm.

Theorem 2.1. *Let $\zeta^H = \{\zeta^H(t), t \geq 0\}$ be a sub-fBm defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t, t \geq 0), P)$. Then the following properties hold.*

(i) The process ζ^H is self-similar, that is, for every $a > 0$,

$$\{\zeta^H(at), t \geq 0\} \triangleq \{a^H \zeta^H(t), t \geq 0\}$$

in the sense that the processes, on both the sides of the equality sign, have the same finite dimensional distributions.

(ii) The process ζ^H is not Markov and it is not a semi-martingale.

(iii) For all $s, t \geq 0$, the covariance function $C_H(s,t)$ of the process ζ^H is positive for all $s > 0, t > 0$. Furthermore

$$C_H(s,t) > D_H(s,t) \ \ \text{if} \ \ H < \frac{1}{2}$$

and

$$C_H(s,t) < D_H(s,t) \ \ \text{if} \ \ H > \frac{1}{2}.$$

(iv) Let $\beta_H = 2 - 2^{2H-1}$. For all $s \geq 0, t \geq 0$,

$$\beta_H (t-s)^{2H} \leq E[\zeta^H(t) - \zeta^H(s)]^2 \leq (t-s)^{2H}, \ \ \text{if} \ \ H > \frac{1}{2}$$

and

$$(t-s)^{2H} \leq E[\zeta^H(t)-\zeta^H(s)]^2 \leq \beta_H(t-s)^{2H}, \quad \textit{if } H<\frac{1}{2}$$

and the constants in the above inequalities are sharp.

(v) The process ζ^H has continuous sample paths almost surely and, for each $0<\epsilon<H$ and $T>0$, there exists a random variable $K_{\epsilon,T}$ such that

$$|\zeta^H(t)-\zeta^H(s)| \leq K_{\epsilon,T}|t-s|^{H-\epsilon}, 0\leq s,t\leq T.$$

Let $f:[0,T]\to R$ be a measurable function and $\alpha>0$, and σ and η be real. Define the Erdeyli-Kober-type fractional integral

$$(I^\alpha_{T,\sigma,\eta}f)(s)=\frac{\sigma s^{\alpha\eta}}{\Gamma(\alpha)}\int_s^T \frac{t^{\sigma(1-\alpha-\eta)-1}f(t)}{(t^\sigma-s^\sigma)^{1-\alpha}}dt, s\in[0,T], \tag{2.2}$$

and the function

$$\begin{aligned} n_H(t,s) &= \frac{\sqrt{\pi}}{2^{H-\frac{1}{2}}} I^{H-\frac{1}{2}}_{T,2,\frac{3-2H}{4}}(u^{H-\frac{1}{2}})I_{[0,t)}(s) \\ &= \frac{2^{1-H}\sqrt{\pi}}{\Gamma(H-\frac{1}{2})}s^{\frac{3}{2}-H}\int_0^t (x^2-s^2)^{H-\frac{3}{2}}dx\, I_{(0,t)}(s). \end{aligned} \tag{2.3}$$

The following theorem is due to Dzhaparidze and Van Zanten [4] (cf. Tudor [25]).

Theorem 2.2. *The following representation holds, in distribution, for a sub-fBm ζ^H:*

$$\zeta^H_t \triangleq c_H\int_0^t n_H(t,s)dW_s, 0\leq t\leq T \tag{2.4}$$

where

$$c_H^2=\frac{\Gamma(2H+1)\ \sin(\pi H)}{\pi} \tag{2.5}$$

and $\{W_t, t\geq 0\}$ is the standard Brownian motion.

Tudor [25] has defined integration of a non-random function $f(t)$ with respect to a sub-fBm ζ^H on an interval $[0,T]$ and obtained a representation of this integral as a Wiener integral for a suitable transformed function $\phi_f(t)$ depending on H and T. For details, see Theorem 3.2 in Tudor [25].

Tudor [23] (cf. Tudor [25], p. 467) obtained the prediction formula for a sub-fBm. For any $0<H<1$, and $0<a<t$,

$$E[\zeta^H_t|\zeta^H_s, 0\leq s\leq a]=\zeta^H_a+\int_0^a \psi_{a,t}(u)d\zeta^H_u \tag{2.6}$$

where

$$\psi_{a,t}(u)=\frac{2\ \sin(\pi(H-\frac{1}{2}))}{\pi}u(a^2-u^2)^{\frac{1}{2}-H}\int_a^t \frac{(z^2-a^2)^{H-\frac{1}{2}}}{z^2-u^2}z^{H-\frac{1}{2}}dz. \tag{2.7}$$

Let

$$M_t^H=d_H\int_0^t s^{\frac{1}{2}-H}dW_s=\int_0^t k_H(t,s)d\zeta^H_s \tag{2.8}$$

where

$$d_H=\frac{2^{H-\frac{1}{2}}}{c_H\Gamma(\frac{3}{2}-H)\sqrt{\pi}}, \tag{2.9}$$

$$k_H(t,s)=d_H s^{\frac{1}{2}-H}\psi_H(t,s), \tag{2.10}$$

and

$$\psi_H(t,s) = \frac{s^{H-\frac{1}{2}}}{\Gamma(\frac{3}{2}-H)}[t^{H-\frac{3}{2}}(t^2-s^2)^{\frac{1}{2}-H} - (H-\frac{3}{2})\int_s^t (x^2-s^2)^{\frac{1}{2}-H}x^{H-\frac{3}{2}}dx]I_{(0,t)}(s).$$

It can be shown that the process $M^H = \{M_t^H, 0 \le t \le T\}$ is a Gaussian martingale (cf. Tudor [25], Diedhiou et al. [3]) and is called the *sub-fractional fundamental martingale.* The filtration generated by this martingale is the same as the filtration $\{\mathcal{F}_t, t \ge 0\}$ generated by the sub-fBm ζ^H and the quadratic variation $< M^H >_s$ of the martingale M^H over the interval $[0,s]$ is equal to $w_s^H = \frac{d_H^2}{2-2H}s^{2-2H} = \lambda_H s^{2-2H}$ (say). For any measurable function $f : [0,T] \to R$ with $\int_0^T f^2(s)s^{1-2H}ds < \infty$, define the probability measure Q_f by

$$\frac{dQ_f}{dP}|_{\mathcal{F}_t} = \exp(\int_0^t f(s)dM_s^H - \frac{1}{2}\int_0^t f^2(s)d < M^H > (s)) = \exp(\int_0^t f(s)dM_s^H - \frac{d_H^2}{2}\int_0^t f^2(s)s^{1-2H}ds)$$

where P is the underlying probability measure. Let

$$(\psi_H f)(s) = \frac{1}{\Gamma(\frac{3}{2}-H)} I^{H-\frac{1}{2}}_{0,2,\frac{1}{2}-H} f(s) \tag{2.11}$$

where, for $\alpha > 0$,

$$(I^{\alpha}_{0,\sigma,\eta} f)(s) = \frac{\sigma s^{-\sigma(\alpha+\eta)}}{\Gamma(\alpha)}\int_0^s \frac{t^{\sigma(1+\eta)-1}f(t)}{(t^\sigma - s^\sigma)^{1-\alpha}}dt, s \in [0,T]. \tag{2.12}$$

Then the following Girsanov type theorem holds for the sub-fBm process (Tudor [25]).

Theorem 2.3. *The process*

$$\zeta_t^H - \int_0^t (\psi_H f)(s)ds, 0 \le t \le T$$

is a sub-fbm with respect to the probability measure Q_f. In particular, choosing the function $f \equiv a \in R$, it follows that the process $\{\zeta_t^H - at, 0 \le t \le T\}$ is a sub-fBm under the probability measure Q_f with $f \equiv a \in R$.

Let $Y = \{Y_t, t \ge 0\}$ be a stochastic process defined on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t, t \ge 0), P)$ and suppose the process Y satisfies the stochastic differential equation

$$dY_t = C(t)dt + d\zeta_t^H, t \ge 0 \tag{2.13}$$

where the process $\{C(t), t \ge 0\}$, adapted to the filtration $\{\mathcal{F}_t, t \ge 0\}$, such that the process

$$R_H(t) = \frac{d}{dw_t^H}\int_0^t k_H(t,s)C(s)ds, t \ge 0 \tag{2.14}$$

is well-defined and the derivative is understood in the sense of absolute continuity with respect to the measure generated by the function w_H. Differentiation with respect to w_t^H is understood in the sense:

$$dw_t^H = \lambda_H(2-2H)t^{1-2H}dt$$

and

$$\frac{df(t)}{dw_t^H} = \frac{df(t)}{dt} / \frac{dw_t^H}{dt}.$$

Suppose the process $\{R_H(t), 0 \leq t \leq T\}$, defined over the interval $[0,T]$ belongs to the space $L^2([0,T], dw_t^H)$. Define

$$\Lambda_H(t) = \exp\{\int_0^t R_H(s) dM_s^H - \frac{1}{2}\int_0^t [R_H(s)]^2 dw_s^H\} \tag{2.15}$$

with $E[\Lambda_H(T)] = 1$ and the distribution of the process $\{Y_t, 0 \leq t \leq T\}$ with respect to the measure $P^Y = \Lambda_H(t)\ P$ coincides with the distribution of the process $\{\zeta_t^H, 0 \leq t \leq T\|$ with respect to the measure P.

We call the process Λ^H as the *likelihood process* or the Radon-Nikodym derivative $\frac{dP^Y}{dP}$ of the measure P^Y with respect to the measure P.

Tudor [25] derived the following Girsanov type formula.

Theorem 2.4. *Suppose the assumptions of Theorem 2.2 hold. Define*

$$\Lambda_H(T) = \exp\{\int_0^T R_H(t) dM_t^H - \frac{1}{2}\int_0^T R_H^2(t) dw_t^H\}. \tag{2.16}$$

Suppose that $E(\Lambda_H(T)) = 1$. Then the measure $P^* = \Lambda_H(T)P$ is a probability measure and the probability measure of the process Y under P^* is the same as that of the process V defined by

$$V_t = \int_0^t d\zeta_s^H, 0 \leq t \leq T. \tag{2.17}$$

3. Main Results

Let us consider the stochastic differential equation

$$dX(t) = \theta\ X(t)dt + d\zeta_t^H, X(0) = 0, t \geq 0 \tag{3.1}$$

where $\theta \in \Theta \subset R, \zeta^H = \{\zeta_t^H, t \geq 0\}$ is a sub-fractional Brownian motion with known Hurst parameter H. In other words $X = \{X(t), t \geq 0\}$ is a stochastic process satisfying the stochastic integral equation

$$X(t) = \theta \int_0^t X(s)ds + \int_0^t d\zeta_s^H, t \geq 0. \tag{3.2}$$

We call such a process as fractional Ornstein-Uhelenbeck type process driven by sub-fractional Brownian motion. Diedhiou et al. [3] and Mendy [13] investigated parametric estimation for such a stochastic differential equation driven by a sub-fBm. We will now obtain a Berry-Esseen type bound for the distribution of the maximum likelihood estimator for the drift parameter for such processes.

Let

$$C(\theta, t) = \theta\ X(t), t \geq 0 \tag{3.3}$$

and assume that the sample paths of the process $\{C(\theta,t), t \geq 0\}$ are smooth enough so that the process

$$R_{H,\theta}(t) = \theta \frac{d}{dw_t^H}\int_0^t k_H(t,s)X(s)ds, t \geq 0 \tag{3.4}$$

is well-defined where w_t^H and $k_H(t,s)$ are as defined in Section 2. Suppose the sample paths of the process $\{R_{H,\theta}(t), 0 \leq t \leq T\}$ belong almost surely to $L^2([0,T], dw_t^H)$. Define

$$Z_t = \int_0^t k_H(t,s) dX_s, t \geq 0. \tag{3.5}$$

Then the process $Z = \{Z_t, t \geq 0\}$ is an $(\mathcal{F}_t)$-semimartingale with the decomposition

$$Z_t = \int_0^t R_{H,\theta}(s) dw_s^H + M_t^H, t \geq 0 \tag{3.6}$$

where M^H is the fundamental martingale defined by the equation (2.8) and the process X admits the representation

$$X_t = \int_0^t K_H(t,s) dZ_s \tag{3.7}$$

where the function

$$K_H(t,s) = \frac{c_H}{d_H} s^{H-\frac{1}{2}} n_H(t,s).$$

Let P_θ^T be the measure induced by the process $\{X_t, 0 \leq t \leq T\}$ when θ is the true parameter. Following Theorem 2.4, we get that the Radon-Nikodym derivative of P_θ^T with respect to P_0^T is given by

$$\frac{dP_\theta^T}{dP_0^T} = \exp[\int_0^T R_{H,\theta}(s) dZ_s - \frac{1}{2} \int_0^T R_{H,\theta}^2(s) dw_s^H]. \tag{3.8}$$

Maximum likelihood estimation

We now consider the problem of estimation of the parameter θ based on the observation of the process $X = \{X_t, 0 \leq t \leq T\}$ and study its asymptotic properties as $T \to \infty$.

Strong consistency:

Let $L_T(\theta)$ denote the Radon-Nikodym derivative $\frac{dP_\theta^T}{dP_0^T}$. The maximum likelihood estimator (MLE) is defined by the relation

$$L_T(\hat{\theta}_T) = \sup_{\theta \in \Theta} L_T(\theta). \tag{3.9}$$

We assume that there exists a measurable maximum likelihood estimator. Sufficient conditions can be given for the existence of such an estimator (cf. Lemma 3.1.2, Prakasa Rao [15]). Note that

$$\begin{aligned} R_{H,\theta}(t) &= \theta \frac{d}{dw_t^H} \int_0^t k_H(t,s) X(s) ds \\ &= \theta J(t).(say) \end{aligned} \tag{3.10}$$

Then

$$\log L_T(\theta) = \theta \int_0^T J(t) dZ_t - \frac{1}{2}\theta^2 \int_0^T J^2(t) dw_t^H \tag{3.11}$$

and the likelihood equation is given by

$$\int_0^T J(t) dZ_t - \theta \int_0^T J^2(t) dw_t^H = 0. \tag{3.12}$$

Hence the MLE $\hat{\theta}_T$ of θ is given by

$$\hat{\theta}_T = \frac{\int_0^T J(t) dZ(t)}{\int_0^T J^2(t) dw_t^H}. \tag{3.13}$$

Let θ_0 be the true parameter. Using the fact that

$$dZ_t = \theta_0 J(t)) dw_t^H + dM_t^H, \tag{3.14}$$

it can be shown that

$$\frac{dP_\theta^T}{dP_{\theta_0}^T} = \exp[(\theta-\theta_0)\int_0^T J(t)dM_t^H - \frac{1}{2}(\theta-\theta_0)^2\int_0^T J^2(t)dw_t^H]. \tag{3.15}$$

Following this representation of the Radon-Nikodym derivative, we obtain that

$$\hat{\theta}_T - \theta_0 = \frac{\int_0^T J(t)dM_t^H}{\int_0^T J^2(t)dw_t^H}. \tag{3.16}$$

We now discuss the problem of estimation of the parameter θ on the basis of the observation of the process X or equivalently the process Z on the interval $[0,T]$.

Theorem 3.1. *The maximum likelihood estimator $\hat{\theta}_T$ is strongly consistent, that is,*

$$\hat{\theta}_T \to \theta_0 \;\; a.s \;\; [P_{\theta_0}] \;\; as \;\; T \to \infty \tag{3.17}$$

provided

$$\int_0^T J^2(t)dw_t^H \to \infty \;\; a.s \;\; [P_{\theta_0}] \;\; as \;\; T \to \infty. \tag{3.18}$$

Proof. This theorem follows by observing that the process

$$\gamma_T \equiv \int_0^T J(t)dM_t^H, t \geq 0 \tag{3.19}$$

is a local continuous martingale with the quadratic variation process

$$<\gamma>_T = \int_0^T J^2(t)dw_t^H \tag{3.20}$$

and applying the Strong law of large numbers (cf. Liptser [11]; Liptser and Shiryayev [12]; Prakasa Rao [16], p. 61) under the condition (3.18) stated above. □

Remark: For the case of sub-fractional Ornstein-Uhlenbeck process investigated here and in Mendy [13], it can be checked that the condition stated in equation (3.18) holds and hence the maximum likelihood estimator $\hat{\theta}_T$ is strongly consistent as $T \to \infty$.

Limiting distribution:

We now discuss the limiting distribution of the MLE $\hat{\theta}_T$ as $T \to \infty$.

Theorem 3.2. *Suppose there exists a norming function $I_t, t \geq 0$ such that*

$$I_T^2 <\gamma_T> = I_T^2 \int_0^T J^2(t)dw_t^H \to \eta^2 \;\; in\; probability \;\; as \;\; T \to \infty \tag{3.21}$$

where $I_T \to 0$ as $T \to \infty$ and η is a random variable such that $P(\eta > 0) = 1$. Then

$$(I_T\gamma_T, I_T^2 <\gamma_T>) \to (\eta Z, \eta^2) \;\; in\; law \;\; as \;\; T \to \infty \tag{3.22}$$

where the random variable Z has the standard normal distribution and the random variables Z and η are independent.

Proof. This theorem follows as a consequence of the central limit theorem for martingales (cf. Theorem 1.49 ; Remark 1.47 , Prakasa Rao [16], p. 65). □

Observe that

$$I_T^{-1}(\hat{\theta}_T - \theta_0) = \frac{I_T\gamma_T}{I_T^2 <\gamma_T>} \tag{3.23}$$

Applying the Theorem 3.2, we obtain the following result.

Theorem 3.3. *Suppose the conditions stated in the Theorem 3.2 hold. Then*

$$I_T^{-1}(\hat{\theta}_T - \theta_0) \to \frac{Z}{\eta} \text{ in law as } t \to \infty \tag{3.24}$$

where the random variable Z has the standard normal distribution and the random variables Z and η are independent.

Remarks: If the random variable η is a constant with probability one, then the limiting distribution of the maximum likelihood estimator is normal with mean 0 and variance η^{-2}. Otherwise it is a mixture of the normal distributions with mean zero and variance η^{-2} with the mixing distribution as that of η.

4. Berry-Esseen Type Bound

Let θ_0 be the true parameter. In addition to the conditions stated in Section 3, suppose that the random variable η is a positive constant with probability one under P_{θ_0}-measure. Theorem 3.3 implies that

$$I_T^{-1}(\hat{\theta}_T - \theta_0) \to N(0, \eta^{-2}) \text{ in law as } T \to \infty \tag{4.1}$$

under P_{θ_0}-measure where $N(0, \sigma^2)$ denoted the Gaussian distribution with mean zero and variance σ^2. We would now like to obtain the rate of convergence in this limit leading to a Berry-Esseen type bound.

Suppose there exists non-random positive functions δ_T and ϵ_T decreasing to zero as $T \to \infty$ such that

$$\delta_T^{-1}\epsilon_T^2 \to \infty \text{ as } T \to \infty \tag{4.2}$$

and

$$\sup_{\theta \in \Theta} P_\theta^T(|\delta_T < \gamma >_T - 1| \geq \epsilon_T) = O(\epsilon_T^{1/2}) \tag{4.3}$$

where the process $\{\gamma_T, T \geq 0\}$ is as defined by equation (3.19). Note that the process $\{\gamma_T, T \geq 0\}$ is a locally square integrable continuous martingale. From the results on the representation of locally square integrable continuous martingales (cf. Ikeda and Watanabe [8], Chapter II, Theorem 7.2), it follows that there exists a standard Wiener process $\{B(t), t \geq 0\}$ adapted to $(\mathcal{F}_t)$ such that $\gamma_t = B(< \gamma >_T), t \geq 0$. In particular

$$\gamma_T \delta_T^{1/2} = B(< \gamma >_T \delta_T) \text{ a.s. } [P_{\theta_0}] \tag{4.4}$$

for all $T \geq 0$.

We use the following lemmas in the sequel.

Lemma 4.1. *Let $(\Omega, \mathcal{F}, P)$ be a probability space and f and g be $\mathcal{F}$-measurable functions. Then, for any $\varepsilon > 0$,*

$$\begin{aligned} &\sup_x |P(\omega : \frac{f(\omega)}{g(\omega)} \leq x) - \Phi(x)| \\ &\leq \sup_y |P(\omega : f(\omega) \leq y) - \Phi(x)| + P(\omega : |g(\omega) - 1| > \varepsilon) + \varepsilon \end{aligned} \tag{4.5}$$

where $\Phi(x)$ is the distribution function of the standard Gaussian distribution.

Proof. See Michael and Pfanzagl [14]. □

Lemma 4.2. *Let $\{B(t), t \geq 0\}$ be a standard Wiener process and V be a nonnegative random variable. Then, for every $x \in R$ and $\varepsilon > 0$,*

$$|P(B(V) \leq x) - \Phi(x)| \leq (2\varepsilon)^{1/2} + P(|V - 1| > \varepsilon). \tag{4.6}$$

Proof. See Hall and Heyde [7], p.85. □

Let us fix $\theta \in \Theta$. It is clear from the earlier remarks that

$$\gamma_T = <\gamma>_T I_T^{-1}(\hat{\theta}_T - \theta) \tag{4.7}$$

under P_θ-measure. Then it follows, from the Lemmas 4.1 and 4.2, that

$$\begin{aligned} (4.8)\quad & P_\theta[\delta_T^{-1/2} I_T^{-1}(\hat{\theta}_T - \theta) \le x] - \Phi(x)| \\ &= |P_\theta[\frac{\gamma_T}{<\gamma>_T}\delta_T^{-1/2} \le x] - \Phi(x)| \\ &= |P_\theta[\frac{\gamma_T/\delta_T^{-1/2}}{<\gamma>_T/\delta_T^{-1}} \le x] - \Phi(x)| \\ &\le \sup_x |P_\theta[\gamma_T \delta_T^{1/2} \le x] - \Phi(x)| \\ &\quad + P_\theta[|\delta_T <\gamma>_T - 1| \ge \varepsilon_T] + \varepsilon_T \\ &= \sup_y |P(B(<\gamma>_T \delta_T) \le y) - \Phi(y)| + P_\theta[|\delta_T <\gamma>_T - 1| \ge \varepsilon_T] + \varepsilon_T \\ &\le (2\varepsilon_T)^{1/2} + 2P_\theta[|\delta_T <\gamma>_T - 1| \ge \varepsilon_T] + \varepsilon_T. \end{aligned}$$

It is clear that the bound obtained above is of the order $O(\varepsilon_T^{1/2})$ under the condition (4.3) and it is uniform in $\theta \in \Theta$. Hence we have the following result giving a Berry-Esseen type bound for the distribution of the MLE.

Theorem 4.3. *Under the conditions (4.2) and (4.3),*

$$\begin{aligned} (4.9)\quad & \sup_{\theta\in\Theta}\sup_{x\in R} |P_\theta[\delta_T^{-1/2} I_T^{-1}(\hat{\theta}_T - \theta) \le x] - \Phi(x)| \\ &\le (2\varepsilon_T)^{1/2} + 2P_\theta[|\delta_T <\gamma>_T - 1| \ge \varepsilon_T] + \varepsilon_T = O(\varepsilon_T^{1/2}). \end{aligned}$$

As a consequence of this result, we have the following theorem giving the rate of convergence of the MLE $\hat{\theta}_T$.

Theorem 4.4. *Suppose the conditions (4.2) and (4.3) hold. Then there exists a constant* $c > 0$ *such that for every* $d > 0$,

$$\sup_{\theta\in\Theta} P_\theta[I_T^{-1}|\hat{\theta}_T - \theta| \ge d] \le c\varepsilon_T^{1/2} + 2P_\theta[|\delta_T <\gamma>_T - 1| \ge \varepsilon_T] = O(\varepsilon_T^{1/2}). \tag{4.10}$$

Proof. Observe that

$$\begin{aligned} (4.11)\quad & \sup_{\theta\in\Theta} P_\theta[I_T^{-1}|\hat{\theta}_T - \theta| \ge d] \\ &\le \sup_{\theta\in\Theta} |P_\theta[\delta_T^{-1/2} I_T^{-1}(\hat{\theta}_T - \theta) \ge d\delta_T^{-1/2}] - 2(1 - \Phi(d\delta_T^{-1/2}))| \\ &\quad + 2(1 - \Phi(d\delta_T^{-1/2})) \\ &\le (2\varepsilon_T)^{1/2} + 2\sup_{\theta\in\Theta} P_\theta[|\delta_T <\gamma>_T - 1| \ge \varepsilon_T] + \varepsilon_T \\ &\quad + 2d^{-1}\delta_T^{1/2}(2\pi)^{-1/2}\exp[-\frac{1}{2}\delta_T^{-1}d^2] \end{aligned}$$

by Theorem 4.3 and the inequality

$$1 - \Phi(x) < \frac{1}{x\sqrt{2\pi}}\exp[-\frac{1}{2}x^2] \tag{4.12}$$

for all $x > 0$ (cf. Feller [6], p.175). Since

$$\delta_T^{-1}\varepsilon_T^2 \to \infty \text{ as } T \to \infty$$

by the condition (4.2), it follows that

$$\sup_{\theta\in\Theta} P_\theta[I_T^{-1}|\hat{\theta}_T - \theta| \ge d] \le c\varepsilon_T^{1/2} + 2\sup_{\theta\in\Theta} P_\theta[|\delta_T <R>_T - 1| \ge \varepsilon_T] \tag{4.13}$$

for some constant $c > 0$ and the last term is of the order $O(\varepsilon_T^{1/2})$ by the condition (4.3). This proves Theorem 4.4. □

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