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GENERALIZED LIKELIHOOD RATIO TEST FOR DETECTION OF MULTIVARIATE DSI PROCESSES

This paper provides a new method in detecting multivariate discrete scale invariant (DSI) processes using an asymptotic generalized likelihood ratio test (GLRT). We consider two hypothesis tests: 1) Is a multivariate process, DSI or is it self-similar? 2) Is a multivariate process, DSI or is it nonstationary?. Then, using the asymptotic GLRT, the DSI behaviour can be detected. In this method, by imposing some flexible sampling scheme, we provide some discretization of continuous time discrete scale invariant (DSI) processes. Then, the relationship between a discrete-time DSI process and a corresponding multidimensional self-similar process, enables us to formulate the problem as a test for covariance structure of the processes. For DSI and self-similar processes, the covariance matrices are as a product of scale matrices to a block-Toeplitz matrix, in which there is no a closed form of maximum likelihood for such matrices. So, by considering the asymptotic case, where the block-Toeplitz matrix converges to a block-circulant matrix, the asymptotic GLRT is derived. To clarify the proposed method, an example as a multivariate simple Brownian motion is presented and its simulations are provided. Also the performance of the method is studied on the S&P500 and Daw Jones indices for some special periods.

1. Introduction

The concept of self-similarity and discrete scale invariance are used as a fundamental property to handle many natural phenomena. Many critical systems, like statistical physics, textures in geophysics, network traffic and image processing can be interpreted by these processes [2]. Scale invariance is often described as a symmetry of the system relatively to a transformation of a scale, that is mainly a dilation or a contraction (up to some re-normalization) of the system parameters [2]. Discrete scale invariance (DSI) is a property which requires invariance by dilation for certain preferred scaling factors [15]. This characteristic feature of such process is the invariance of its finite dimensional distributions by certain dilation for specific scaling factor. Burnecki et.al. [4] and Borgnat et.al. [3] have studied the property of DSI and its relation to periodically correlated processes by means of the Lamperti transformation.

Detecting of discrete scale invariance in a process is one of the most important problems. If a process is DSI, then this fact can usually be exploited in applications to improve estimation performance. Treating a process as DSI, when in fact it is not, generally leads to very poor performance. Also, the presence or absence of discrete scale invariance can be used to adopt other actions. But, most of the proposed methods in detection of DSI behaviour, have presented for scalar time series, even though some of the scalar detectors could easily be extended to multivariate time series. In this work, we consider vector-valued stochastic processes, and the proposed detector, consider testing discrete scale invariance vs. self-similarity, and also, testing discrete scale invariance vs. nonstationarity. So, we provide a new method to detect discrete scale invariance in a continuous-time multivariate process. The detection method is based

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on the work of Ramirez et.al [18] in detecting periodically correlated (PC) processes. Let $\{X(t), t \in \mathbb{R}^+\}$ be a DSI process with scale $\lambda > 1$. We consider the flexible sampling proposed by Rezakhah and Modarresi [11] which enables one to have samples at q arbitrary points $s_0 < s_1 < \cdots < s_{q-1}$ in the first scale interval $[1, \lambda)$ and follow the sampling at corresponding points $\{\lambda^n s_j, n \in \mathbb{N}, j = 0, ..., q-1\}$ in the other scale intervals. The sequence of samples of DSI processes provided by this scheme is called sampled DSI process. Embedding the sampled DSI process in q columns, provided an embedded multi-dimensional self-similar process, denoted by $U(\lambda^n) = (U^0(\lambda^n), \cdots, U^{q-1}(\lambda^n))$, where $U^j(\lambda^n) = X(\lambda^n s_j)$ [11]. This arrangement provides a suitable platform to extend analytic property of discrete time periodically correlated processes to the sampled DSI processes. By this method, we can investigate the DSI behaviour of the process, using studing the covariance structure of multivariate sampled DSI processes.

This paper is organized as follows: Section 2 presented some background on sampled DSI process, embedded multi-dimensional self-similar process and the modified Lamperti transformation. In Section 3, we provide the detection problem and formulates it as a test for the covariance structure of the observations. Section 4, derives the asymptotic GLRT. In Section 5, the performance of the estimation method is studied via simulation. Finally this method is applied to the real data of S&P500 and Dow Jones indices for some special periods.

2. Preliminaries

In this section, definitions of sampled DSI processes and the modified Lamperti transformation are provided. Also, the structure of the covariance matrix of multi-dimensional self-similar processes is reviewed.

Definition 2.1. A process $\{X(t), t \in \mathbb{R}^+\}$ is said to be self-similar of index $H > 0$, if for any $\lambda > 0$

$$\{X(\lambda t), t \in \mathbb{R}^+\} \equiv \{\lambda^{-H} X(t), t \in \mathbb{R}^+\}, \tag{1}$$

where $\equiv$ means equality in finite-dimensional distributions. The process is said to be DSI of index H and scaling factor $\lambda_0 > 0$ if (1) holds for $\lambda = \lambda_0$.

2.1. Sampled DSI Process and Modified Lamperti Transform.

Following the special scheme of sampling, the sampled DSI process and the modified Lamperti transform [11] are defined in this section.

Remark 2.1. Let $\{X(t), t \in \mathbb{R}^+\}$ be a DSI process with scale $\lambda > 1$. Considering the flexible sampling of this process at points of set $\breve{T} = \{\lambda^n s_j : n \in \mathbb{W}, j = 0, \cdots, q-1, 1 \le s_0 < \cdots < s_{q-1} < \lambda\}$, where $\mathbb{W} = \{0, 1, 2, \cdots\}$. Then $X(\cdot)$ with parameter space $\breve{T}$ is called sampled DSI process. If we consider sampling of $X(\cdot)$ at points $\breve{T} = \{\lambda^n s_j : n \in \mathbb{W}$, for fixed $1 \le s_j < \lambda\}$, then $X(\cdot)$ with parameter space $\breve{T}$ is called sampled self-similar process [11].

Definition 2.2. The modified Lamperti transform [11] with Hurst index $H > 0$, denoted by L_H^* and its inverse L_H^{*-1} provides a correspondence between a DSI process $\{X(t), t \in \breve{T}\}$ with scale $\lambda > 1$ and parameter space $\breve{T} = \{\lambda^n s_i : n \in \mathbb{W}, 1 \le s_0 < \cdots < s_{q-1}\}$, and a discrete time PC process $\{Y(i), i \in \mathbb{W}\}$ with period q by

$$X(\lambda^n s_k) = L_H^* Y(\lambda^n s_k) := \lambda^{nH} Y(nq+k), \;\; Y(nq+k) = L_H^{*-1} X(nq+k) = \lambda^{-nH} X(\lambda^n s_k). \tag{2}$$

One can easily verify that $X(\cdot)$ is a self-similar process if and only if $Y(\cdot)$ is a stationary process, and also $X(\cdot)$ is a DSI process with scale λ if and only if $Y(\cdot)$ is a PC process with period q for some $H > 0$.

Definition 2.3. The process $\mathbf{U}(t) = (U^0(t), U^1(t), \cdots, U^{q-1}(t))'$ with parameter space $\breve{T} = \{\lambda^n, n \in \mathbb{W}\}$ is a multi-dimensional self-similar process [10], where
(a) $\{U^j(\cdot)\}$ for every $j = 0, \cdots, q-1$ is self-similar process with parameter space $\breve{T} = \{\lambda^n, n \in \mathbb{W}\}$.
(b) For every $n, \tau \in \mathbb{Z}, j, k = 0, \cdots, q-1$

$$Cov(U^j(\lambda^{n+\tau}), U^k(\lambda^n)) = \lambda^{2nH} Cov(U^j(\lambda^\tau), U^k(1)).$$

Remark 2.2. Let $\{X(t), t \in \breve{T}\}$ be the sampled DSI process with scale $\lambda > 1$ and parameter space $\breve{T}$, as defined in Remark 2.1. Then $\mathbf{U}(\lambda^n) = (U^0(\lambda^n), \cdots, U^{q-1}(\lambda^n))'$ is called an embedded multi-dimensional self-similar process, where $\{U^j(\lambda^n) \equiv X(\lambda^n s_j)\}$ for fixed $j = 0, \cdots, q-1$ and $1 \leq s_0 < \cdots < s_{q-1} < l$, is a self-similar process [11].

3. Problem Formulation

We consider a multivariate process $\mathbf{X}(t) = (X^0(t), \cdots, X^{D-1}(t))'$ of dimension D, with parameter space $\breve{T} = \{\lambda^n s_j, j = 0, \cdots, q-1, n \in \mathbb{W}\}$. We also assume that the process is zero-mean circularly symmetric Gaussian. The question is whether the process $\mathbf{X}(\cdot)$ is self-similar, DSI with known scale λ, or nonstationary. That is, we are interested in the following three hypothesis tests:

$$\begin{aligned} \mathcal{H}_0 &: \quad \mathbf{X} \; is \; self-similar, \\ \mathcal{H}_1 &: \quad \mathbf{X} \; is \; DSI \; with \; scale \lambda, \\ \mathcal{H}_2 &: \quad \mathbf{X} \; is \; nonstationary. \end{aligned} \tag{3}$$

To answer the question, we consider Nq samples of the process $\mathbf{X}$, into the vector $\mathbf{y}$ as:

$$\mathbf{y} = \big(\mathbf{X}(s_0), \cdots, \mathbf{X}(s_{q-1}), \mathbf{X}(\lambda s_0), \cdots, \mathbf{X}(\lambda s_{q-1}), \cdots, \mathbf{X}(\lambda^{N-1} s_{q-1})\big)' \in \mathbb{C}^{DNq}. \tag{4}$$

Thus, the hypothesis in (3) may be formulated as

$$\begin{aligned} \mathcal{H}_0 &: \quad \mathbf{y} \sim N(0, \widetilde{\mathbf{C}}_{\mathbf{0}}^{\mathbf{H}}), \\ \mathcal{H}_1 &: \quad \mathbf{y} \sim N(0, \widetilde{\mathbf{C}}_{\mathbf{1}}^{\mathbf{H}}), \\ \mathcal{H}_2 &: \quad \mathbf{y} \sim N(0, \widetilde{\mathbf{C}}_{\mathbf{2}}^{\mathbf{H}}), \end{aligned} \tag{5}$$

where $\widetilde{\mathbf{C}}_{\mathbf{i}}^{\mathbf{H}} := E[\mathbf{y}\mathbf{y}^*] \in \mathbb{C}^{DNq \times DNq}$ is the covariance matrix under the ith hypothesis. So, the hypothesis test is based on the structure of $\widetilde{\mathbf{C}}_{\mathbf{i}}^{\mathbf{H}}$.

3.1. **Structure of Covariance Matrices.** For a self-similar process $\{\mathbf{X}(\lambda^n s_j), j = 0, \cdots, q-1, n \in \mathbb{W}\}$, the structure of the covariance matrix is

$$\widetilde{\mathbf{C}}_{\mathbf{0}}^{\mathbf{H}} = \begin{pmatrix} \mathbf{R}_0^H(0) & \mathbf{R}_0^H(-1) & \cdots & \lambda^{(N-1)H}\mathbf{R}_0^H(-Nq+1) \\ \mathbf{R}_0^H(1) & \mathbf{R}_0^H(0) & \cdots & \lambda^{(N-1)H}\mathbf{R}_0^H(-Nq+2) \\ \vdots & \vdots & \ddots & \vdots \\ \lambda^{(N-1)H}\mathbf{R}_0^H(Nq-1) & \lambda^{(N-1)H}\mathbf{R}_0^H(Nq-2) & \cdots & \lambda^{2(N-1)H}\mathbf{R}_0^H(0) \end{pmatrix},$$

where $\widetilde{\mathbf{C}}_{\mathbf{0}}^{\mathbf{H}} = \{\widetilde{\mathbf{C}}_{\mathbf{0},j,k}^{\mathbf{H}}(n,\tau)\}_{j,k=0,\cdots,q-1}$, is obtained using the modified Lamperti transform (2) as

$$\begin{aligned} \widetilde{\mathbf{C}}_{\mathbf{0},j,k}^{\mathbf{H}}(n,\tau) = E[\mathbf{X}(\lambda^{n+\tau}s_j)\mathbf{X}^*(\lambda^n s_k)] &= \lambda^{(2n+\tau)H} E[\mathbf{Y}((n+\tau)q+j)\mathbf{Y}^*(nq+k)] \\ &= \lambda^{(2n+\tau)H}\mathbf{R}_0^H(\tau q + j - k), \end{aligned}$$

where $\mathbf{R}_0^H \in \mathbb{C}^{D\times D}$, is the covariance matrix of a multivariate stationary process $\mathbf{Y}$. Moreover, the covariance matrix $\widetilde{\mathbf{C}}_\mathbf{0}^\mathbf{H}$ can also be represented as product of a scale matrix $\mathbf{\Lambda}$ to a block-Toeplitz matrix $\mathbf{C}_\mathbf{0}^\mathbf{H} \in \mathbb{C}^{DNq\times DNq}$ with block-size D, as

$$\widetilde{\mathbf{C}}_\mathbf{0}^\mathbf{H} = \mathbf{\Lambda}\mathbf{C}_\mathbf{0}^\mathbf{H}\mathbf{\Lambda}', \tag{6}$$

where $\mathbf{\Lambda} = \mathbf{\Lambda}^* \otimes \mathbf{I}_{Dq}$, $\mathbf{\Lambda}^*$ is a $N\times N$ diagonal matrix as

$$\mathbf{\Lambda}^* = Diag[1, \lambda^H, \lambda^{2H}, \cdots, \lambda^{(N-1)H}],$$

$\mathbf{I}_{Dq}$ is identical matrix of size Dq, and

$$\mathbf{C}_\mathbf{0}^\mathbf{H} = \begin{pmatrix} \mathbf{R}_0^H(0) & \mathbf{R}_0^H(-1) & . \; . \; . & \mathbf{R}_0^H(-Nq+1) \\ \mathbf{R}_0^H(1) & \mathbf{R}_0^H(0) & . \; . \; . & \mathbf{R}_0^H(-Nq+2) \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ \mathbf{R}_0^H(Nq-1) & \mathbf{R}_0^H(Nq-2) & . \; . \; . & \mathbf{R}_0^H(0) \end{pmatrix}_{DNq\times DNq}.$$

To construct the structure of the covariance matrix $\widetilde{\mathbf{C}}_\mathbf{1}^\mathbf{H}$ under DSI assumption, $\mathcal{H}_1$, we shall proceed by considering the data matrix $\widetilde{\mathbf{X}}$ as

$$\begin{aligned} \widetilde{\mathbf{X}} &= (\mathbf{X}(\lambda^n s_0), \cdots, \mathbf{X}(\lambda^n s_{q-1})) = \\ &= \begin{pmatrix} X^0(\lambda^n s_0) & X^0(\lambda^n s_1) & . \; . \; . & X^0(\lambda^n s_{q-1}) \\ X^1(\lambda^n s_0) & X^1(\lambda^n s_1) & . \; . \; . & X^1(\lambda^n s_{q-1}) \\ . & . & & . \\ . & . & & . \\ . & . & & . \\ X^{D-1}(\lambda^n s_0) & X^{D-1}(\lambda^n s_1) & . \; . \; . & X^{D-1}(\lambda^n s_{q-1}) \end{pmatrix}, \end{aligned}$$

where the ith row, is a q-dimensional self-similar process [10]. By considering the vector $\mathbf{V}(n) = vec(\widetilde{\mathbf{X}})$, which stacks the columns of $\widetilde{\mathbf{X}}$, we have that

$$\begin{aligned} \mathbf{V}(n) &= (\mathbf{V}^0(n), \cdots, \mathbf{V}^{q-1}(n))' \\ &\equiv (\mathbf{X}(\lambda^n s_0), \cdots, \mathbf{X}(\lambda^n s_{q-1}))' \in \mathbb{C}^{Dq}, \end{aligned}$$

which is self-similar, and $\mathbf{V}^j(n) \equiv \mathbf{X}(\lambda^n s_j)$. Thus, the vector $\mathbf{y}$ in (4) would be as a stack of N realizations of $\mathbf{V}(n)$ as

$$\mathbf{y} = (\mathbf{V}(0), \cdots, \mathbf{V}(N-1))' \in \mathbb{C}^{DNq}. \tag{7}$$

Now, using the modified Lamperti transform (2), and the correspondence between multidimensional stationary and self-similar processes, $\mathbf{V}(n) = \lambda^{nH}\mathbf{W}(n)$ where $\mathbf{W}(n) = (\mathbf{W}^0(n), \cdots, \mathbf{W}^{q-1}(n))'$ and $\mathbf{W}^j(n) = \mathbf{Y}(nq+j)$. Thus, the covariance matrix under $\mathcal{H}_1$ is obtained as

$$\widetilde{\mathbf{C}}_\mathbf{1}^\mathbf{H} = \begin{pmatrix} \mathbf{R}_1^H(0) & \lambda^H\mathbf{R}_1^H(-1) & . \; . \; . & \lambda^{(N-1)H}\mathbf{R}_1^H(-N+1) \\ \lambda^H\mathbf{R}_1^H(1) & \lambda^{2H}\mathbf{R}_1^H(0) & . \; . \; . & \lambda^{NH}\mathbf{R}_1^H(-N+2) \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ \lambda^{(N-1)H}\mathbf{R}_1^H(N-1) & \lambda^{NH}\mathbf{R}_1^H(N-2) & . \; . \; . & \lambda^{2(N-1)H}\mathbf{R}_1^H(0) \end{pmatrix}$$

The covariance matrix components $\widetilde{\mathbf{C}}_\mathbf{1}^\mathbf{H} = \{\widetilde{\mathbf{C}}_\mathbf{1}^\mathbf{H}(n,\tau)\}_{n,\tau=0,\cdots,N-1}$, are computed by

$$\begin{aligned} \widetilde{\mathbf{C}}_\mathbf{1}^\mathbf{H}(n,\tau) &= E[\mathbf{V}(n+\tau)\mathbf{V}^*(n)] \\ &= \lambda^{(2n+\tau)H}E[\mathbf{W}(n+\tau)\mathbf{W}^*(n)] \\ &= \lambda^{(2n+\tau)H}\mathbf{R}_1^H(\tau), \end{aligned}$$

where $\mathbf{R}_1^H \in \mathbb{C}^{Dq\times Dq}$. Moreover, $\widetilde{\mathbf{C}}_{\mathbf{1}}^{\mathbf{H}}$ can be represented as product of a scale matrix $\mathbf{\Lambda} = \mathbf{\Lambda}^* \otimes \mathbf{I}_{Dq}$ to a block-Toeplitz matrix $\mathbf{C}_{\mathbf{1}}^{\mathbf{H}}$ with block-size Dq as:

$$\widetilde{\mathbf{C}}_{\mathbf{1}}^{\mathbf{H}} = \mathbf{\Lambda}\mathbf{C}_{\mathbf{1}}^{\mathbf{H}}\mathbf{\Lambda}', \tag{8}$$

and

$$\mathbf{C}_{\mathbf{1}}^{\mathbf{H}} = \begin{pmatrix} \mathbf{R}_1^H(0) & . \; . \; . & \mathbf{R}_1^H(-N+1) \\ . & . & . \\ . & . & . \\ . & & . \\ \mathbf{R}_1^H(N-1) & . \; . \; . & \mathbf{R}_1^H(0) \end{pmatrix}.$$

For the nonstationary case, that is the simplest one, because the covariance matrix does not have any particular structure beyond being positive definite, we have that

$$\widetilde{\mathbf{C}}_{\mathbf{2}}^{\mathbf{H}} = \begin{pmatrix} \mathbf{R}_{2,0,0}^H(0,0) & \mathbf{R}_{2,0,1}^H(0,0) & . \; . \; . & \mathbf{R}_{2,0,q-1}^H(0,-N+1) \\ \mathbf{R}_{2,1,0}^H(0,0) & \mathbf{R}_{2,1,1}^H(0,0) & . \; . \; . & \mathbf{R}_{2,1,q-1}^H(0,-N+1) \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ \mathbf{R}_{2,q-1,0}^H(0,N-1) & \mathbf{R}_{2,q-1,1}^H(0,N-1) & . \; . \; . & \mathbf{R}_{2,q-1,q-1}^H(N-1,0) \end{pmatrix},$$

where $\mathbf{R}_{2,j,k}^H(n,\tau) = E[\mathbf{X}(\lambda^{n+\tau}s_j)\mathbf{X}^*(\lambda^n s_k)] \in \mathbb{C}^{D\times D}$.

Thus, it is shown that under the three hypothesis tests, $\mathcal{H}_0, \mathcal{H}_1, \mathcal{H}_2$, only the covariance matrix structures are known and the matrix-valued covariance sequences are unknown. Since the covariance matrices are unknown, so the hypotheses are composite and the GLRT would be as one of the typical approaches for binary tests [18], [23].

3.2. **Asymptotic Generalized Likelihood Ratio Test.** For generalized likelihood ratio tests, the ML estimate of covariance matrices are needed. But, under self-similarity and discrete scale invariance hypotheses, $\mathcal{H}_0$ and $\mathcal{H}_1$, we have seen that the covariance matrices are constructed as a product of scale matrices to a block-Toeplitz matrix. Since there is no closed-form solution for ML estimates of block-Toeplitz matrices, such matrices are approximated by block-circulant ones [18].

We consider $M \geq D$ independent and identically distributed realizations $\{\mathbf{y}_j\}_{j=1}^M$ of the vector $\mathbf{y}$, the likelihood of theses observations under $\mathcal{H}_i$ is

$$f_{\mathbf{y}_0,\cdots,\mathbf{y}_{M-1}}(\widetilde{\mathbf{C}}_{\mathbf{i}}^{\mathbf{H}}) = \prod_{m=0}^{M-1} f_{\mathbf{y}_m}(\widetilde{\mathbf{C}}_{\mathbf{i}}^{\mathbf{H}})$$

$$= \prod_{m=0}^{M-1} \frac{1}{(2\pi)^{DNq/2}|\widetilde{\mathbf{C}}_{\mathbf{i}}^{\mathbf{H}}|^{1/2}} \exp\left\{-M/2tr\left((\widetilde{\mathbf{C}}_{\mathbf{i}}^{\mathbf{H}})^{-1}\widehat{\mathbf{C}_{\mathbf{i}}^{\mathbf{H}}}\right)\right\}, \tag{9}$$

where $\widehat{\mathbf{C}_{\mathbf{i}}^{\mathbf{H}}} = \frac{1}{M}\sum_{m=0}^{M-1}\mathbf{y}_m\mathbf{y}_m^*$ is the sample covariance matrix.

As it can be seen by (6) and (8), under self-similarity and discrete scale invariance hypotheses, $\mathbf{C}_{\mathbf{0}}^{\mathbf{H}}$ and $\mathbf{C}_{\mathbf{1}}^{\mathbf{H}}$ are block-Toeplitz matrices. But it is shown by the following Theorem [19], [20] that, block-Toeplitz matrices are asymptotically equivalent to block-circulant matrices. So, the ML estimate can be achieved asymptotically.

Theorem 3.1. *As the number of samples, N, tends to infinity, the log-likelihood random variable parameterized by a block-Toeplitz covariance matrix, converges in mean-square sense to the log-likelihood random variable parameterized by a particular covariance matrix, i.e.,*

$$\lim_{N\to\infty} E\left[\frac{1}{N^2}\left|\log f_{\mathbf{y}_0,\cdots,\mathbf{y}_{M-1}}(\mathbf{C}) - \log f_{\mathbf{y}_0,\cdots,\mathbf{y}_{M-1}}(\mathbf{Q})\right|^2\right] = 0,$$

where $\mathbf{y}_m \in \mathbb{C}^{NB}$ *and* $\mathbf{C} \in \mathbb{C}^{NB\times NB}$ *is the block-Toeplitz covariance matrix with block size* B*,*

$$\mathbf{C} = \begin{pmatrix} \mathbf{R}(0) & . & . & . & \mathbf{R}(-N+1) \\ . & & . & & . \\ . & & & . & . \\ . & & & & . \\ \mathbf{R}(N-1) & . & . & . & \mathbf{R}(0) \end{pmatrix}.$$

The matrix valued covariance sequence that generates $\mathbf{C}$ *is* $\mathbf{R}(m) \in \mathbb{C}^{B\times B}$*, and* $\mathbf{Q} \in \mathbb{C}^{NB\times NB}$ *is the block-circulant covariance matrix whose* (j,k)*th block is* $\mathbf{R}(j-k \; mod \; N)$ [17]*. Equivalently, the block-circulant matrix may be factored as*

$$\mathbf{Q} = (\mathbf{F}_N \otimes \mathbf{J}_B)\mathbf{S}(\mathbf{F}_N \otimes \mathbf{J}_B)^H, \tag{10}$$

where $\mathbf{F}_N$ *is the Fourier matrix of dimension* N *with elements given by* $[\mathbf{F}_N]_{jk} = e^{-i2\pi jk/N}/\sqrt{N}$*,* $\mathbf{J}_B$ *is identity matrix of size* B*, the superscript* $(\cdot)^H$ *denotes Hermitian, and* $\mathbf{S}$ *is a block-diagonal matrix, whose* k*th block is given by the discrete Fourier transform of the covariance sequence*

$$\mathbf{S}(\theta_k) = \sum_{m=0}^{N} \mathbf{R}(m) e^{-i\theta_k m}, \tag{11}$$

with $\theta_k = \frac{2\pi k}{N}$*, and* $\mathbf{S}(\theta_k)$ *is the cross-spectral matrix (CSM) at frequency* θ_k [17]*,* [18]*.*

Proof. See [17]. □

Now, by Theorem 3.1, the hypothesis in (5) are asymptotically equivalent to

$$\begin{aligned} \mathcal{H}_0 &: \quad \mathbf{y} \sim N(0, \widetilde{\mathbf{Q}}_\mathbf{0}^\mathbf{H}), \\ \mathcal{H}_1 &: \quad \mathbf{y} \sim N(0, \widetilde{\mathbf{Q}}_\mathbf{1}^\mathbf{H}), \\ \mathcal{H}_2 &: \quad \mathbf{y} \sim N(0, \widetilde{\mathbf{Q}}_\mathbf{2}^\mathbf{H}). \end{aligned} \tag{12}$$

Thus, for self-similar processes, by (6) and (10), $\widetilde{\mathbf{Q}}_\mathbf{0}^\mathbf{H} = \mathbf{\Lambda}\mathbf{Q}_\mathbf{0}^\mathbf{H}\mathbf{\Lambda}'$, where $\mathbf{Q}_\mathbf{0}^\mathbf{H}$ is a block-circulant covariance matrix with block size D, with the structure as [18]

$$\mathbf{Q}_\mathbf{0}^\mathbf{H} = (\mathbf{F}_{Nq} \otimes \mathbf{J}_D)\mathbf{S}_\mathbf{0}^\mathbf{H}(\mathbf{F}_{Nq} \otimes \mathbf{J}_D)^H, \tag{13}$$

and $\mathbf{S}_\mathbf{0}^\mathbf{H}$ is a positive definite block-diagonal matrix of block size D. Also, for discrete scale invariant processes, $\widetilde{\mathbf{Q}}_\mathbf{1}^\mathbf{H} = \mathbf{\Lambda}\mathbf{Q}_\mathbf{1}^\mathbf{H}\mathbf{\Lambda}'$, where $\mathbf{Q}_\mathbf{1}^\mathbf{H}$ is block-circulant covariance matrix with block size Dq and the structure [18]

$$\mathbf{Q}_\mathbf{1}^\mathbf{H} = (\mathbf{F}_N \otimes \mathbf{J}_{Dq})\mathbf{S}_\mathbf{1}^\mathbf{H}(\mathbf{F}_N \otimes \mathbf{J}_{Dq})^H, \tag{14}$$

and $\mathbf{S}_\mathbf{1}^\mathbf{H}$ is an unkown positive definite block-diagonal matrix of block-size Dq. Finally, for nonstationary processes, $\widetilde{\mathbf{Q}}_\mathbf{2}$ is positive definite without further structure.

Now, as a particular reordering of the frequencies in the discrete Fourier transform of multivariate process $\mathbf{X}(\cdot)$, we consider a transform of observations $\mathbf{y}$ as [18]

$$\mathbf{z} = (\mathbf{L}_{\mathbf{Nq},\mathbf{N}} \otimes \mathbf{J}_\mathbf{D})(\mathbf{F}_{\mathbf{Nq}} \otimes \mathbf{J}_\mathbf{D})^H\mathbf{y}, \tag{15}$$

where $\mathbf{L}_{Nq,N}$ is the commutation (or stride permutation) matrix [21], which fulfills $vec(\mathbf{A}) = \mathbf{L}_{\mathbf{Nq},\mathbf{N}} \; vec(\mathbf{A}')$, and A is a $q \times N$ matrix.

Using (15), the hypothesis tests (12) should be formulated in terms of $\mathbf{z}$ instead of $\mathbf{y}$. Thus, we should obtain the covariance matrix $\mathbf{z}$ under these hypotheses. For self-similar processes, the covariance matrix of the transformed observations is

$$\begin{aligned} \widetilde{\mathbf{G}}_\mathbf{0} = E[\mathbf{z}\mathbf{z}^*|\mathcal{H}_0] &= (\mathbf{L}_{Nq,N}\mathbf{F}_{Nq}^H \otimes \mathbf{J}_D)\widetilde{\mathbf{Q}}_\mathbf{0}^\mathbf{H}(\mathbf{F}_{Nq}\mathbf{L}'_{Nq,N} \otimes \mathbf{J}_D) \\ &= \mathbf{\Lambda}(\mathbf{L}_{Nq,N} \otimes \mathbf{J}_D)\mathbf{S}_\mathbf{0}^\mathbf{H}(\mathbf{L}_{\mathbf{Nq},\mathbf{N}} \otimes \mathbf{J}_\mathbf{D})'\mathbf{\Lambda}', \end{aligned} \tag{16}$$

where we have used (13) and $(\mathbf{F}_{Nq} \otimes \mathbf{J}_D)^H(\mathbf{F}_{Nq} \otimes \mathbf{J}_D) = \mathbf{J}_{DNq}$. The covariance matrix $\widetilde{\mathbf{G}}_\mathbf{0}$ is a scaled and permuted version of the blocks $\mathbf{S}_\mathbf{0}^\mathbf{H}$ [18]. Also, $\widetilde{\mathbf{G}}_\mathbf{0}$ can be represented as $\widetilde{\mathbf{G}}_\mathbf{0} = \mathbf{\Lambda G_0 \Lambda'}$, where $\mathbf{G_0} = (\mathbf{L}_{Nq,N} \otimes \mathbf{J}_D)\mathbf{S}_\mathbf{0}^\mathbf{H}(\mathbf{L_{Nq,N}} \otimes \mathbf{J_D})'$ is a positive definite block-diagonal matrix.

For DSI processes, the covariance matrix under $\mathcal{H}_1$ is given by

$$\widetilde{\mathbf{G}}_\mathbf{1} = (\mathbf{L}_{Nq,N}\mathbf{F}_{Nq}^H \otimes \mathbf{J}_D)\widetilde{\mathbf{Q}}_\mathbf{1}^\mathbf{H}(\mathbf{F}_{Nq}\mathbf{L}'_{Nq,N} \otimes \mathbf{J}_D).$$

Using (14), and by the fact that $\mathbf{F}_N \otimes \mathbf{J}_{Dq} = (\mathbf{F}_N \otimes \mathbf{J}_q) \otimes \mathbf{J}_D$, the covariance matrix $\widetilde{\mathbf{G}}_\mathbf{1}$ becomes

$$\begin{aligned}
\widetilde{\mathbf{G}}_\mathbf{1} &= \mathbf{\Lambda}(\mathbf{L}_{Nq,N}\mathbf{F}_{Nq}^H \otimes \mathbf{J}_D)(\mathbf{F}_N \otimes \mathbf{J}_{Dq})\mathbf{S}_\mathbf{1}^\mathbf{H}(\mathbf{F}_N \otimes \mathbf{J}_{Dq})^H(\mathbf{F}_{Nq}\mathbf{L}'_{Nq,N} \otimes \mathbf{J}_D)\mathbf{\Lambda}' \\
&= \mathbf{\Lambda}(\mathbf{L}_{Nq,N}\mathbf{F}_{Nq}^H \otimes \mathbf{J}_D)\Big((\mathbf{F}_N \otimes \mathbf{J}_q) \otimes \mathbf{J}_D\Big)\mathbf{S}_\mathbf{1}^\mathbf{H}\Big((\mathbf{F}_N \otimes \mathbf{J}_q) \otimes \mathbf{J}_D\Big)^H \\
&\qquad \cdot (\mathbf{F}_{Nq}\mathbf{L}'_{Nq,N} \otimes \mathbf{J}_D)\mathbf{\Lambda}' \\
&= \mathbf{\Lambda}\Big(\mathbf{L}_{Nq,N}\mathbf{F}_{Nq}^H(\mathbf{F}_N \otimes \mathbf{J}_q)\Big) \otimes \mathbf{J}_D\ \mathbf{S}_\mathbf{1}^\mathbf{H}\ \Big((\mathbf{F}_N \otimes \mathbf{J}_q)^H\mathbf{F}_{Nq}\mathbf{L}'_{Nq,N}\Big) \otimes \mathbf{J}_D\ \mathbf{\Lambda}'.
\end{aligned} \tag{17}$$

In this case, the derivation of $\widetilde{\mathbf{G}}_\mathbf{1}$ is more involved and based on the following Cooley-Tukey theorem [18], [22].

Theorem 3.2. *The Fourier matrix may be factored as*

$$\mathbf{F}_{Nq} = (\mathbf{F}_N \otimes \mathbf{J}_q)\mathbf{T}_{Nq,q}(\mathbf{J}_N \otimes \mathbf{F}_q)\mathbf{L}_{Nq,N}, \tag{18}$$

where $\mathbf{T}_{Nq,q}$ *is a diagonal matrix of twiddle factors.*

Proof. See [22]. □

Now, by applying (18) on terms in (17), we have that

$$\begin{aligned}
\mathbf{L}_{Nq,N}\mathbf{F}_{Nq}^H(\mathbf{F}_N \otimes \mathbf{J}_q) &= (\mathbf{J}_N \otimes \mathbf{F}_q)^H\mathbf{T}^*_{Nq,q}, \\
(\mathbf{F}_N \otimes \mathbf{J}_q)^H\mathbf{F}_{Nq}\mathbf{L}'_{Nq,N} &= \mathbf{T}_{Nq,q}(\mathbf{J}_N \otimes \mathbf{F}_q),
\end{aligned}$$

[18]. Thus, the covariance matrix $\widetilde{\mathbf{G}}_\mathbf{1}$ would be as:

$$\begin{aligned}
\widetilde{\mathbf{G}}_\mathbf{1} &= \mathbf{\Lambda}\Big((\mathbf{J}_N \otimes \mathbf{F}_q)^H\mathbf{T}^*_{Nq,q}\Big) \otimes \mathbf{J}_D\ \mathbf{S}_\mathbf{1}^\mathbf{H}\ \Big(\mathbf{T}_{Nq,q}(\mathbf{J}_N \otimes \mathbf{F}_q)\Big) \otimes \mathbf{J}_D\mathbf{\Lambda}' \\
&= \mathbf{\Lambda G_1 \Lambda'},
\end{aligned} \tag{19}$$

where $\mathbf{G_1} = \Big((\mathbf{J}_N \otimes \mathbf{F}_q)^H\mathbf{T}^*_{Nq,q}\Big) \otimes \mathbf{J}_D\ \mathbf{S}_\mathbf{1}^\mathbf{H}\ \Big(\mathbf{T}_{Nq,q}(\mathbf{J}_N \otimes \mathbf{F}_q)\Big) \otimes \mathbf{J}_D$ is a block-diagonal matrix with block size Dq.

For nonstationary processes, the covariance matrix is

$$\widetilde{\mathbf{G}}_\mathbf{2} = (\mathbf{L}_{Nq,N}\mathbf{F}^*_{Nq} \otimes \mathbf{J}_D)\widetilde{\mathbf{Q}}_\mathbf{2}^\mathbf{H}(\mathbf{F}_{Nq}\mathbf{L}'_{Nq,N} \otimes \mathbf{J}_D),$$

which is just a positive definite matrix.

Therefore, the hypothesis (12) would be as

$$\begin{aligned}
\mathcal{H}_0 &: \quad \mathbf{z} \sim N(0, \widetilde{\mathbf{G}}_\mathbf{0}), \\
\mathcal{H}_1 &: \quad \mathbf{z} \sim N(0, \widetilde{\mathbf{G}}_\mathbf{1}), \\
\mathcal{H}_2 &: \quad \mathbf{z} \sim N(0, \widetilde{\mathbf{G}}_\mathbf{2}),
\end{aligned} \tag{20}$$

where $\widetilde{\mathbf{G}}_\mathbf{0} = \mathbf{\Lambda G_0 \Lambda'}$, and $\mathbf{G_0}$ is a positive definite block-diagonal matrix with block size D; and $\widetilde{\mathbf{G}}_\mathbf{1} = \mathbf{\Lambda G_1 \Lambda'}$, where $\mathbf{G_1}$ is also a positive definite block-diagonal matrix with block size Dq, and $\widetilde{\mathbf{G}}_\mathbf{2}$ is a positive definite matrix without further structure. In fact $\widetilde{\mathbf{G}}_\mathbf{2}$, is a block-diagonal matrix with block size DNq. So, all covariance matrices are block-diagonal, and this point simplifies the derivation of the GLRT test.

4. Asymptotic GLRT for Detecting DSI Processes

In this section, we derive the asymptotic GLRT for the tests discrete scale invariant vs. self-similar processes, and discrete scale invariant vs. nonstationary processes. To this end, first we consider the GLRT for testing the hypothesis

$$\begin{aligned} \mathcal{H}_0 &: \quad \mathbf{z} \sim N(0, \mathbf{D_0}) \\ \mathcal{H}_1 &: \quad \mathbf{z} \sim N(0, \mathbf{D_1}), \end{aligned} \tag{21}$$

where $\mathbf{D_0}$ and $\mathbf{D_1}$ are block-diagonal matrices with block size B_0 and B_1, respectively. It is shown in [18] that the GLRT for the test in (21), which is given by

$$\mathcal{G} = \frac{max f_{\mathbf{z}_0, \cdots, \mathbf{z}_{M-1}}(\mathbf{D_0})}{max f_{\mathbf{z}_0, \cdots, \mathbf{z}_{M-1}}(\mathbf{D_1})}, \tag{22}$$

is carried out by

$$\mathcal{G}^{1/M} = \frac{\left| diag_{B_1}(\hat{\mathbf{G}}) \right|}{\left| diag_{B_0}(\hat{\mathbf{G}}) \right|} = \left| \hat{\mathbf{A}}_{B_0}^{B_1} \right|, \tag{23}$$

where $diag_{B_i}(\hat{\mathbf{G}}), i = 0, 1$, builds a block-diagonal matrix from the $B_i \times B_i$ blocks on the main diagonal of $\hat{\mathbf{G}}$ by setting the off-diagonal blocks, equal to zero,

$$\hat{\mathbf{A}}_{B_0}^{B_1} = [diag_{B_0}(\hat{\mathbf{G}})]^{-1/2} diag_{B_1}(\hat{\mathbf{G}}) [diag_{B_i}(\hat{\mathbf{G}})]^{-1/2}$$

is a coherence matrix, and $\hat{\mathbf{G}}$ is the sample covariance matrix of $\mathbf{z}_0, \cdots, \mathbf{z}_{M-1}$.

4.1. GLRT for Testing DSI vs. Self-Similarity.

Now, consider the following hypothesis test

$$\begin{aligned} \mathcal{H}_0 &: \quad \mathbf{z} \sim N(0, \widetilde{\mathbf{G}}_\mathbf{0}), \\ \mathcal{H}_1 &: \quad \mathbf{z} \sim N(0, \widetilde{\mathbf{G}}_\mathbf{1}), \end{aligned} \tag{24}$$

for testing discrete scale invariance vs. self-similarity of a process $\{\mathbf{X}(t), t \in \widetilde{T}\}$, $\widetilde{T} = \{\lambda^n s_j, j = 0, \cdots, q-1, n \in \mathbb{W}\}$. The GLRT for testing $\mathcal{H}_0$ vs. $\mathcal{H}_1$ is

$$\mathcal{G}_{0:1} = \frac{max f_{\mathbf{z}_0, \cdots, \mathbf{z}_{M-1}}(\widetilde{\mathbf{G}}_\mathbf{0})}{max f_{\mathbf{z}_0, \cdots, \mathbf{z}_{M-1}}(\widetilde{\mathbf{G}}_\mathbf{1})}. \tag{25}$$

By the fact that $\widetilde{\mathbf{G}}_\mathbf{0}$ and $\widetilde{\mathbf{G}}_\mathbf{1}$ are block-diagonal matrices with block size D and Dq, respectively. So, by(23), the solution of (25) is provided in the following Theorem.

Theorem 4.1. *As $N \to \infty$, asymptotically the GLR for testing discrete scale invariance vs. self-similarity of a process $\{\mathbf{X}(T), t \in \widetilde{T}\}$, $\widetilde{T} = \{\lambda^n s_j, j = 0, \cdots, q-1, n \in \mathbb{W}\}$ is*

$$\mathcal{G}_{0:1}^{1/M} = \frac{\left| diag_{Dq}(\hat{\mathbf{G}}) \right|}{\left| diag_{D}(\hat{\mathbf{G}}) \right|} = \left| \hat{\mathbf{A}}_q^{Dq} \right| = \prod_{k=1}^{N} \left| \hat{\mathbf{A}}_k \right|, \tag{26}$$

where $\hat{\mathbf{A}}_q^{Dq} = [diag_D(\hat{\mathbf{G}})]^{-1/2} diag_{Dq}(\hat{\mathbf{G}}) [diag_D(\hat{\mathbf{G}})]^{-1/2}$ is a coherence matrix, and the kth $Dq \times Dq$ block on the diagonal of $\hat{\mathbf{A}}_q^{Dq}$ is denoted by $\hat{\mathbf{A}}_k$.

Proof. The proof is a direct application of Eq. 23. □

4.2. **GLRT for Testing DSI vs. Nonstationarity.** Consider the hypothesis test $\mathcal{H}_1$ against $\mathcal{H}_2$ for testing discrete scale invariance vs. nonstationarity of a process $\{\mathbf{X}(t), t \in \widetilde{T}\}$. The GLRT is

$$\mathcal{G}_{1:2} = \frac{max f_{\mathbf{z}_0,\cdots,\mathbf{z}_{M-1}}(\widetilde{\mathbf{G}}_\mathbf{1})}{max f_{\mathbf{z}_0,\cdots,\mathbf{z}_{M-1}}(\widetilde{\mathbf{G}}_\mathbf{2})}. \tag{27}$$

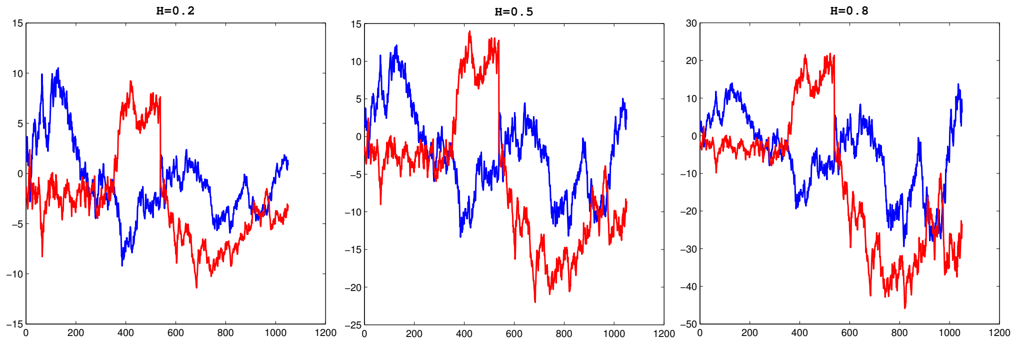

FIGURE 1. Sample paths of bivariate simple Brownian motion with scale $\lambda = 1.1$, different Hurst indices $H = 0.2, 0.5, 0.8$, $\gamma = 0.7$ and $\sigma_1 = \sigma_2 = 1$.

The solution of (27) is provided in the following Theorem.

Theorem 4.2. *As $N \to \infty$, asymptotically the GLR for testing discrete scale invariance vs. nonstationarity of a process $\{\mathbf{X}(t), t \in \widetilde{T}\}$ is*

$$\mathcal{G}_{1:2}^{1/M} = \frac{\left|\hat{\mathbf{G}}\right|}{\left|diag_{Dq}(\hat{\mathbf{G}})\right|} = \left|\hat{\mathbf{A}}_{Dq}^{DNq}\right|, \tag{28}$$

where $\hat{\mathbf{A}}_{Dq}^{DNq} = [diag_{Dq}(\hat{\mathbf{G}})]^{-1/2}\hat{\mathbf{G}}[diag_{Dq}(\hat{\mathbf{G}})]^{-1/2}$ is a coherence matrix.

Proof. The proof is a direct application of Eq. 23. □

Also, for testing self-similarity vs. nonstationarity, the GLRT is $\left|\hat{\mathbf{A}}_{D}^{DNq}\right|$. But we do not consider it in details, because it is not in the scope of the paper.

Example 1. The simple Brownian motion (SBM) was first introduced in [10] as a DSI process with scale λ and Hurst index H. Now, we consider a multivariate version of it, where in this case, moving of D particles in different environments $A_1, A_2, \cdots$ is considered based on Brownian motion with different rates, and we show these movements by a multivariate process $\mathbf{X}(t)$ as

$$\mathbf{X}(t) = \sum_{n=1}^{\infty} \lambda^{n(H-1/2)} I_{[\lambda^{n-1},\lambda^n)}(t)\mathbf{B}(t), \tag{29}$$

where $\mathbf{B}(\cdot)$ is a multivariate standard Brownian motion of dimension D, $I(\cdot)$ indicator function, $H > 0$, $\lambda > 0$. Such a process is a multivariate Brownian motion inside each scale interval $[\lambda^{n-1}, \lambda^n)$, and in general is a multivariate discrete scale invariant process with scale λ and Hurst index H. For $H = 0.5$, this process is a multivariate Brownian motion, and for $H \neq 0.5$, we call $\mathbf{X}(t)$ a multivariate simple Brownian motion (MSBM). It is shown in [10] that a simple Brownian motion is DSI and Markov with Hurst index H and scale λ. By sampling of the process $\mathbf{X}(\cdot)$ at points $\{\lambda^n s_j, j = 0, \cdots, q-1, n \in \mathbb{W}\}$, we have a discrete-time MSBM (DT-MSBM) process. Now, for checking the DSI property of the process, we follow a similar method in [10] for a multivariate process $\mathbf{X}(\mathbf{t})$. The covariance function of the process for $t \in A_n$ and $s \in A_m$ $(t \leq s)$ is

$$Cov(\mathbf{X}(t), \mathbf{X}(s)) = \lambda^{(n+m)(H-1/2)} Cov(\mathbf{B}(t), \mathbf{B}(s)) = \lambda^{(n+m)(H-1/2)} t\, \mathbf{\Gamma}, \tag{30}$$

where $Cov(\mathbf{B}(t), \mathbf{B}(s)) = min\{t, s\}\mathbf{\Gamma}$, and $\mathbf{\Gamma}$ is a covariance matrix with components $\gamma_{jk} = E[\mathbf{X}^j(t)\mathbf{X}^{*k}(s)]$. Now, if $t \in (\lambda^{n-1}, \lambda^n]$, then $\lambda t \in (\lambda^n, \lambda^{n+1}]$. So,

$$\begin{aligned} Cov(\mathbf{X}(\lambda t), \mathbf{X}(\lambda s)) &= \lambda^{(n+m+2)(H-1/2)} Cov(\mathbf{B}(\lambda t), \mathbf{B}(\lambda s)) = \lambda^{(n+m+2)(H-1/2)} \lambda t\, \mathbf{\Gamma} \\ &= \lambda^{2H} \lambda^{(n+m)(H-1/2)} t\, \mathbf{\Gamma} = \lambda^{2H} Cov(\mathbf{X}(t), \mathbf{X}(s)), \end{aligned}$$

which verifies the DSI property of $\mathbf{X}$.

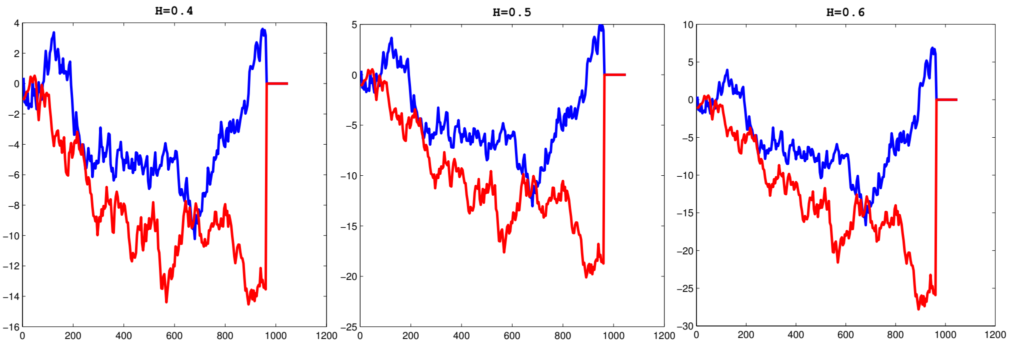

FIGURE 2. Sample paths of bivariate simple Brownian motion with scale $\lambda = 1.5$, different Hurst indices $H = 0.4, 0.5, 0.6$, $\gamma = 0.7$ and $\sigma_1 = \sigma_2 = 1$.

5. Simulation

In this section, we evaluate the performance of our detector using computer simulations, and it is shown that the introduced detector can exploit discrete scale invariant processes with a known scale λ. To this end, we used Matlab program to simulate $\mathbf{X}(t) = \sum_{n=1}^{M} \lambda^{n(H-1/2)} I_{[\lambda^{n-1}, \lambda^n)}(t)\mathbf{B}(t)$ for $M = 35$. Also, we consider $q = 30$ samples in each scale interval $[\lambda^{n-1}, \lambda^n)$ as $\lambda^n s_j$, $j = 0, \cdots, q-1$. In this case, we choose the sample points s_j such that they will be equally spaced in each scale interval, i.e. $s_j = 1 + j(\lambda - 1)/q$. Figure 1 consist of three curves of MSBM, all with scale $\lambda = 1.1$ and Hurst indices $H = 0.2, 0.5, 0.8$. It is worthy to note that, we have simulated only one discrete time multivariate Brownian motion $\mathbf{B}(\lambda^n s_j)$ of Example 1 for these three curves. In the middle plot, the Hurst index is $H = 0.5$, so it is a discrete-time multivariate Brownian motion which is a self-similar process. The left curve has scale $H = 0.2$, so in compare with the middle curve, the coefficients at the begining of the $n-$th scale interval, decrease to $\lambda^{n(0.2-1/2)}$, and it has less variation than multivariate Brownian motion at the begining of each scale interval. Also, in the right curve, the Hurst index is $H = 0.8$, where the enlargemant at the begining of scale intervals $[\lambda^{n-1}, \lambda^n)$, in compare with multivariate Brownian motion, caused by the growth of coefficients $\lambda^{n(0.8-1/2)}$ [11]. Figure 2 is also shows three curves of MSBM for $\lambda = 1.5$ and $H = 0.4, 0.5, 0.6$. Again only one multivariate Brownian motion is generated for three curves. In the middle plot, the Hurst index is $H = 0.5$, which is a multivariate Brownian motion. Also, the left and the right plots are MSBM with $H = 0.4$ and 0.6, respectinely, and the enlargmant of their variations are compared with the middle plot, multivariate Brownian motion. The left curve, where $H = 0.4$, has less variation in compare with the multivariate Brownian motion, which caused by the rate $\lambda^{n(0.4-1/2)}$ at the beginning of scale interval $[\lambda^{n-1}, \lambda^n)$. Also, in the right curve, where $H = 0.6$, the size of variations at the beginning of scale intervals, increased by the rate $\lambda^{n(0.6-1/2)}$.

Now, to investigate the accuracy of the proposed method in detecting DSI processes, we have simulated the multivariate simple Brownian with scale $\lambda = 1.1$, motion for different Hurst indices, by considering 50 scale intervals and $q = 30$ equally spaced samples in each scale interval with 100 repetitions. By applying our detector, we have computed and plotted the mean-square of errors (MSE) for different Hurst indices in Figure 3.

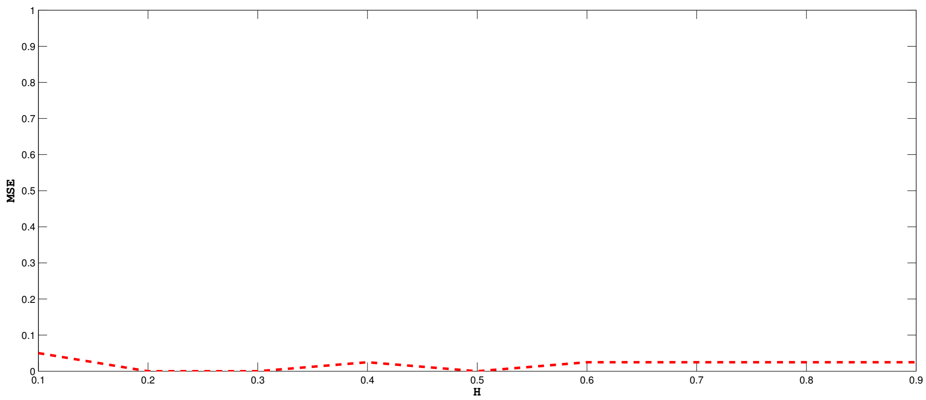

FIGURE 3. Mean-square of errors for the proposed GLRT detector in 100 repetitions of simulated multivariate simple Brownian with scale $\lambda = 1.1$, by considering 50 scale intervals and $q = 30$ equally spaced samples in each scale interval, for different Hurst indices.

Empirical Data. The superiority of the proposed detector, is also investigated for empirical data. To this end, we study the daily indices of two stock markets: S&P500 and Dow Jones, for some special periods. First, we consider daily indices of S&P500 from the first January 2000 till the end of 2004. As there is not any index on Saturdays, Sundays and holidays, the available data for the selected period are 1256 days. The process was studied by Bartolozzi et.al. [1], Rezakhah and Maleki [14], where the existence of a DSI behavior, in some periods of data has been justified. The indices from 16th October 2000 until 23th July 2002, which the DSI behavior can be seen in four scale intervals, was considered by the author in [14], and the preferred scaling factor of the process for the periods was evaluated approximately with 1.66, Figure 4 (a). Now, to investigate the efficiency of the proposed detector in multivariate DSI processes, we consider maximum and minimum of prices in each day to have a bivariate process. Then, by imposing the flexible sampling, we apply the proposed method to detect DSI behaviour of the process for selected periods. As a result, the propose method can detect DSI behaviour, for given scale, well.

As an another example, we consider daily indices of Dow Jones from 25th October 2001 till 28th May 2014. Same as the S&P500 indices, there is not any index on Saturdays, Sundays and holidays, the available data for the selected period are 3168 days. The existence of a DSI behavior in a period from 6th March 2009 until 14th November 2012 has been justified by the author [14] in four scale intervals, and the scale parameter λ was evaluated approximately with 1.493, Figure 4 (b) [14]. For these indices again, we consider maximum and minimum of the prices in each day, to have a bivariate DSI process. Then, by imposing the flexible sampling and applying the proposed method, it has be seen that the DSI process can be detected for the selected period.

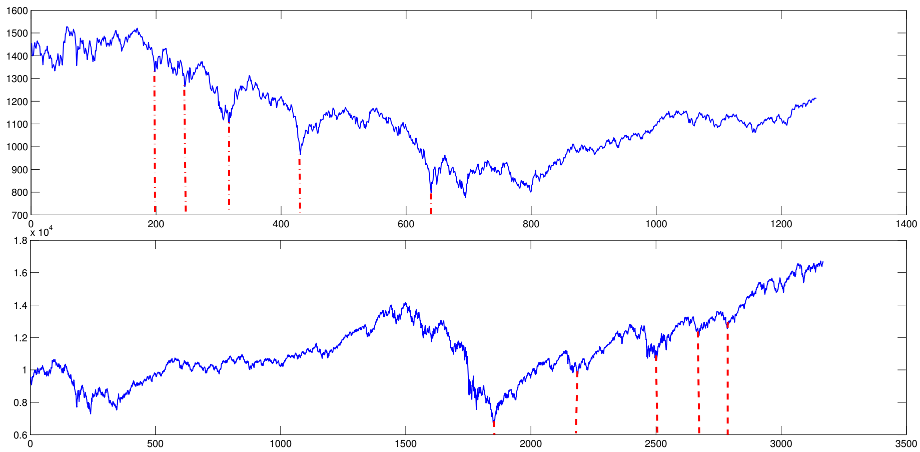

FIGURE 4. (Top) S&P500 indices from 1/1/2000 until 31/12/2004. The existence of a DSI behavior is justified from 16/10/2000 until 23/7/2002 in four scale intervals which are indicated with red dashed lines. (Bottom) Dow Jones indices from 25/10/2001 until 28/5/2014. The existence of a DSI behavior is justified from 6/3/2009 until 14/11/2012 in four scale intervals which are indicated with red dashed lines. The scale of the process for the periods is evaluated approximately with 1.493.

6. Conclusion

For testing whether a multivariate process is DSI, we have used the generalized likelihood ratio test. To this end, first we considered a flexible sampling scheme which provides some discritization of a continuous DSI process. Then, by a correspondence between a discrete-time DSI process and a multi-dimensional self-similar process, obtained by arranging the sampled DSI process in blocks of size given by the number of sample points in scale intervals, we obtained the covariance structure of processes. It is shown that, for DSI and self-similar processes, the covariance matrices are as a product of scale matrices to a block-Toeplitz matrix. But, the maximum likelihood does not have a closed form for block-Toeplitz matrices. So, we have used the asymptotic case, in which the block-Toeplitz matrix is replaced by a block-circulant matrix and the GLRT can be derived asymptotically. Simulations and numerical evaluations clarified the performance of the proposed method. The method is also applied for real data of S&P500 and Dow Jones indices for some special periods.

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