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ON CONSTRUCTING A STICKY MEMBRANE LOCATED ON A GIVEN SURFACE FOR A SYMMETRIC α-STABLE PROCESS

For a symmetric α-stable stochastic process with $\alpha \in (1,2)$ in a Euclidean space, a membrane located on a fixed bounded closed surface S is constructed in such a way that the points of the surface possess the property of delaying the process with some given positive coefficient $(p(x))_{x\in S}$. In other words, the points of S are *sticky* for the process constructed. We show that this process is associated with some initial-boundary value problem for pseudo-differential equations related to a symmetric α-stable process.

Introduction

Let $(x(t), \mathcal{M}_t, \mathbb{P}_x)$ be a standard Markov process in a d-dimensional Euclidean space $\mathbb{R}^d$ (we assume that $d \geq 2$ in this article) whose transition probability density g (with respect to Lebesgue measure on $\mathbb{R}^d$) is given by the equality

$$g(t,x,y) = (2\pi)^{-d} \int_{\mathbb{R}^d} \exp\{i(x-y,\xi) - ct|\xi|^\alpha\}\, d\xi, \quad t>0,\ x\in\mathbb{R}^d,\ y\in\mathbb{R}^d, \tag{1}$$

where $c > 0$ and $\alpha \in (1,2)$ are fixed parameters. We use Dynkin's notation from [3]. Theorem 3.14 there implies the existence of a standard Markov process in $\mathbb{R}^d$ with transition probability density g. This process is called a symmetric (more precisely, rotationally invariant) α-stable process. Its generator is denoted by $\mathbf{A}$ and this is a pseudo-differential operator whose symbol is given by the function $(-c|\xi|^\alpha)_{\xi\in\mathbb{R}^d}$. For any unit vector $l \in \mathbb{R}^d$ we denote by $\mathbf{B}_l$ a pseudo-differential operator with the function $(2ic|\xi|^{\alpha-2}(\xi,l))_{\xi\in\mathbb{R}^d}$ as its symbol.

The operator $\mathbf{A}$ acts on a bounded real-valued function $(\varphi(x))_{x\in\mathbb{R}^d}$ with bounded Lipschitzean gradient according to the formula

$$\mathbf{A}\varphi(x) = cq_\alpha \int_{\mathbb{R}^d} (\varphi(x+z) - \varphi(x) - (z, \nabla\varphi(x)))\frac{dz}{|z|^{d+\alpha}}, \quad x\in\mathbb{R}^d, \tag{2}$$

where $q_\alpha = \dfrac{\Gamma(\alpha+1)\Gamma((d+\alpha)/2)\sin(\pi\alpha/2)}{\pi^{(d+1)/2}\Gamma((\alpha+1)/2)}$. As for the operator $\mathbf{B}_l$, its action on a bounded Lipschitzean function $(\varphi(x))_{x\in\mathbb{R}^d}$ is given by the equality

$$\mathbf{B}_l\varphi(x) = \frac{2c}{\alpha} q_\alpha \int_{\mathbb{R}^d} (\varphi(x+z) - \varphi(x))(z,l)\frac{dz}{|z|^{d+\alpha}}, \quad x\in\mathbb{R}^d.$$

These formulae for $\mathbf{A}\varphi(x)$ and $\mathbf{B}_l\varphi(x)$ can be verified by immediate calculations resulting in the following equalities $\mathbf{A}\varphi_\xi(x) = -c|\xi|^\alpha\varphi_\xi(x)$ and $\mathbf{B}_l\varphi_\xi(x) = 2ic|\xi|^{\alpha-2}(\xi,l)\varphi_\xi(x)$ valid for all $x\in\mathbb{R}^d$ and $\xi\in\mathbb{R}^d$, where $\varphi_\xi(x) = e^{i(\xi,x)}$.

Let a bounded closed surface S in $\mathbb{R}^d$ be given such that it separates $\mathbb{R}^d$ into two open parts: the interior and exterior. We assume that S is the surface of the class $H^{1+\gamma}$ for some constant $\gamma\in(0,1)$ (see [7, Ch. IV, §4] and, also, [8]). Then there exists a tangent

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hyperplane to S at each point $x \in S$. The unit outer normal vector to S at $x \in S$ will be denoted by $\nu(x)$.

Let $(p(x))_{x\in S}$ be a given continuous function with positive values. We show that there exists a W-functional $(\eta_t(p))_{t\geq 0}$ of the process $(x(t))_{t\geq 0}$ such that its characteristic is given by $\mathbb{E}_x\eta_t(p) = \int_0^t d\tau \int_S g(\tau, x, y)p(y)\, d\sigma_y, \quad t \geq 0,\ x \in \mathbb{R}^d$, where the inner integral is a surface one.

For $t \geq 0$, we put $\zeta_t = \inf\{s \geq 0 : s + \eta_s(p) \geq t\}$, $\hat{x}(t) = x(\zeta_t)$, $\hat{\mathcal{M}}_t = \mathcal{M}_{\zeta_t}$. It is well-known that the process $(\hat{x}(t), \hat{\mathcal{M}}_t, \mathbb{P}_x)$ is also a standard Markov process in $\mathbb{R}^d$ (see, for example, [3, Ch. 10]).

Denote by $\mathbb{C}_b(\mathbb{R}^d)$ the Banach space of all continuous bounded functions on $\mathbb{R}^d$ with real values and the norm $\|\varphi\| = \sup_{x\in\mathbb{R}^d} |\varphi(x)|$.

We will use the following notation: for a function $(f(y))_{y\in\mathbb{R}^d}$ the symbol $f(x+)$ (respectively, $f(x-)$) for $x \in S$ means the limit value of $f(y)$, as y approaches x along any curve lying in a finite closed cone $\mathcal{K}$ in $\mathbb{R}^d$ with vertex at x such that $\mathcal{K} \subset \{y \in \mathbb{R}^d : (y, \nu(x)) > 0\} \cup \{x\}$ (respectively, $\mathcal{K} \subset \{y \in \mathbb{R}^d : (y, \nu(x)) < 0\} \cup \{x\}$).

The main result of this article is formulated as follows: for each bounded Hölder continuous function φ, the function $\hat{U}(\lambda, x, \varphi) = \int_0^{+\infty} e^{-\lambda t}\mathbb{E}_x\varphi(\hat{x}(t))\, dt, \quad \lambda > 0,\ x \in \mathbb{R}^d$ is continuous and satisfies the following equalities

$$\lambda\hat{U}(\lambda, x, \varphi) - \varphi(x) = \mathbf{A}\hat{U}(\lambda, \cdot, \varphi)(x) \tag{3}$$

for all $\lambda > 0$, $x \in \mathbb{R}^d \setminus S$;

$$\lambda\hat{U}(\lambda, x, \varphi) - \varphi(x) = \frac{1}{2p(x)}\left[\mathbf{B}_{\nu(x)}\hat{U}(\lambda, \cdot, \varphi)(x+) - \mathbf{B}_{\nu(x)}\hat{U}(\lambda, \cdot, \varphi)(x-)\right] \tag{4}$$

for all $\lambda > 0$, $x \in S$.

Note that equalities (3) and (4) are the Laplace transforms of the ones

$$\frac{\partial\hat{u}(t, x, \varphi)}{\partial t} = \mathbf{A}\hat{u}(t, \cdot, \varphi)(x), \quad t > 0,\ x \in \mathbb{R}^d \setminus S, \tag{5}$$

and

$$\frac{\partial\hat{u}(t, x, \varphi)}{\partial t} = \frac{1}{2p(x)}\left[\mathbf{B}_{\nu(x)}\hat{u}(t, \cdot, \varphi)(x+) - \mathbf{B}_{\nu(x)}\hat{u}(t, \cdot, \varphi)(x-)\right], \quad t > 0,\ x \in S, \tag{6}$$

respectively, if the condition $\hat{u}(0+, x, \varphi) = \varphi(x)$ is valid for all $x \in \mathbb{R}^d$ and

$$\hat{u}(t, x, \varphi) = \mathbb{E}_x\varphi(\hat{x}(t)), \quad t > 0,\ x \in \mathbb{R}^d. \tag{7}$$

Remark. It is curious to notice that if $\varphi \in D(\hat{\mathbf{A}})$ ($\hat{\mathbf{A}}$ denotes the generator of the process $(\hat{x}(t))_{t\geq 0}$), then the function (7) is differentiable with respect to $t > 0$ and therefore, it satisfies the equations (5) and (6). The interesting problem of describing the domain of $\hat{\mathbf{A}}$ will be out of our attention in this article.

In our article [9], the problem of constructing a sticky membrane, as well as an elastic screen located on a given hyperplane in $\mathbb{R}^d$ was considered.

In the limit case of $\alpha = 2$ (and $c = \frac{1}{2}$), our process is a standard Brownian motion and the operator $\mathbf{A}$ coincides with $\frac{1}{2}\Delta$ (Δ is the Laplace operator) and $\mathbf{B}_l$ coincides with $\frac{\partial}{\partial l}$ (the derivative in the direction l). Similar problems (in this case) were considered in the books [3, 5] and also in [1, 2, 6] and many others.

The article is organized as follows. In Section 1 some auxiliary results are presented. Section 2 is devoted to solving the main problem (3), (4).

1. Some auxiliary results

1.1. The function g defined by (1) is continuous in the region $t>0$, $x\in\mathbb{R}^d$ and $y\in\mathbb{R}^d$. Moreover, it is uniformly continuous in any region of the form $(t,x,y)\in[\tau,+\infty)\times\mathbb{R}^d\times\mathbb{R}^d$ for a fixed $\tau>0$.

As follows from [4, Ch. 4], the following inequalities (D^k means any partial derivative of the order $k=0,1,2,\dots$ in spatial arguments)

$$|D^k g(t,\cdot,y)(x)|\le N_k\frac{t}{(t^{1/\alpha}+|y-x|)^{d+\alpha+k}},\quad t>0,\ x\in\mathbb{R}^d,\ y\in\mathbb{R}^d, \tag{8}$$

are fulfilled, where N_k is some positive constant. Similar estimates are established in [4] for fractional derivatives of the function g, in particular, the inequality

$$|\mathbf{A}g(t,\cdot,y)|\le\frac{\tilde N}{(t^{1/\alpha}+|y-x|)^{d+\alpha}} \tag{9}$$

holds true for all $t>0$, $x\in\mathbb{R}^d$ and $y\in\mathbb{R}^d$ with some constant $\tilde N>0$.

One can easily see from (8) for $k=0$ that the following estimate

$$\int_S g(t,x,y)\,d\sigma_y\le Ct^{-1/\alpha} \tag{10}$$

holds true for all $x\in\mathbb{R}^d$ and $t>0$ with some constant $C>0$. Here and below we will denote by C all specific constants whose values are not important for us.

It is well-known that for each $\varphi\in\mathbb{C}_b(\mathbb{R}^d)$ the function $u(t,x,\varphi)\overset{def}{=}\int_{\mathbb{R}^d}g(t,x,y)\varphi(y)\,dy$, $t\ge0,\ x\in\mathbb{R}^d$ satisfies the equation $\dfrac{\partial u(t,x,\varphi)}{\partial t}=\mathbf{A}u(t,\cdot,\varphi)$ in the region $t>0$, $x\in\mathbb{R}^d$ and the initial condition $u(0+,x,\varphi)=\varphi(x)$ for $x\in\mathbb{R}^d$ (see [4, Ch. 4]).

1.2. Let $(\psi(t,x))_{t>0,x\in S}$ be a continuous function with real values satisfying the inequality $|\psi(t,x)|\le Ct^{-\beta}$ for all $t>0$ and $x\in S$ with some constants $C>0$ and $\beta<1$. We put $v(t,x,\psi)\overset{def}{=}\int_0^t d\tau\int_S g(t-\tau,x,y)\psi(\tau,y)\,d\sigma_y,\quad t>0,\ x\in\mathbb{R}^d$. Inequality (10) implies the fact that this function is well-defined. It is called a single-layer potential.

The following properties of the function v are established in [8].

1.2.A. The function v is continuous in the region $t>0$, $x\in\mathbb{R}^d$ and it satisfies the inequality

$$|v(t,x,\psi)|\le Ct^{1-\beta-1/\alpha} \tag{11}$$

for all $t>0$, $x\in\mathbb{R}^d$ with some constant $C>0$.

1.2.B. The function v is a solution of the equation $\dfrac{\partial v(t,x,\psi)}{\partial t}=\mathbf{A}v(t,\cdot,\psi)(x)$ in the region $t>0$ and $x\in\mathbb{R}^d\setminus S$.

1.2.C. The following relations

$$\mathbf{B}_{\nu(x)}v(t,\cdot,\psi)(x\pm)=\int_0^t d\tau\int_S g^{\nu(x)}(t-\tau,x,y)\psi(\tau,y)\,d\sigma_y\mp\psi(t,x) \tag{12}$$

hold true for all $t>0$ and $x\in S$, where $g^{\nu(x)}(t,x,y)=\mathbf{B}_{\nu(x)}g(t,\cdot,y)(x)$. The first item (denote it by $\mathbf{B}^{(d.v.)}_{\nu(x)}v(t,\cdot,\psi)(x)$) on the right-hand side of relations (12) is called the direct value of the expression $\mathbf{B}_{\nu(x)}v(t,\cdot,\psi)(x)$ for fixed $x\in S$ and $t>0$.

Note that the following estimation

$$\left|\mathbf{B}^{(d.v.)}_{\nu(x)}v(t,\cdot,\psi)(x)\right|\le Ct^{-\beta+\gamma/\alpha}(1\vee t^{1-\gamma/\alpha}),\quad t>0,\ x\in S, \tag{13}$$

is valid with some constant $C > 0$. This estimate was actually established in the process of proving Lemma 2 in [8].

1.3. Let $(\varphi(x))_{x\in\mathbb{R}^d}$ be a bounded Hölder continuous function. We now prove that the function

$$U(\lambda, x, \varphi) \stackrel{def}{=} \int_0^{+\infty} e^{-\lambda t} u(t, x, \varphi)\, dt, \quad \lambda > 0,\ x \in \mathbb{R}^d \tag{14}$$

is well-defined and it satisfies the following equation

$$\lambda U(\lambda, x, \varphi) - \varphi(x) = \mathbf{A} U(\lambda, \cdot, \varphi)(x), \quad \lambda > 0,\ x \in \mathbb{R}^d. \tag{15}$$

The obvious inequality $|u(t, x, \varphi)| \leq \|\varphi\|$, $t > 0$, $x \in \mathbb{R}^d$ shows that the integral in (14) is well-defined. In addition, we have the equality $U(\lambda, x, \varphi) = \int_{\mathbb{R}^d} G(\lambda, x, y)\varphi(y)\, dy$ valid for all $\lambda > 0$, $x \in \mathbb{R}^d$, where $G(\lambda, x, y) = \int_0^{+\infty} e^{-\lambda t} g(t, x, y)\, dt$, $\lambda > 0$, $x \in \mathbb{R}^d$, $y \in \mathbb{R}^d$.

Note that the following equality $\mathbf{A}u(t, \cdot, \varphi)(x) = \int_{\mathbb{R}^d} \mathbf{A}g(t, \cdot, y)(x)\varphi(y)\, dy$, $t > 0$, $x \in \mathbb{R}^d$ holds true for all $\varphi \in \mathbb{C}_b(\mathbb{R}^d)$ (see [8]). This fact, inequality (9) and the Hölder continuity of the function φ lead us to the following inequalities

$$|\mathbf{A}u(t, \cdot, \varphi)(x)| \leq \int_{\mathbb{R}^d} |\mathbf{A}g(t, \cdot, y)(x)||\varphi(y) - \varphi(x)|\, dy \leq C \cdot t^{-1+\theta/\alpha}, \quad t > 0,\ x \in \mathbb{R}^d$$

with some constant $C > 0$, where $\theta \in (0, 1)$ is the Hölder exponent of the function φ. This means that the integral $\int_0^{+\infty} e^{-\lambda t} \mathbf{A}u(t, \cdot, \varphi)(x)\, dt$ exists for all $\lambda > 0$ and $x \in \mathbb{R}^d$.

Taking into account formula (2) and inequality (8) we can change the order of the integration on the right-hand side of the following equality

$$\begin{aligned}\int_0^{+\infty} e^{-\lambda t} \mathbf{A}u(t, \cdot, \varphi)(x)\, dt =& c q_\alpha \int_0^{+\infty} e^{-\lambda t}\, dt \int_{\mathbb{R}^d} (\varphi(y) - \varphi(x))\, dy \times \\ &\times \int_{\mathbb{R}^d} (g(t, x+z, y) - g(t, x, y) - (z, \nabla g(t, \cdot, y)(x))) \frac{dz}{|z|^{d+\alpha}}.\end{aligned}$$

As a consequence, we obtain the equality $\mathbf{A}U(\lambda, \cdot, \varphi)(x) = \int_0^{+\infty} e^{-\lambda t} \mathbf{A}u(t, \cdot, \varphi)(x)\, dt$ valid for all $\lambda > 0$, $x \in \mathbb{R}^d$ and each bounded Hölder continuous function φ.

So, equation (15) for the function U follows now from the last of the statements in Subsection 1.1.

1.4. Formulae (11) and (13) show that the following equalities for the Laplace transforms of the functions $(v(t, x, \psi))_{t\geq 0, x\in\mathbb{R}^d}$ and $(\mathbf{B}^{(d.v.)}_{\nu(x)} v(t, \cdot, \psi)(x))_{t>0, x\in S}$ are valid:

$$V(\lambda, x, \psi) \stackrel{def}{=} \int_0^{+\infty} e^{-\lambda t} v(t, x, \psi)\, dt = \int_S G(\lambda, x, y)\Psi(\lambda, y)\, d\sigma_y, \quad \lambda > 0,\ x \in \mathbb{R}^d,$$

$$\int_0^{+\infty} e^{-\lambda t} \mathbf{B}^{(d.v.)}_{\nu(x)} v(t, \cdot, \psi)(x)\, dt = \int_S G^{\nu(x)}(\lambda, x, y)\Psi(\lambda, y)\, d\sigma_y, \lambda > 0,\ x \in S,$$

where $\Psi(\lambda, x) = \int_0^{+\infty} e^{-\lambda t} \psi(t, x)\, dt$ and $G^{\nu(x)}(\lambda, x, y) = \int_0^{+\infty} e^{-\lambda t} g^{\nu(x)}(t, x, y)\, dt$.

Moreover, the following relation

$$\mathbf{A}V(\lambda, \cdot, \psi)(x) = \lambda V(\lambda, x, \psi), \quad \lambda > 0,\ x \in \mathbb{R}^d \setminus S$$

holds true for any function ψ satisfying the inequality $|\psi(t,x)| \le Ct^{-\beta}$, $t>0$, $x\in S$ (see Subsection 1.2). If we additionally assume that the function $(\psi(t,x))_{t\ge0,x\in S}$ is continuous and bounded then the relations

$$\mathbf{B}_{\nu(x)}V(\lambda,\cdot,\psi)(x\pm) = \int_S G^{\nu(x)}(\lambda,x,y)\Psi(\lambda,y)\,d\sigma_y \mp \Psi(\lambda,x), \quad \lambda>0,\ x\in S$$

are valid. The proofs of these relations are similar to those proving the properties of the single-layer potential (see, [8]).

1.5. Let a positive continuous function $(p(x))_{x\in S}$ be given. One can easily verify that the function $f_t(x) = \int_0^t d\tau \int_S g(\tau,x,y)p(y)\,d\sigma_y$, $t>0$, $x\in\mathbb{R}^d$ is a W-function for the process $(x(t))_{t\ge0}$ (see [3, Ch. 6, §3]) satisfying the inequality $f_t(x) \le C\frac{\alpha}{\alpha-1}t^{1-1/\alpha}\|p\|$ for all $t\ge0$ and $x\in\mathbb{R}^d$ (see (10)), where $\|p\| = \sup_{x\in S} p(x)$ and C is the constant from (10). Therefore, according to Theorem 6.6 from [3], a W-functional $(\eta_t(p))_{t\ge0}$ of the process $(x(t))_{t\ge0}$ exists such that $\mathbb{E}_x\eta_t(p) = f_t(x)$ for all $t\ge0$ and $x\in\mathbb{R}^d$. The functional $\eta_t(p_0)$ with $p_0(x)\equiv1$ is called the local time on S for the process $(x(t))_{t\ge0}$. It is evident that $\eta_t(p) = \int_0^t p(x(s))\,d\eta_s(p_0)$.

Making use of Theorem 6.4 from [3], one can approximate the functional $(\eta_t(p))_{t\ge0}$ by the following ones $\eta_t^{(h)}(p) = \int_0^t d\tau \int_S g(h,x(\tau),y)p(y)\,d\sigma_y$, $t\ge0$ in the sense that

$$\mathbb{E}_x[\eta_t^{(h)}(p) - \eta_t]^2 \to 0, \quad h\to0+ \tag{16}$$

for all $t\ge0$, $x\in\mathbb{R}^d$.

As was proved in [10], this approximation and the Feynman-Kac formula allow one to write down the equation (with $\lambda>0$, $\varphi\in\mathbb{C}(\mathbb{R}^d)$)

$$Q_\lambda(t,x,\varphi) = \int_{\mathbb{R}^d} g(t,x,y)\varphi(y)\,dy - \lambda\int_0^t d\tau\int_S g(t-\tau,x,y)Q_\lambda(\tau,y,\varphi)p(y)\,d\sigma_y. \tag{17}$$

for the function $Q_\lambda(t,x,\varphi) = \mathbb{E}_x\varphi(x(t))\exp\{-\lambda\eta_t(p)\}$, $t>0$, $x\in\mathbb{R}^d$.

2. Solving the main problem

Consider the Markov process $(\hat{x}(t),\hat{\mathcal{M}}_t,\mathbb{P}_x)$ defined in Introduction.

Theorem. *For each bounded Hölder continuous function $(\varphi(x))_{x\in\mathbb{R}^d}$ the function*

$$\hat{U}(\lambda,x,\varphi) = \int_0^{+\infty} e^{-\lambda t}\mathbb{E}_x\varphi(\hat{x}(t))\,dt, \quad \lambda>0,\ x\in\mathbb{R}^d,$$

solves the problem (3), (4).

Proof. The resolvent operator of the process $(\hat{x}(t))_{t\ge0}$ can be calculated in the following way (see [5, Ch. II, §6])

$$\begin{aligned}\mathbb{E}_x\int_0^{+\infty} e^{-\lambda t}\varphi(\hat{x}(t))\,dt &= \mathbb{E}_x\int_0^{+\infty} e^{-\lambda t}\varphi(x(\zeta_t))\,dt = \\ = \mathbb{E}_x\int_0^{+\infty} e^{-\lambda(t+\eta_t(p))}\varphi(x(t))\,dt &+ \mathbb{E}_x\int_0^{+\infty} e^{-\lambda(t+\eta_t(p))}\varphi(x(t))\,d\eta_t(p),\end{aligned} \tag{18}$$

where $x\in\mathbb{R}^d$, $\lambda>0$, $\varphi\in\mathbb{C}_b(\mathbb{R}^d)$.

The first item on the right-hand side of (18) can be found from equation (17). Multiplying both sides of that equation by $e^{-\lambda t}$ and integrating with respect to t over $(0,+\infty)$, we get the equation

$$U_1(\lambda,x,\varphi)=\int_{\mathbb{R}^d}G(\lambda,x,y)\varphi(y)\,dy-\lambda\int_S G(\lambda,x,y)U_1(\lambda,y,\varphi)p(y)\,d\sigma_y, \tag{19}$$

where $G(\lambda,x,y)=\displaystyle\int_0^{+\infty}g(t,x,y)e^{-\lambda t}\,dt$ and

$$U_1(\lambda,x,\varphi)=\int_0^{+\infty}Q_\lambda(t,x,\varphi)e^{-\lambda t}\,dt=\mathbb{E}_x\int_0^{+\infty}e^{-\lambda(t+\eta_t(p))}\varphi(x(t))\,dt.$$

To calculate the second item on the right-hand side of (18), we observe that in accordance with (16), the relation

$$\mathbb{E}_x\int_0^{+\infty}e^{-\lambda(t+\eta_t(p))}\varphi(x(t))\,d\eta_t(p)=\lim_{h\to 0+}\mathbb{E}_x\int_0^{+\infty}e^{-\lambda(t+\eta_t(p))}\varphi(x(t))v_h(x(t))\,dt$$

is held, where $v_h(x)=\displaystyle\int_S g(h,x,y)p(y)\,d\sigma_y$, $h>0$, $x\in\mathbb{R}^d$.

Since $\mathbb{E}_x\displaystyle\int_0^{+\infty}e^{-\lambda(t+\eta_t(p))}\varphi(x(t))v_h(x(t))\,dt=U_1(\lambda,x,\varphi\cdot v_h)$, we have from (19) the following equation for $U_2(\lambda,x,\varphi)=\lim_{h\to 0+}U_1(\lambda,x,\varphi\cdot v_h)$

$$U_2(\lambda,x,\varphi)=\int_S G(\lambda,x,y)\varphi(y)p(y)\,d\sigma_y-\lambda\int_S G(\lambda,x,y)U_2(\lambda,y,\varphi)p(y)\,d\sigma_y. \tag{20}$$

We now put $\hat{U}(\lambda,x,\varphi)=U_1(\lambda,x,\varphi)+U_2(\lambda,x,\varphi)$. Then $\mathbb{E}_x\displaystyle\int_0^{+\infty}e^{-\lambda t}\varphi(\hat{x}(t))\,dt=\hat{U}(\lambda,x,\varphi)$. As follows from equations (19), (20) (see, also, Subsections 1.3 and 1.4), the function $\hat{U}$ satisfies the equation $\mathbf{A}\hat{U}(\lambda,\cdot,\varphi)(x)=\lambda\hat{U}(\lambda,x,\varphi)-\varphi(x)$ in the region $x\in\mathbb{R}^d\setminus S$ for each $\lambda>0$, that is equation (3).

As a consequence of the statements in Subsection 1.4 we have the following relations $(x\in S,\ \lambda>0)$

$$\begin{aligned}\mathbf{B}_{\nu(x)}U_1(\lambda,\cdot,\varphi)(x\pm)&=\int_{\mathbb{R}^d}G^{\nu(x)}(\lambda,x,y)\varphi(y)\,dy-\\&\quad-\lambda\int_S G^{\nu(x)}(\lambda,x,y)U_1(\lambda,y,\varphi)p(y)\,d\sigma_y\pm\lambda p(x)U_1(\lambda,x,\varphi),\\ \mathbf{B}_{\nu(x)}U_2(\lambda,\cdot,\varphi)(x\pm)&=\int_S G^{\nu(x)}(\lambda,x,y)\varphi(y)p(y)\,dy\mp p(x)\varphi(x)-\\&\quad-\lambda\int_S G^{\nu(x)}(\lambda,x,y)U_2(\lambda,y,\varphi)p(y)\,d\sigma_y\pm\lambda p(x)U_2(\lambda,x,\varphi).\end{aligned}$$

Hence, the function $\hat{U}$ satisfies the condition $(\lambda>0,\ x\in S)$

$$\mathbf{B}_{\nu(x)}\hat{U}(\lambda,\cdot,\varphi)(x+)-\mathbf{B}_{\nu(x)}\hat{U}(\lambda,\cdot,\varphi)(x-)=2p(x)(\lambda\hat{U}(\lambda,x,\varphi)-\varphi(x)),$$

that is condition (4). The theorem is proved. □

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