In this issue of Theory of Stochastic Processes we publish a paper of a former member of our editorial board Salah Mohammed with a co-author. It gives us a good reason to remember this great mind, and to say a few words in his memory. Professor Salah Eldin Mohammed passed away on December 21, 2016. Since 1998, almost for twenty years, he had maintained close relations with the Department of the Random Processes of the Institute of Mathematics of the National Academy of Sciences of Ukraine. That period of time was marked by active scientific cooperation, mutual visits, interesting and fruitful discussions. His knowledge of stochastic analysis was so deep and versatile that he could always offer a new and unexpected approach to the problem. Professor Mohammed was a contributing and very responsible member of our editorial board. Despite his numerous duties he always found time to formulate his opinion about certain article or to invite new authors.

Besides professional activities, Salah Mohammed was staying socially active. He could not remain indifferent to pain and injustice in the world. In 2007 he became one of the initiators of idea of organizing a mathematical institute in North Africa that made a huge difference in lives of many talented students. Also Salah was a very kind person. He always had a good joke or funny story to tell, to make people around him feel better and happier. We will remember him as a prominent mathematician, the man of duty and character.

Editorial board of the journal Theory of Stochastic Processes



Salah Mohammed with his wife and Andrey Dorogovtsev in Cambridge, 2010

FLAVIA SANCIER AND SALAH MOHAMMED

ON THE SOLUTION OF STOCHASTIC FUNCTIONAL DIFFERENTIAL EQUATIONS VIA MEMORY GAP

We present an alternative proof for the existence of solutions of stochastic functional differential equations satisfying a global Lipschitz condition. The proof is based on an approximation scheme in which the continuous path dependence does not go up to the present: there is a *memory gap*. Strong convergence is obtained by *closing the gap*. Such approximation is particularly useful when extending stochastic models with discrete delay to models with continuous full finite memory.

1. INTRODUCTION

Stochastic systems with memory are processes whose evolution in time is governed by random forces as well as an intrinsic dependence of the state on a finite part of its past history. Such systems are described by stochastic functional differential equations (SFDEs) [6]. The existence and uniqueness of a general class of SFDEs was first derived by Mohammed in [6] (Chapter II, sec. 2) through a successive approximation scheme, and conditions on the coefficients have been improved by others (e.g. [7]).

Our results are based upon approximating stochastic systems with full finite memory by stochastic systems with a *memory gap*. Solutions of systems with a memory gap are processes in which the intrinsic dependence of the state on its history goes only up to a specific time in the past. In this way, there is a *gap* between the past and present states.

The paper is outlined as follows. In section 2, we introduce the framework necessary for the development of our results. In section 3, we describe an approximation scheme and show the properties necessary to obtain convergence. In section 4, we show strong convergence of the approximation scheme to the solution of an SFDE with full finite memory.

2. Framework

Let $|\cdot|$ stand for the (*d*-dimensional) Euclidean norm, and let $|\cdot|_{d \times m}$ stand for the Frobenius norm in the space of real $d \times m$ matrices, $\mathbb{R}^{d \times m}$. Denote by $C_d := C([-L,0],\mathbb{R}^d)$ the separable Banach space of all continuous paths $\eta: [-L,0] \to \mathbb{R}^d$ given the supremum norm

$$\|\eta\|_{C_d} := \sup_{v \in [-L,0]} |\eta(v)|.$$

Consider a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0,T]}, P)$ satisfying the usual conditions, viz. the filtration $(\mathcal{F}_t)_{t \in [0,T]}$ is right-continuous, and each $\mathcal{F}_t, t \in [0,T]$, contains all *P*-null sets in \mathcal{F} . For a Banach space *E*, denote by $L^2(\Omega, E)$ the Banach space of all (equivalent classes of) $(\mathcal{F}, \text{Borel} E)$ -measurable maps $\Omega \to E$ which are L^2 (in the Bochner sense). A Bochner-measurable map $\eta : \Omega \to E$ is a map such that $\eta(\omega) = \lim_{n \to \infty} \eta_n(\omega)$ a.s., where each $\eta_n : \Omega \to E$ has a countable range and for which $\eta^{-1}\{x\}$ is \mathcal{F} -measurable for each x in *E*.

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The norm in $L^2(\Omega, E)$ is given by

$$\|\eta\|_{L^2(\Omega,E)} := \left[\int_{\Omega} \|\eta(\omega)\|_E^2 dP(\omega)\right]^{1/2},$$

for any $\eta \in L^2(\Omega, E)$.

Let $F : [0,T] \times L^2(\Omega, C_d) \to L^2(\Omega, \mathbb{R}^d)$ and $G : [0,T] \times L^2(\Omega, C_d) \to L^2(\Omega, \mathbb{R}^{d \times m})$ be jointly continuous and uniformly Lipschitz in the second variable, viz.

(1)
$$\|F(t,\psi_1) - F(t,\psi_2)\|_{L^2(\Omega,\mathbb{R}^d)} + \|G(t,\psi_1) - G(t,\psi_2)\|_{L^2(\Omega,\mathbb{R}^d \times m)}$$

(1) $\leq \alpha \|\psi_1 - \psi_2\|_{L^2(\Omega, C_d)}$

for all $t \in [0, T]$ and $\psi_1, \psi_2 \in L^2(\Omega, C_d)$. The Lipschitz constant α is independent of $t \in [0, T]$. For each $(\mathcal{F}_t)_{t \in [0, T]}$ -adapted process $y : [0, T] \to L^2(\Omega, C_d)$, the processes $F(\cdot, y(\cdot))$ and $G(\cdot, y(\cdot))$ are also $(\mathcal{F}_t)_{t \in [0, T]}$ -adapted. Let W be an m-dimensional Brownian Motion on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]}, P)$ and let $\theta \in L^2(\Omega, C_d)$ be an \mathcal{F}_0 -measurable process.

Remark 2.1. The functionals F and G also satisfy the linear growth property

(2)
$$\|F(t,\psi)\|_{L^{2}(\Omega,\mathbb{R}^{d})} + \|G(t,\psi)\|_{L^{2}(\Omega,\mathbb{R}^{d\times m})} \leq D(1+\|\psi\|_{L^{2}(\Omega,C_{d})}),$$

where D is a positive constant independent of $t \in [0, T]$. To see this, set $\psi_1 = \psi$ and $\psi_2 = 0$ in the Lipschitz condition (1), and use the joint continuity of F and G.

Consider the stochastic functional differential equation

(3)
$$\begin{cases} dx(t) = F(t, x_t)dt + G(t, x_t)dW(t), & t \in [0, T] \\ x(t) = \theta(t), & t \in [-L, 0], \end{cases}$$

where the segment $x_t \in L^2(\Omega, C_d)$ is given by the relation $x_t(\omega)(s) := x(\omega)(t+s)$, $s \in [-L, 0]$, a.a. $\omega \in \Omega$. A solution of (3) is a process $x \in L^2(\Omega, C([-L, T], \mathbb{R}^d))$ adapted to $(\mathcal{F}_t)_{t \in [0,T]}$, with initial process θ , which satisfies the Itô integral equation

$$x(t) = \begin{cases} \theta(\cdot)(0) + \int_0^t F(u, x_u) du + \int_0^t G(u, x_u) dW(u), & t \in [0, T] \\ \theta(\cdot)(t), & t \in [-L, 0], \end{cases}$$

almost surely. In order to prove existence of solutions of the SFDE (3), we will make frequent use of a martingale-type inequality for the Ito integral, which we state below.

Lemma 2.1. Let $W : [a,b] \times \Omega \to \mathbb{R}^m$ be an m-dimensional Brownian Motion on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [a,b]}, P)$. Suppose $g : [a,b] \times \Omega \to L(\mathbb{R}^m, \mathbb{R}^d)$ is measurable, $(\mathcal{F}_t)_{t \in [a,b]}$ -adapted and $\int_a^b E|g(t,\cdot)|^{2k} dt < \infty$, for a positive integer $k \ge 1$. Then

$$E \sup_{t \in [a,b]} \left| \int_{a}^{t} g(u,\cdot) dW(u) \right|^{2k} \le A_{k} (b-a)^{k-1} \int_{a}^{b} E|g(u,\cdot)|^{2k} du,$$

where

$$A_k := d^{k-1} \left(\frac{4k^3m^2}{2k-1}\right)^k.$$

Proof. For a proof, the reader may refer to Mohammed [6] (pg. 27).

3. The approximation scheme

In this section, we introduce an approximation scheme for the linear SFDE (3), and show the properties necessary to obtain convergence in $L^2(\Omega, C([-1 - L, T], \mathbb{R}^d))$.

Let $\hat{\theta}$ be given by

(4)
$$\hat{\theta}(t) := \begin{cases} \theta(t), \quad t \in [-L, 0], \\ \theta(-L), \quad t \in [-1 - L, -L], \end{cases}$$

and define the sequence of processes, for any positive integer $k \geq 1$,

(5)
$$x^{k}(t) = \begin{cases} \theta(0) + \int_{0}^{t} F(u, x_{u-1/k}^{k}) du + \int_{0}^{t} G(u, x_{u-1/k}^{k}) dW(u), & t \in [0, T] \\ \hat{\theta}(t), & t \in [-1 - L, 0], \end{cases}$$

where for $t \in [-1,0]$, x_t^k is given by $x_t^k(s) = \hat{\theta}_t(s) := \hat{\theta}(t+s)$, $s \in [-L,0]$, a.a. $\omega \in \Omega$, $k \geq 1$. For $t \in [-1,0]$ define $\mathcal{F}_t := \mathcal{F}_0$. In what follows, we show existence of the x^k 's by integrating forward over steps of length 1/k.

Proposition 3.1. Each x^k satisfies the following properties for any $t \in [-1/k, T]$: (i) $x^k(t)$ and x_t^k are well defined and \mathcal{F}_t -measurable. (*ii*) $(x^k(u))_{u \in [-1/k,t]} \in L^2(\Omega, C([-1/k,t], \mathbb{R}^d))$ and $x^k_t \in L^2(\Omega, C_d)$, with

$$E\left[\sup_{v\in[-1/k,t]}|x^{k}(v)|^{2}\right]+\|x_{t}^{k}\|_{L^{2}(\Omega,C_{d})}^{2}\leq K,$$

where K is a constant independent of k.

Proof. We use induction with forward steps of length 1/k. For simplicity, consider T a positive integer. For $t \in [-1/k, 0]$, properties (i) and (ii) hold trivially. Now assume that properties (i) and (ii) hold for any $t \in [(n-1)/k, n/k]$, where n is an integer satisfying $0 \le n < kT$. Then we show that properties (i) and (ii) hold for any $t \in [n/k, (n+1)/k]$. For any $t \in [n/k, (n+1)/k]$, one can write

(6)
$$x^{k}(t) = x^{k}(n/k) + \int_{n/k}^{t} F(u, x_{u-1/k}^{k}) du + \int_{n/k}^{t} G(u, x_{u-1/k}^{k}) dW(u).$$

Since $(x^k(t))_{t \in [(n-1)/k, n/k]}$ is well defined, $(\mathcal{F}_t)_{t \in [(n-1)/k, n/k]}$ -adapted and continuous

(induction hypothesis), then $(x_t^k)_{t \in [(n-1)/k, n/k]}$ is well defined, $(\mathcal{F}_t)_{t \in [(n-1)/k, n/k]}$ -adapted and continuous (Lemma II-2.1 in Mohammed [6]). This implies that $x_{t-1/k}^k$ is $\mathcal{F}_{t-1/k}$ -measurable for any $t \in [n/k, (n+1)/k]$. Therefore, by the assumptions in section 2 and the previous discussion, the processes $[n/k,(n+1)/k] \ni t \mapsto$ $F(t, x_{t-1/k}^k)$ and $[n/k, (n+1)/k] \ni t \mapsto G(t, x_{t-1/k}^k)$ are continuous and $(\mathcal{F}_t)_{t \in [n/k, (n+1)/k]}$ adapted. Hence, from (6), $(x^k(t))_{t \in [n/k,(n+1)/k]}$ is a well defined $(\mathcal{F}_t)_{t \in [n/k,(n+1)/k]}$ adapted continuous semimartingale. This shows that property (i) holds for any $t \in$ [n/k, (n+1)/k].

Next, notice that for any $t \in [-1/k, (n+1)/k]$,

$$E\left[\sup_{s\in[-L,0]}|x_t^k(s)|^2\right] \le \|\theta\|_{L^2(\Omega,C_d)}^2 + E\left[\sup_{s\in[0,t]}|x^k(s)|^2\right].$$

Now applying lemma 2.1 and the linear growth property (2) of F and G, we have for any $t \in [0, (n+1)/k]$:

$$\begin{split} E \left[\sup_{v \in [0,t]} |x^{k}(\cdot)(v)|^{2} \right] \\ &= E \left[\sup_{v \in [0,t]} \left| \theta(0) + \int_{0}^{v} F(u, x_{u-1/k}^{k}) du + \int_{0}^{v} G(u, x_{u-1/k}^{k}) dW(u) \right|^{2} \right] \\ &\leq E \left[3|\theta(0)|^{2} \right] + 3t \left[\int_{0}^{t} \|F(u, x_{u-1/k}^{k})\|_{L^{2}(\Omega, \mathbb{R}^{d})}^{2} du \right] \\ &+ 3(4m^{2}) \left[\int_{0}^{t} \|G(u, x_{u-1/k}^{k})\|_{L^{2}(\Omega, \mathbb{R}^{d \times m})}^{2} du \right] \\ &\leq 3 \|\theta\|_{L^{2}(\Omega, C_{d})}^{2} + 6TD^{2}(T + 4m^{2})(1 + \|\theta\|_{L^{2}(\Omega, C_{d})}^{2}) \end{split}$$

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(7) +
$$6D^2(T+4m^2) \int_0^t E\left[\sup_{v\in[0,u]} |x^k(v)|^2\right] du.$$

Applying Gronwall's inequality to the above inequality, we obtain:

$$E\left[\sup_{v\in[0,t]}|x^k(v)|^2\right] \le C_1 e^{C_2 T}$$

where $C_1 := 3 \|\theta\|_{L^2(\Omega, C_d)}^2 + 6TD^2(T + 4m^2)(1 + \|\theta\|_{L^2(\Omega, C_d)}^2)$ and $C_2 := 6D^2(T + 4m^2)$. Hence, for each $t \in [0, (n+1)/k]$,

$$E\left[\sup_{v\in[-1/k,t]}|x^{k}(v)|^{2}\right] + \|x_{t}^{k}\|_{L^{2}(\Omega,C)} \le 2C_{1}e^{C_{2}T} + 2\|\theta\|_{L^{2}(\Omega,C)} =: K.$$

Notice that the constant K does not depend on k or t. This concludes the induction argument. \Box

Proposition 3.2. For any integer $\gamma \geq 1$, each x^k satisfies $E|x^k(t)-x^k(s)|^{2\gamma} \leq B_{\gamma}|t-s|^{\gamma}$ for all $s, t \in [0, T]$, where B_{γ} is a constant independent of k.

Proof. By lemma 2.1, proposition 3.1 (iii), and the linear growth property of F and G (2), we obtain for any $0 \le s < t \le T$:

$$\begin{split} E\left[\left|x^{k}(t)-x^{k}(s)\right|^{2\gamma}\right] &= E\left[\left|\int_{s}^{t}F(u,x_{u-1/k}^{k})du+\int_{s}^{t}G(u,x_{u-1/k}^{k})dW(u)\right|^{2\gamma}\right]\\ &\leq 2^{2\gamma-1}|t-s|^{2\gamma-1}\int_{s}^{t}\|F(u,x_{u-1/k}^{k})\|_{L^{2}(\Omega,\mathbb{R}^{d})}^{2\gamma}du\\ &+ 2^{2\gamma-1}A_{\gamma}|t-s|^{\gamma-1}\int_{s}^{t}\|G(u,x_{u-1/k}^{k})\|_{L^{2}(\Omega,\mathbb{R}^{d\times m})}^{2\gamma}du\\ &\leq 2^{2\gamma-1}D^{2\gamma}(T^{\gamma}+A_{\gamma})|t-s|^{\gamma-1}(1+\sqrt{K})^{2\gamma}|t-s| = B_{\gamma}|t-s|^{\gamma}, \end{split}$$

where $A_{\gamma} := d^{\gamma-1} (\frac{4\gamma^3 m^2}{2\gamma-1})^{\gamma}$ and $B_{\gamma} := 2^{2\gamma-1} D^{2\gamma} (T^{\gamma} + A_{\gamma}) (1 + \sqrt{K})^{2\gamma}$. By symmetry in t and s, the proposition holds for any $t, s \in [0, T]$.

Next, we state Kolmogorov's continuity criterion for a sequence of Banach-valued stochastic processes. The theorem will be used in the proof of proposition 3.3.

Theorem 3.1. Kolmogorov's continuity criterion for a sequence of stochastic processes. Let $\{X^k(t)\}_{k=1}^{\infty}, t \in [0,T]$, be a sequence of stochastic processes with values in a Banach space E. Assume that there exist positive constants ρ_1 , c and $\rho_2 > 1$, all independent of k, satisfying

(8)
$$E[\|X^{k}(t) - X^{k}(s)\|_{E}^{\rho_{1}}] \leq c|t - s|^{\rho_{2}},$$

for every $s,t \in [0,T]$. Then each X^k has a continuous modification \tilde{X}^k . Further, let b be an arbitrary positive number less than $\frac{\rho_2-1}{\rho_1}$. Then there exists a positive random variable ξ_k with $E[\xi_k^{\rho_1}] < H$, where H is a constant independent of k, such that

$$\|\tilde{X}^{k}(t) - \tilde{X}^{k}(s)\|_{E} \le \xi_{k}|t-s|^{b},$$

for every $s, t \in [0, T]$ and a.s..

Proof. The reader may refer to Kunita [5], pg. 31, for a proof. Although the author does not consider a sequence of stochastic processes satisfying (8), it is easy to check that the theorem indeed holds for such a sequence.

Proposition 3.3. Let $\beta \in (0, 1/2)$ be a fixed constant. Each x^k satisfies

$$\begin{aligned} &(i) \quad |x^{k}(t) - x^{k}(s)| \leq c_{k}|t - s|^{\beta} \quad for \ all \ s, t \in [0, T] \ a.s.; \\ &(ii) \quad \|x^{k}_{t} - x^{k}_{s}\|^{2}_{L^{2}(\Omega, C_{d})} \leq 3\tilde{c}|t - s|^{2\beta} + \\ & 2E \sup_{v \in (-(t \wedge L) \wedge 0, -(s \wedge L) \wedge 0]} |\hat{\theta}(0) - \hat{\theta}(s + v)|^{2} + \\ & E \sup_{v \in [-L, -(t \wedge L) \wedge 0]} |\hat{\theta}(t + v) - \hat{\theta}(s + v)|^{2} \quad for \ all \ -1 \leq s < t \leq T, \ a.s., \end{aligned}$$

where \tilde{c} is a constant independent of k and c_k is a positive random variable satisfying $E(c_k^{\gamma}) \leq \tilde{c}$.

Proof. Let $\rho > \frac{1}{1-2\beta}$ be an integer. From proposition 3.2, $E|x^k(t) - x^k(s)|^{2\rho} \leq B_{\rho}|t-s|^{\rho}$, for any $s, t \in [0, T]$. Since $\beta < \frac{\rho-1}{2\rho}$, then it follows from Kolmogorov's continuity criterion (theorem 3.1) that there exists a positive random variable c_k such that $|x^k(t) - x^k(s)| \leq c_k |t-s|^{\beta}$ a.s., with $E(c_k^{\gamma}) \leq \tilde{c}$, where \tilde{c} is a constant independent of k. This proves part (i).

We now proceed to prove part (ii). For any $-1 \le s < t \le T$,

$$\|x_t^k - x_s^k\|_{L^2(\Omega, C_d)}^2 = E \sup_{v \in [-L, 0]} |x^k(t+v) - x^k(s+v)|^2$$

(9)
$$\leq E \sup_{v \in (-(s \wedge L) \wedge 0, 0]} |x^{k}(t+v) - x^{k}(s+v)|^{2}$$

(10) +
$$E \sup_{v \in (-(t \wedge L) \wedge 0, -(s \wedge L) \wedge 0]} |x^k(t+v) - x^k(s+v)|^2$$

(11)
$$+ E \sup_{v \in [-L, -(t \wedge L) \wedge 0]} |x^{k}(t+v) - x^{k}(s+v)|^{2}.$$

Using part (i), we obtain the estimate for (9):

$$E \sup_{v \in (-(s \wedge L) \wedge 0, 0]} |x^{k}(t+v) - x^{k}(s+v)|^{2} \le E \sup_{v \in (-(s \wedge L) \wedge 0, 0]} (c_{k}|t-s|^{\beta})^{2}$$

= $E(c_{k}^{2}|t-s|^{2\beta}) = E(c_{k}^{2})|t-s|^{2\beta} \le \tilde{c}|t-s|^{2\beta}.$

Moreover, from part (i), we also estimate (10):

$$\begin{split} & E \sup_{v \in (-(t \wedge L) \wedge 0, -(s \wedge L) \wedge 0]} |x^{k}(t+v) - x^{k}(s+v)|^{2} \\ & \leq 2E \sup_{v \in (-(t \wedge L) \wedge 0, -(s \wedge L) \wedge 0]} \{|x^{k}(t+v) - x^{k}(0)|^{2} + |\hat{\theta}(0) - \hat{\theta}(s+v)|^{2}\} \\ & \leq 2E \sup_{v \in (-(t \wedge L) \wedge 0, -(s \wedge L) \wedge 0]} \{c_{k}^{2}|t+v|^{2\beta} + |\hat{\theta}(0) - \hat{\theta}(s+v)|^{2}\} \\ & \leq 2\tilde{c}|t-s|^{2\beta} + 2E \sup_{v \in (-(t \wedge L) \wedge 0, -(s \wedge L) \wedge 0]} |\hat{\theta}(0) - \hat{\theta}(s+v)|^{2}. \end{split}$$

Finally, (11) becomes:

$$\begin{split} E \sup_{v \in [-L, -(t \wedge L) \wedge 0]} |x^k(t+v) - x^k(s+v)|^2 &= E \sup_{v \in [-L, -(t \wedge L) \wedge 0]} |\hat{\theta}(t+v) - \hat{\theta}(s+v)|^2. \\ \text{Hence, for } -1 \leq s < t \leq T, \\ \|x^k_t - x^k_s\|^2_{L^2(\Omega, C_d)} &\leq 3\tilde{c} |t-s|^{2\beta} + 2E \sup_{v \in (-(t \wedge L) \wedge 0, -(s \wedge L) \wedge 0]} |\hat{\theta}(0) - \hat{\theta}(s+v)|^2 \\ &+ E \sup_{v \in [-L, -(t \wedge L) \wedge 0]} |\hat{\theta}(t+v) - \hat{\theta}(s+v)|^2. \quad \Box \end{split}$$

Proposition 3.3(i) shows that each x^k is pathwise β -Hölder continuous on [0,T]. Notice that if t > s > L, part (ii) can be written as $\|x_t^k - x_s^k\|_{L^2(\Omega, C_d)}^2 \leq 3\tilde{c}|t - s|^{2\beta}$, which

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implies that the memory process $t \mapsto x_t^k$ is β -Hölder continuous (as an $L^2(\Omega, C)$ -valued function) only for $t \in [L, T]$. On the other hand, if the initial process θ is pathwise β -Hölder continuous, then $t \mapsto x_t^k$ is β -Hölder continuous (as an $L^2(\Omega, C)$ -valued function) for all $t \in [0, T]$. We prove this in theorem 4.1.

4. Closing the gap

In this section, we show that the sequence $(x^k)_{k=1}^{\infty}$ converges to the solution of (3). The proof is divided into the following steps:

- 1. The sequence $(x^k)_{k=1}^{\infty}$ converges to a limit $x \in L^2(\Omega, C([-1-L, T], \mathbb{R}^d))$.
- 2. The process $(x(t))_{t \in [-L,T]}$ satisfies the SFDE (3).
- 3. The process $(x(t))_{t\in[-L,T]}$ is the unique solution of the SFDE (3).

4. If the initial process θ is pathwise β -Hölder continuous for a fixed $\beta \in (0, 1/2)$, then the rate at which x^k converges to x is β .

Proposition 4.1. The sequence $(x^k)_{k=1}^{\infty}$ converges to a limit $x \in L^2(\Omega, C([-1-L, T], \mathbb{R}^d))$. *Proof.* We first notice that for any $t \in [-1, T]$,

$$\|x_t^l - x_t^k\|_{L^2(\Omega, C_d)}^2 \leq E \sup_{v \in [-1-L, t]} |x^l(v) - x^k(v)|^2 = E \sup_{v \in [0, t]} |x^l(v) - x^k(v)|^2.$$

Now, from lemma 2.1, the Lipschitz condition (1), and inequality (12), we have that for any $t \in [0, T]$ and l > k:

$$\begin{aligned} (12) \quad & E \sup_{v \in [0,t]} |x^{l}(v) - x^{k}(v)|^{2} = E \sup_{v \in [0,t]} \left| \int_{0}^{v} [F(u, x_{u-1/l}^{l}) - F(u, x_{u-1/k}^{k})] du \right. \\ & + \int_{0}^{v} [G(u, x_{u-1/l}^{l}) - G(u, x_{u-1/k}^{k})] dW(u) \Big|^{2} \\ & \leq 2t \int_{0}^{t} \|F(u, x_{u-1/l}^{l}) - F(u, x_{u-1/k}^{k})\|_{L^{2}(\Omega, \mathbb{R}^{d})}^{2} du \\ & + 2(4m^{2}) \int_{0}^{t} \|G(u, x_{u-1/l}^{l}) - G(u, x_{u-1/k}^{k})\|_{L^{2}(\Omega, \mathbb{C}_{d})}^{2} du \\ & \leq 4\alpha^{2}(t + 4m^{2}) \int_{0}^{t} \left(\|x_{u-1/l}^{l} - x_{u-1/k}^{l}\|_{L^{2}(\Omega, C_{d})}^{2} + \|x_{u-1/k}^{l} - x_{u-1/k}^{k}\|_{L^{2}(\Omega, C_{d})}^{2} \right) du \\ & \leq 4\alpha^{2}(T + 4m^{2}) \int_{0}^{t} \|x_{u-1/l}^{l} - x_{u-1/k}^{l}\|_{L^{2}(\Omega, C_{d})}^{2} du \\ & + 4\alpha^{2}(T + 4m^{2}) \int_{0}^{t} E \sup_{v \in [0,u]} |x^{l}(v) - x^{k}(v)|^{2} du. \end{aligned}$$

Hence, by Gronwall's inequality,

(13)
$$E \sup_{v \in [0,t]} |x^{l}(v) - x^{k}(v)|^{2} \leq 4\alpha^{2} (T + 4m^{2}) e^{4\alpha^{2} (T + 4m^{2})t} \int_{0}^{t} ||x_{u-1/l}^{l} - x_{u-1/k}^{l}||_{L^{2}(\Omega,C)}^{2} du$$

From proposition 3.3(ii) and the uniform continuity of $\hat{\theta}$, it follows that $\int_0^t \|x_{u-1/l}^l - x_{u-1/k}^l\|_{L^2(\Omega,C_d)}^2 du \to 0$ as $l, k \to \infty$. Therefore, from inequality (13),

$$E \sup_{v \in [0,t]} |x^l(v) - x^k(v)|^2 \to 0 \quad \text{as} \quad l, k \to \infty.$$

This shows that the sequence $(x^k)_{k=1}^{\infty}$ is a Cauchy sequence in $L^2(\Omega, C([-1-L, T], \mathbb{R}^d))$ and therefore convergent to a limit $x \in L^2(\Omega, C([-1-L, T], \mathbb{R}^d))$. From (12), it also follows that for each $t \in [-1, T]$, $(x_t^k)_{k=1}^{\infty}$ converges to x_t in $L^2(\Omega, C_d)$. \Box **Proposition 4.2.** The process $(x(t))_{t \in [-L,T]}$ satisfies the SFDE (3).

Proof. To show this, we take limits as $k \to \infty$ in both sides of (5). The left-hand side of (5) converges to x in $L^2(\Omega, C([-1-L, T], \mathbb{R}^d))$. Furthermore, $(x(t))_{t \in [0,T]}$ is $(\mathcal{F}_t)_{t \in [0,T]}$ -adapted, since each $(x^k(t))_{t \in [0,T]}$ is. Moreover, in a calculation similar to (12), we have that

$$E \sup_{v \in [0,t]} \left| \int_0^v [F(u, x_u) - F(u, x_{u-1/k}^k)] du + \int_0^v [G(u, x_u) - G(u, x_{u-1/k}^k)] dW(u) \right|^2$$

$$\leq 4\alpha^2 (t + 4m^2) \int_0^t \|x_u - x_{u-1/k}\|_{L^2(\Omega, C_d)}^2 du$$

(14) $+4\alpha^{2}(t+4m^{2})\|x_{u-1/k} - x_{u-1/k}^{k}\|_{L^{2}(\Omega,C_{d})}^{2}du.$

The continuity of $[-1,T] \ni t \mapsto x_t \in L^2(\Omega, C_d)$ implies that

$$\int_{0}^{s} \|x_{u} - x_{u-1/k}\|_{L^{2}(\Omega, C_{d})}^{2} du \to 0 \text{ as } k \to \infty.$$

Also,

$$\|x_{u-1/k} - x_{u-1/k}^k\|_{L^2(\Omega, C_d)}^2 \le E \sup_{v \in [-L, T]} |x(v) - x^k(v)|^2 \to 0 \text{ as } k \to \infty.$$

Hence, (14) converges to 0 as $k \to \infty$. This shows that the right-hand side of (5) converges to $\theta(0) + \int_0^{(\cdot)} F(u, x_u) du + \int_0^{(\cdot)} G(u, x_u) dW(u)$ in $L^2(\Omega, C([-L, T], \mathbb{R}^d))$ as $k \to \infty$. Therefore, $(x(t))_{t \in [-L,T]}$ satisfies the sfde (3). \Box

Proposition 4.3. (Uniqueness) If \tilde{x} is an $(\mathcal{F}_t)_{t \in [0,T]}$ -adapted process satisfying (3), then $\tilde{x} = x|_{[-L,T]}$ a.s..

Proof. In a calculation similar to (12), we find the difference

$$\begin{split} \|\tilde{x} - x|_{[-L,T]}\|_{L^{2}(\Omega,C([-L,T],\mathbb{R}))}^{2} &= E \sup_{v \in [0,T]} |\tilde{x}(v) - x(v)|^{2} \\ \leq & 4\alpha^{2}(T + 4m^{2}) \int_{0}^{t} \|\tilde{x}_{u} - x_{u}\|_{L^{2}(\Omega,C_{d})}^{2} du \\ & + 4\alpha^{2}(T + 4m^{2}) \int_{0}^{t} E \sup_{v \in [0,u]} |\tilde{x}(v) - x(v)|^{2} du \\ \leq & 8\alpha^{2}(T + 4m^{2}) \int_{0}^{t} E \sup_{v \in [0,u]} |\tilde{x}(v) - x(v)|^{2} du. \end{split}$$

Hence, from Gronwall's inequality, it follows that

$$E \sup_{v \in [0,T]} |\tilde{x}(v) - x(v)|^2 = 0 \Rightarrow \tilde{x} = x|_{[-L,T]} \ a.s.. \ \Box$$

Thus far, we have proven existence and uniqueness of a strong solution x to the SFDE (3). Moreover, we have proven that the SFDE (3) can be approximated by the sequence $(x^k)_{k=1}^{\infty}$ described in (5). The following theorem gives the rate in which this sequence converges to x, when the initial process satisfies a Hölder continuity condition.

Theorem 4.1. Let
$$\beta \in (0, 1/2)$$
 be a fixed constant. If the initial process θ satisfies
(15) $E|\theta(t) - \theta(s)|^{2\gamma} \leq C_{\theta}|t-s|^{\gamma},$

for any $\gamma > 1$, where C_{θ} is a positive constant, then θ is pathwise β -Hölder continuous and

$$E \sup_{v \in [0,t]} |x(v) - x^k(v)|^2 \le c \left(\frac{1}{k}\right)^{2\beta}$$

where c is a constant independent of k.

Proof. Let $\rho > \frac{1}{1-2\beta}$ be an integer. From (15) and the fact that $\beta < \frac{\rho-1}{2\rho}$, Kolmogorov's continuity criterion (theorem 3.1) implies that there exists a positive random variable c_{θ} such that $|\theta(t) - \theta(s)| \le c_{\theta}|t - s|^{\beta}$ a.s., with $E(c_{\theta}^{\gamma}) \le \tilde{c}_{\theta}$, where \tilde{c}_{θ} is a positive constant. That is, θ is pathwise β -Hölder continuous.

Then notice that $\hat{\theta}$ is also pathwise β -Hölder continuous. Indeed, for $-1 - L \leq s < t \leq 0$,

$$|\hat{\theta}(t) - \hat{\theta}(s)| = \begin{cases} |\theta(t) - \theta(s)| \le c_{\theta} |t - s|^{\beta}, & -L \le s, t \le 0\\ |\theta(t) - \theta(-L)| \le c_{\theta} |t + L|^{\beta} \le c_{\theta} |t - s|^{\beta}, & s < -L, t > -L;\\ |\theta(-L) - \theta(-L)| = 0 \le c_{\theta} |t - s|^{\beta}, & -1 - L \le s, t < -L. \end{cases}$$

Then proposition 3.3(ii) and the pathwise β -Hölder continuity of $\hat{\theta}$ imply that

$$\begin{aligned} \|x_{t}^{k} - x_{s}^{k}\|_{L^{2}(\Omega, C_{d})}^{2} &\leq 3\tilde{c}|t - s|^{2\beta} + 2E \sup_{v \in (-(t \wedge L) \wedge 0, -(s \wedge L) \wedge 0]} c_{\theta}^{2}|s + v|^{2\beta} \\ &+ E \sup_{v \in [-L, -(t \wedge L) \wedge 0]} c_{\theta}^{2}|t - s|^{2\beta} \\ &\leq 3\tilde{c}|t - s|^{2\beta} + 2E(c_{\theta}^{2})|t - s|^{2\beta} + E(c_{\theta}^{2})|t - s|^{2\beta} \\ &\leq 3(\tilde{c} + \tilde{c}_{\theta})|t - s|^{2\beta}, \end{aligned}$$

for any $s, t \in [-1, T]$. Hence, for l > k > 0,

$$\|x_{u-1/l}^l - x_{u-1/k}^l\|_{L^2(\Omega, C_d)}^2 \leq 3(\tilde{c} + \tilde{c_\theta})|1/k - 1/l|^{2\beta},$$

which implies that $\int_0^t \|x_{u-1/l}^l - x_{u-1/k}^l\|_{L^2(\Omega,C_d)}^2 du \leq 3T(\tilde{c} + \tilde{c_\theta})|1/k - 1/l|^{2\beta}, t \in [0,T].$ Therefore, from inequality 13, we obtain

$$E \sup_{v \in [0,t]} |x^{l}(v) - x^{k}(v)|^{2} \le 4\alpha^{2} (T + 4m^{2}) e^{4\alpha^{2} (T + 4m^{2})t} 3(\tilde{c} + \tilde{c}_{\theta}) T |1/k - 1/l|^{2\beta}.$$

Finally, letting $l \to \infty$ and $c := 12T\alpha^2(\tilde{c} + \tilde{c}_{\theta})(T + 4m^2)e^{4\alpha^2(T + 4m^2)T}$, we obtain

$$E \sup_{v \in [0,t]} |x(v) - x^k(v)|^2 \le c \left(\frac{1}{k}\right)^{2\beta}, \quad t \in [0,T]. \quad \Box$$

5. Remarks

5.1. Order of convergence. Theorem 4.1 gives the order of convergence for the approximation scheme (5), when the initial process θ is pathwise β -Hölder continuous. Since the quantity 1/k is the length of a small time interval, the order of convergence is given with respect to time increments. This result is comparable with the 0.5 order of convergence of the Strong Euler-Maruyama scheme for SFDEs investigated in [3, 1].

5.2. Existence of SODEs. If we set L = 0 in (3) and (5), the approximation scheme reduces to the one in [2], which provides an alternative existence theorem for (non-delayed) stochastic ordinary differential equations (SODEs).

5.3. Alternative scheme. Alternatively, one could use the approximation scheme

(16)
$$x^{k}(t) = \theta(0) + \int_{0}^{t} F(u - 1/k, x_{u-1/k}^{k}) du + \int_{0}^{t} G(u - 1/k, x_{u-1/k}^{k}) dW(u),$$

for $t \in [0, T]$, and $x^k(t) = \hat{\theta}(t)$, for $t \in [-1 - L, 0]$. The process $\hat{\theta}$ is given by (4). In this case, the functionals F and G need to satisfy the additional regularity condition:

$$\|F(t,\eta) - F(s,\eta)\|_{L^{2}(\Omega,\mathbb{R}^{d})} + \|G(t,\eta) - G(s,\eta)\|_{L^{2}(\Omega,\mathbb{R}^{d\times m})}$$

(17)
$$\leq \alpha_1 (1 + \|\eta\|_{C_d}) |t - s|,$$

for any $\eta \in C_d$, $s, t \in [0, T]$, where α_1 is a positive constant, independent of η , s and t.

5.4. Numerical simulation. The approximation scheme (5) provides a way of numerically simulating the SFDE (3), with the help a numerical method that can approximate Wiener integrals. More specifically, for each $n/k < t \leq (n+1)/k$ with $0 \leq n < kT$, x^k can be written as

$$x^{k}(t) = x^{k}(n/k) + \int_{n/k}^{t} F(u, x_{u-1/k}^{k}) du + \int_{n/k}^{t} G(u, x_{u-1/k}^{k}) dW(u).$$

The integral $\int_{n/k}^{t} F(u, x_{u-1/k}^{k}) du$ is a Riemann integral and can be easily approximated. The integral $\int_{n/k}^{t} G(u, x_{u-1/k}^{k}) dW(u)$ is a Wiener integral, since $G(u, x_{u-1/k}^{k})$ is $\mathcal{F}_{n/k}$ -measurable for each $u \in (n/k, (n+1)/k]$. Wiener integrals can be approximated in the Riemann-Stieltjes sense. In [4], the authors provide several numerical schemes for SODEs, which include the less general case of a Wiener integral.

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