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OPTIMAL ESTIMATION OF A SIGNAL PERTURBED BY A MIXED FRACTIONAL BROWNIAN MOTION

We consider the problem of optimal estimation of the vector parameter θ of the drift term in a mixed fractional Brownian motion. We obtain the maximum likelihood estimator as well as the Bayesian estimator when the prior distribution is Gaussian.

1. Introduction

Fractional Brownian motion $W^H = \{W_t^H, t \geq 0\}$ (fBm) has been used for modeling stochastic phenomena with long-range dependence. It is a centered Gaussian process with the covariance function

$$R_H(s,t) = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H})$$

where $0 < H < 1$ and the constant H is called the Hurst index. The case $H = 1/2$ corresponds to the Brownian motion. The process fBm is the only Gaussian process which is self-similar and has stationary increments. For properties of fBm, see Samorodnitsky and Taqqu [24], Mishura [9] and Prakasa Rao [16]. Geometric Brownian motion has been widely used for modeling fluctuations of share prices in a stock market. Recently there has been an interest to study the problem of estimation of parameters for processes driven by processes which are mixtures of independent Brownian and fractional Brownian motions starting from the work of Cheridito [3], Rudomino-Dusyatska [23] and more recently in Prakasa Rao [15], [17], [18], [20], [22] among others. Mixed fractional Brownian models were studied in Mishura [9] and Prakasa Rao [16]. Cai et al. [2] present a new approach via filtering for analysis of mixed processes of type $\{X_t = B_t + G_t, 0 \leq t \leq T\}$ where $\{B_t, 0 \leq t \leq T\}$ is a Brownian motion and $\{G_t, 0 \leq t \leq T\}$ is an independent Gaussian process. Statistical Analysis of mixed fractional Ornstein-Uhlenbebeck process was investigated in Chigansky and Kleptsyna [4]. Large deviations for drift parameter estimator of mixed fractional Ornstein-Uhlenbeck process were studied by Marushkevych [8]. Optimal estimation of a signal perturbed by a mixed fractional Brownian motion over a finite time horizon is discussed here. Parameter estimation for linear stochastic differential equations driven by a mixed fractional Brownian motion and the asymptotic properties of the maximum likelihood and Bayes estimators are investigated in Prakasa Rao [20], [22].

2. Preliminaries

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ be a stochastic basis satisfying the usual conditions. The natural filtration of a stochastic process is understood as the P-completion of the filtration generated by this process. Let $\{W_t, t \geq 0\}$ be a standard Wiener process and

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$W^H = \{W_t^H, t \geq 0\}$ be an independent normalized fractional Brownian motion with Hurst parameter $H \in (0,1)$, that is a Gaussian process with continuous sample paths such that $W_0^H = 0, E(W_t^H) = 0$ and

$$E(W_s^H W_t^H) = \frac{1}{2}[s^{2H} + t^{2H} - |s-t|^{2H}], t \geq 0, s \geq 0. \tag{2.1}$$

Let

$$\tilde{W}_t^H = W_t + W_t^H, t \geq 0.$$

The process $\{\tilde{W}_t^H, t \geq 0\}$ is called the mixed fractional Brownian motion (mfBm) with Hurst index H. We assume here after that Hurst index H is known. Following the results in Cheridito [3], it is known that the process $\tilde{W}^H$ is a semimartingale in its own filtration if and only if either $H = 1/2$ or $H \in (\frac{3}{4}, 1)$. We will assume hereafter that $H \in (\frac{3}{4}, 1)$.

Let us consider a stochastic process $Y = \{Y_t, t \geq 0\}$ defined by the stochastic integral equation

$$Y_t = \int_0^t C(s)ds + \tilde{W}_t^H, t \geq 0 \tag{2.2}$$

where the process $C = \{C(t), t \geq 0\}$ is an $(\mathcal{F}_t)$-adapted process. For convenience, we write the above integral equation in the form of a stochastic differential equation

$$dY_t = C(t)dt + d\tilde{W}_t^H, t \geq 0, Y_0 = 0 \tag{2.3}$$

driven by the mixed fractional Brownian motion $\tilde{W}^H$. Following the recent works by Cai et al. [2] and Chigansky and Kleptsyna [4], one can construct an integral transformation that transforms the mixed fractional Brownian motion $\tilde{W}^H$ into a martingale M^H. Let $g_H(s,t)$ be the solution of the integro-differential equation

$$g_H(s,t) + H\frac{d}{ds}\int_0^t g_H(r,t)|s-r|^{2H-1} sign(s-r)dr = 1, 0 < s < t. \tag{2.4}$$

Cai et al. [2] proved that the process

$$M_t^H = \int_0^t g_H(s,t)d\tilde{W}_s^H, t \geq 0 \tag{2.5}$$

is a Gaussian martingale with quadratic variation

$$< M^H >_t = \int_0^t g_H(s,t)ds, t \geq 0. \tag{2.6}$$

Furthermore the natural filtration of the martingale M^H coincides with that of the mixed fractional Brownian motion $\tilde{W}^H$. Suppose that, for the martingale M^H defined by the equation (2.5), the sample paths of the process $\{C(t), t \geq 0\}$ are smooth enough in the sense that the process

$$Q_t = \frac{d}{d < M^H >_t}\int_0^t g_H(s,t)C(s)ds, t \geq 0 \tag{2.7}$$

is well defined. Define the process

$$Z_t = \int_0^t g_H(s,t)dY_s, t \geq 0. \tag{2.8}$$

As a consequence of the results in Cai et al. [2], it follows that the process Z is a fundamental semimartingale associated with the process Y in the following sense.

Theorem 2.1: *Let $g_H(s,t)$ be the solution of the equation (2.4). Define the process Z as given in the equation (2.8). Then the following relations hold.*

(i) The process Z is a semimartingale with the decomposition

$$Z_t = \int_0^t Q_s d < M^H >_s + M_t^H, t \geq 0 \tag{2.9}$$

where M^H is the martingale defined by the equation (2.5).
(ii) The process Y admits the representation

$$Y_t = \int_0^t \hat{g}_H(s,t) dZ_s, t \geq 0 \tag{2.10}$$

where

$$\hat{g}_H(s,t) = 1 - \frac{d}{d < M^H >_s} \int_0^t g_H(r,s) dr. \tag{2.11}$$

(iii) The natural filtrations $(\mathcal{Y}_t)$ and $(\mathcal{Z}_t)$ of the processes Y and Z respectively coincide.

Applying Corollary 2.9 in Cai et al. [2], it follows that the probability measures μ_Y and $\mu_{\tilde{W}^H}$ generated by the processes Y and $\tilde{W}^H$ on an interval $[0,T]$ are absolutely continuous with respect to each other and the Radon-Nikodym derivative is given by

$$\Lambda_T^H = \frac{d\mu_Y}{d\mu_{\tilde{W}^H}}(Y) = \exp[\int_0^T Q_s dZ_s - \frac{1}{2}\int_0^T Q_s^2 d < M^H >_s] \tag{2.12}$$

which is also the likelihood function based on the observation $\{Y_s, 0 \leq s \leq T.\}$ Since the filtrations generated by the processes Y and Z are the same, the information contained in the families of σ-algebras $(\mathcal{Y}_t)$ and $(\mathcal{Z}_t)$ is the same and hence the problem of the estimation of the parameters involved based on the observation $\{Y_s, 0 \leq s \leq T\}$ and $\{Z_s, 0 \leq s \leq T\}$ are equivalent.

We call the process Λ^H as the *likelihood process* or the Radon-Nikodym derivative $\frac{dP^Y}{dP}$ of the measure P^Y with respect to the measure P.

Let $\xi = \{\xi_t, t \geq 0\}$ be a stochastic process defined on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t, t \geq 0), P)$ satisfying the stochastic differential equation

$$d\xi_t = (\sum_{i=1}^k \theta_i \phi_i(t)) dt + \sigma(t) d\tilde{W}_t^H, t \geq 0. \tag{2.13}$$

We discuss the problem of estimation of the unknown vector parameter $\theta = (\theta_1, \ldots, \theta_k)$ based on the observation of the process ξ over the interval $[0,t]$. Here $\tilde{W}^H$ is a mixed fractional Brownian motion as defined above with the Hurst index $H \in (\frac{3}{4}, 1)$, the drift coefficient is given by

$$a(t) = \sum_{i=1}^k \theta_i \phi_i(t) \tag{2.14}$$

where the vector parameter $\theta = (\theta_1, \ldots, \theta_k)$ is *unknown* but the function

$$\phi(t) = (\phi_1(t), \ldots, \phi_k(t))$$

and the function $\sigma(t)$ are assumed to be *known.* We assume further that the functions $\phi_i(t), 1 \leq i \leq k$ are Liptshitz continuous and satisfy linear growth conditions and the function $\sigma(t)$ is positive so that the stochastic differential equation (2.13) has a unique solution $\{\xi(t), 0 \leq t \leq T\}$ (cf. Guerra and Nualart [7]; Mishura and Shevchenko [10]; da Silva, J.L., Erraoui, M. and Essaky, E.H. [6]). The problem of maximum likelihood estimation of the parameter θ, given the observations $\{\xi_s, 0 \leq s \leq t\}$, has been investigated and its asymptotic properties such as strong consistency and asymptotic mixed normality, as $t \to \infty$, have been obtained in Prakasa Rao [22].

The problem of sequential estimation consists of choosing an optimal sequential plan to stop sampling and construct an estimator based on this sampled data based on minimization of expectation of cost of sampling plus loss incurred up to the time the sampling is stopped. We now consider the problem of sequential estimation of the vector parameter θ given the observations $\{\xi_s, 0 \leq s \leq t\}$ available up to time t using the maximum likelihood and Bayesian methods when the loss function is the squared error loss function. Sequential estimation and testing for parameters, for processes driven by a fractional Brownian motion, were investigated in Prakasa Rao [12], [13], [14]. For a survey of problems of estimation for fractional diffusion processes, see Prakasa Rao [16]. Optimal estimation of a signal perturbed by a fractional Brownian noise has been recently discussed by Artemov and Burnaev [1] and optimal estimation of a signal perturbed by a sub-fractional Brownian motion is studied in Prakasa Rao [21]. Bayesian sequential estimation of the drift parameter of fractional Brownian motion is also investigated in Cetin et al. [5].

3. Maximum likelihood estimation of the drift parameter

We will now investigate the maximum likelihood estimation of the parameter $\theta = (\theta_1, \ldots, \theta_k)$ based on the observation of the process $\{\xi_t, 0 \leq t \leq T\}$. Since the filtrations generated by the processes $\{\xi_t, 0 \leq t \leq T\}$, $\{\zeta_t^H, 0 \leq t \leq T\}$ and $\{M_t^H, 0 \leq t \leq T\}$ are the same, the information contained in the three sets of observations is the same and hence the problem of estimation of the parameter θ based on the observations $\{\xi_t, 0 \leq t \leq T\}$ is equivalent to the problem of estimation based on the the process $\{M_t^H, 0 \leq t \leq T\}$. Following the general form of the process $Q_H(t)$ defined in the previous section, we define

$$Q_H(t) = \sum_{i=1}^{k} \theta_i \frac{d}{d < M^H >_t} \int_0^t g_H(t,s) \frac{\phi_i(s)}{\sigma(s)} ds = \sum_{i=1}^{k} \theta_i \psi_i(t) \tag{3.1}$$

where

$$\psi_i(t) = \frac{d}{d < M^H >_t} \int_0^t g_H(t,s) \frac{\phi_i(s)}{\sigma(s)} ds, 1 \leq i \leq k. \tag{3.2}$$

and the function $g_H(t,s)$ is as defined in the previous section. We assume that the functions $\psi_i(.), 1 \leq i \leq k$ are square integrable over the interval $[0,t], t \geq 0$ with respect to the measure induced by the function $< M^H >_t, t \geq 0$. Then the likelihood process Λ^H is given by the equation

$$\Lambda_t^H(\theta) = \exp\{\sum_{i=1}^{k} \theta_i \int_0^t \psi_i(s) dM_s^H - \frac{1}{2} \int_0^t [\sum_{i=1}^{k} \theta_i \psi_i(s)]^2 d < M^H >_s\}. \tag{3.3}$$

Let $J_H(t)$ denote the matrix of order $k \times k$ with the (i,j)-th element

$$(J_H(t))_{i,j} = \int_0^t \psi_i(s) \psi_j(s) d < M^H >_s \tag{3.4}$$

and let $\psi^H = \{\psi_t^H, t \geq 0\}$ be a k-dimensional process with the i-th component of ψ_t^H as

$$(\psi_t^H)_i = \int_0^t \psi_i(s) dM_s^H, 1 \leq i \leq k. \tag{3.5}$$

Following the notation defined above, the likelihood process can be written in the form

$$\Lambda_t^H(\theta) = \exp\{\theta' \psi_t^H - \frac{1}{2} \theta' J_H(t) \theta\}. \tag{3.6}$$

The maximum likelihood estimator $\hat{\theta}_t$ of the parameter θ is a maximizer of the likelihood $\Lambda_s^H(\theta)$ over the interval $[0,t]$ and can be obtained as a solution of the system of linear equations

$$\int_0^t \psi_i(s) dM_s^H - \sum_{j=1}^k \theta_j \int_0^t \psi_i(s)\psi_j(s) d < M^H >_s = 0, 1 \leq i \leq k \tag{3.7}$$

which, in turn, can be written in the form

$$\psi_t^H - J_H(t)\theta = 0. \tag{3.8}$$

The matrix $J_H(t)$ is a covariance matrix and is positive semidefinite. If the matrix $J_H(t)$ is positive definite, then the maximum likelihood estimator (MLE) of the vector parameter θ is given by the equation

$$\hat{\theta}_t = J_H^{-1}(t)\psi_t^H. \tag{3.9}$$

where $J_H^{-1}(t)$ is the inverse of the matrix $J_H(t)$. Let θ^0 be the true mean vector. Note that the martingale M^H is a zero mean Gaussian martingale and hence the random vector ψ_t^H has the multivariate normal distribution. This will imply that the random vector $\hat{\theta}_t - \theta_0$ has the multivariate normal distribution with mean zero and the covariance matrix $J_H^{-1}(t)$ which in turn will imply that the estimator $\hat{\theta}_t$ is an unbiased estimator of the vector θ component wise. Consistency of the i-th component of the estimator $\hat{\theta}_t$ to the i-th component θ_i of the vector parameter θ as $t \to \infty$ follows if the function $\psi_i(t)$ satisfies the condition

$$\int_0^\infty \psi_i^2(s) d < M^H >_s ds = \infty.$$

Non-singularity of the matrix $J_H^{-1}(t)$ will hold if the functions $\psi_i(.), 1 \leq i \leq k$ are linearly independent in the space $L^2([0,t], d < M^H >)$. Note that the functions $\psi_i(.), 1 \leq i \leq k$ in turn depend on the functions $\phi_i(.), 1 \leq i \leq k$ and the functions $\sigma(.)$ and $< M^H >$ by the equation (3.4). If the matrix $J_H(t)$ is singular, then one can construct a generalized inverse $J_H^*(t)$ of the matrix and construct a solution of the differential equation (3.8). However, this solution is not unique and depends on the type of generalized inverse.

4. Bayes estimation of the drift parameter

We now consider the problem of Bayes estimation of the parameter $\theta \in R^k$ assuming that the parameter θ has a prior probability measure with density $p^\theta(.)$ with respect to the Lebesgue measure on R^k and the loss function is the squared error loss function. It is well known that the Bayes estimator is the conditional expectation of the parameter given the observed data, that is, it is the mean or expectation of the posterior distribution of the parameter θ given the observed data. The posterior density of θ given the observed data $\{\xi_s, 0 \leq s \leq t\}$ or equivalently the information $\mathcal{F}_t$, the σ-algebra generated by the family $\{\xi_s, 0 \leq s \leq t\}$, is given by

$$p^\theta(z|\mathcal{F}_t) = \frac{p^\theta(z)\Lambda_t^H(z)}{\int_{R^k} p^\theta(y)\Lambda_t^H(y)dy}, z \in R^k \tag{4.1}$$

where $\Lambda_t^H(z)$ is the likelihood process defined earlier. We will also consider the problem of finding the optimal sequential Bayes estimation rule $\tilde{\delta} = (\tilde{\tau}, \tilde{\theta}_\tau)$ for estimation of the parameter θ in the sense that

$$\inf_{\delta \in \mathcal{D}} E[c\tau + ||\theta_\tau^* - \theta||^2] = E[c\tilde{\tau} + ||\theta - \tilde{\theta}_{\tilde{\tau}}||^2] \tag{4.2}$$

where $\mathcal{D} = \{\delta : (\tau, \theta_\tau^*)\}$ is a class of stopping rules with finite stopping time $\tau \leq T < \infty$ with respect to the filtration $\{\mathcal{F}_t, 0 \leq s \leq t\}$and estimate the parameter θ by θ_τ^*. Here $\mathcal{F}_t$ is the σ-algebra generated by the process $\{\xi_s, 0 \leq s \leq t\}$. The constant $c > 0$ can be interpreted as the cost per unit of observation and the Bayes sequential estimation consists in stopping sampling at time $\tilde{\tau}$ and declaring $\tilde{\theta}_{\tilde{\tau}}$ as the optimal estimator of θ.

Special case: Suppose the vector θ has a multivariate normal prior density with the mean vector $\mathbf{m}$ and the non-singular covariance matrix Σ. Following the standard methods, it can be shown that the optimal Bayes estimator $\tilde{\theta}_t$, under the squared error loss function based on the observations up to time t, is given by

$$\tilde{\theta}_t = E[\theta|\mathcal{F}_t] = (J_H(t) + \Sigma^{-1})^{-1}(\psi_t^H + \Sigma^{-1}\mathbf{m}) \tag{4.3}$$

and the mean squared error $E[||\theta - \tilde{\theta}_t||^2|\mathcal{F}_t]$ is the trace of the posterior covariance matrix given by

$$Cov[\theta|\mathcal{F}_t] = (J_H(t) + \Sigma^{-1})^{-1}. \tag{4.4}$$

This can be checked by the arguments similar to those given in the proof of Theorem 3 in Artemov and Burnaev [1]. We omit the details. The optimal stopping time in this special case is given by

$$\tilde{\tau} = \arg\inf_{\tau\in\mathcal{D}} E[c\tau + E(||\theta - \tilde{\theta}_\tau||^2|\mathcal{F}_\tau)] = \arg\inf_{t\in[0,T]} F_H(t) \tag{4.5}$$

where

$$F_H(t) = ct + E(||\theta - \tilde{\theta}||^2|\mathcal{F}_t) = ct + tr((J_H(t) + \Sigma^{-1})^{-1}), 0 \leq t \leq T. \tag{4.6}$$

It can be seen that the function $F_H(t)$ is continuous over the interval $[0, T]$. Here $\arg\inf_{0\leq t\leq T} f(t)$ denotes the argument of a continuous function $f(t)$ at which the function f is minimum over the interval $[0, T]$ and if there are more than one value of t at which f is minimum over the interval $[0, T]$, then $\arg\inf_{0\leq t\leq T}$ stands for infimum over all such $t \in [0, T]$.

Note that the function $F_H(t)$ is deterministic and hence the optimal stopping rule is deterministic in this special case.

Suppose the observation process $\xi = \{\xi_t, t \geq 0\}$ satisfies the stochastic differential equation

$$d\xi_t = \theta dt + \sigma d\tilde{W}_t^H, t \geq 0 \tag{4.7}$$

where θ is a scalar and is normally distributed a priori with mean m and variance γ^2, then the posterior distribution of θ given the observed data $\{\xi_s, 0 \leq s \leq t\}$ is normal with the mean

$$\frac{(M_t^H/\sigma) + (m/\gamma^2)}{< M^H >_t /\sigma^2 + 1/\gamma^2}$$

and the variance

$$\frac{1}{(< M^H >_t /\sigma^2) + (1/\gamma^2)}.$$

From the general results on Bayes estimation for squared error loss function, it follows that the Bayes estimator for the parameter θ is given by

$$\tilde{\theta} = E[\theta|\mathcal{F}_t] = \frac{(M_t^H/\sigma) + (m/\gamma^2)}{< M^H >_t /\sigma^2 + 1/\gamma^2} \tag{4.8}$$

and the variance of this estimator is

$$E[(\theta - \tilde{\theta})^2|\mathcal{F}_t] = \frac{1}{(< M^H >_t /\sigma^2) + (1/\gamma^2)}. \tag{4.9}$$

Remarks: It is possible to investigate the problem of Bayes estimation for the vector parameter $\theta \in R^k$ when it has a uniform prior on the k-dimensional cube $\Pi_{i=1}^k[a_i, b_i]$ following the arguments in Artemov and Burnaev [1].

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