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MOMENT MEASURES AND STABILITY FOR GAUSSIAN INEQUALITIES

Let γ be the standard Gaussian measure on \mathbb{R}^n and let \mathcal{P}_γ be the space of probability measures that are absolutely continuous with respect to γ . We study lower bounds for the functional $\mathcal{F}_\gamma(\mu) = \text{Ent}(\mu) - \frac{1}{2}W_2^2(\mu, \nu)$, where $\mu \in \mathcal{P}_\gamma, \nu \in \mathcal{P}_\gamma$, $\text{Ent}(\mu) = \int \log\left(\frac{d\mu}{d\gamma}\right)d\mu$ is the relative Gaussian entropy, and W_2 is the quadratic Kantorovich distance. The minimizers of \mathcal{F}_γ are solutions to a dimension-free Gaussian analog of the (real) Kähler–Einstein equation. We show that $\mathcal{F}_\gamma(\mu)$ is bounded from below under the assumption that the Gaussian Fisher information of ν is finite and prove a priori estimates for the minimizers. Our approach relies on certain stability estimates for the Gaussian log-Sobolev and Talagrand transportation inequalities.

1. INTRODUCTION

Given a probability measure $\nu = \varrho dx$ one can try to find a log-concave measure $\mu = e^{-\Phi} dx$ (i.e., Φ is a convex function) satisfying the following remarkable property: ν is the image of μ under the mapping T generated by the logarithmic gradient of μ :

$$T(x) = \nabla\Phi(x), \quad \nu = \mu \circ T^{-1}.$$

Following the terminology from [11], we say that ν is a moment measure if such a function Φ exists.

There are many motivations to study moment measures. The associated equation on Φ

$$e^{-\Phi} = \varrho(\nabla\Phi) \det D^2\Phi$$

is a non-linear elliptic PDE of the Monge–Ampère type. After a suitable complexification it turns out to be a particular case of the complex Monge–Ampère equation. The case where ν is Lebesgue measure on a polytope with rational coordinates is of special interest in differential and algebraic geometry because of its relation to the theory of toric varieties. First results on the well-posedness of this equation have been established in a series of geometric papers (see [28], [2], [11], and the references therein). The most general result on existence of the moment measure has been obtained in [11] under fairly general assumptions. It is known that Φ is a maximum point of the following functional:

$$(1.1) \quad J(f) = \log \int e^{-f^*} dx - \int f d\nu,$$

where f^* is the Legendre transform of f . This functional has deep relations to the classical Brunn–Minkowski theory. In particular, J is concave under the usual addition and this fact is a particular form of the famous Prékopa–Leindler inequality. The measure μ is unique up to translations. To determine it uniquely we always assume that the barycenter (mean) of μ equals zero: $\int x d\mu = 0$.

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An alternative viewpoint was suggested in [27], where another natural functional was proposed. It was shown in [27] that $\rho = e^{-\Phi}$ gives a minimum to the functional

$$(1.2) \quad \mathcal{F}(\rho) = -\frac{1}{2}W_2^2(\nu, \rho dx) + \frac{1}{2} \int x^2 \rho dx + \int \rho \log \rho dx.$$

Here W_2 is the Kantorovich distance for the cost function $c(x, y) = |x - y|^2$. Unlike the approach of [11], the moment measure problem is viewed here as a problem on the space of probability measures equipped with the quadratic Kantorovich distance. We emphasize that the mass transportation problem is very relevant here. Indeed, the mapping $x \rightarrow \nabla \Phi(x)$ is the optimal transportation taking $e^{-\Phi} dx$ to ν . However, since μ depends on Φ explicitly, there is no simple way to find Φ as a solution to a Monge–Kantorovich problem.

Following the idea from [27] (see also [17] for further developments), we are looking for the minima of the Gaussian analog of (1.2)

$$\mathcal{F}_\gamma(\rho) = -\frac{1}{2}W_2^2(g \cdot \gamma, \rho \cdot \gamma) + \text{Ent} \rho,$$

where $\gamma = \frac{1}{(\sqrt{2\pi})^n} e^{-\frac{|x|^2}{2}} dx$,

$$\text{Ent} \rho = \int \rho \log \rho d\gamma$$

is the Gaussian entropy of g .

This question is motivated by the following infinite-dimensional analog of the moment measure problem. Let γ be the standard Gaussian product measure on \mathbb{R}^∞ and let $\nu = g \cdot \gamma$ be a probability measure such that

$$\int x_i g d\gamma = 0 \quad \text{for every } i \in \mathbb{N}.$$

The problem is to find a log-concave measure $\mu = e^{-\varphi} \cdot \gamma$ such that ν is the image of μ under the mapping

$$T(x) = x + \nabla \varphi,$$

where $\nabla \varphi$ is the Cameron–Martin gradient.

There exists a rich theory of optimal transportation on the Wiener space with a number of interesting results (see [6], [7], [8], [10], [13], [15], and [20]). So the well-posedness of the moment measure problem on the Wiener space is a natural and interesting question. We emphasize that the finite-dimensional estimates obtained in this paper are the first crucial step towards infinite-dimensional spaces. However, the infinite-dimensional moment measure problem seems to be delicate and requires hard technical work. This will be done in a forthcoming paper of the authors.

The following theorem is our main result (see Theorem 3.3).

Theorem 1.1. *Assume that g is a probability density satisfying $I(g) < \infty$, where*

$$I(g) = \int \frac{|\nabla g|^2}{g} d\gamma$$

is the Gaussian Fisher information of g . Then there exists a constant $C > 0$ depending only on $I(g)$ such that

$$\mathcal{F}_\gamma \geq -C$$

and

$$W_2^2(g \cdot \gamma, \rho \cdot \gamma) \leq C,$$

where $\rho \cdot \gamma$ is the minimum point of \mathcal{F}_γ satisfying the condition $\int x \rho d\gamma = 0$.

Our approach is based on certain stability results for the log-Sobolev and the Talagrand transportation inequalities:

$$\begin{aligned} \frac{1}{2}I(g) - \text{Ent}g &\geq \delta_1(g), \\ \text{Ent}g - \frac{1}{2}W_2^2(\gamma, g \cdot \gamma) &\geq \delta_2(g), \end{aligned}$$

where δ_1, δ_2 are some non-negative functionals defined on probability densities.

The stability of the Euclidean isoperimetric inequality (see the survey paper [16]) and the Gaussian inequalities (see [18], [3], [14], [12], and [24]) has been recently studied by many researchers. In this paper we establish several new results in this direction and give new simple proofs of some previously known inequalities.

Finally, we obtain a priori estimates for the (centered) minimum point $\rho \cdot \gamma = e^{-\varphi} \cdot \gamma$ of \mathcal{F}_γ . In particular, applying the approach developed in [5] for the standard Monge–Kantorovich problem, we establish new bounds for the entropy- and information-type functionals

$$\int \rho |\log \rho|^p d\gamma, \quad \int \rho \left| \frac{\nabla \rho}{\rho} \right|^p d\gamma, \quad p \geq 1,$$

and certain exponential moments.

2. STABILITY RESULTS

2.1. Notation. We shall use some standard results and terminology from Gaussian analysis (see [4]) and optimal transportation theory (see [8]).

Let γ be the standard Gaussian measure on \mathbb{R}^n :

$$\gamma = \frac{1}{(\sqrt{2\pi})^n} e^{-\frac{|x|^2}{2}} dx.$$

We denote by T the optimal transportation taking $g \cdot \gamma$ to γ . Recall that T gives a minimum to the functional

$$F \rightarrow \int |F(x) - x|^2 g d\gamma$$

considered on the mappings taking $g \cdot \gamma$ to γ .

Moreover, T is the gradient of a convex function. It can be written in the form

$$T(x) = x + \nabla \varphi(x),$$

where the potential φ satisfies the estimate

$$D^2\varphi \geq -\text{Id}.$$

The corresponding Kantorovich distance $W_2(\gamma, g \cdot \gamma)$ for the cost function $c(x, y) = |x - y|^2$ can be computed as follows:

$$W_2^2(\gamma, g \cdot \gamma) = \int |\nabla \varphi|^2 g d\gamma.$$

The notation

$$\|A\| = \sqrt{\text{Tr}(AA^T)}$$

will be used for the Hilbert–Schmidt norm of the matrix A and $\|A\|_{op}$ will denote the operator norm. We also use the standard notation for the (Gaussian) entropy

$$\text{Ent}g = \int g \log g d\gamma$$

and information

$$I(g) = \int \frac{|\nabla g|^2}{g} d\gamma.$$

2.2. Stability for the logarithmic Sobolev inequality. The celebrated Gaussian logarithmic Sobolev inequality

$$(2.1) \quad \frac{1}{2}I(g) \geq \text{Ent}g$$

is one of the central results in Gaussian analysis. Here g is a sufficiently regular probability density. For the proofs and the history of (2.1), see [4], [23], and [1].

Throughout the paper we assume that g has finite information.

Assumption I.

$$I(g) < \infty.$$

It is known that (2.1) is sharp and the corresponding minimizers have the form $g = e^l$, where l is an affine function. This has been proved by Carlen in [9]. He has shown that the so-called log-Sobolev deficit

$$\frac{1}{2}I(g) - \text{Ent}g$$

is bounded from below by a non-negative term, which is a functional involving certain integral transform of g .

Yet another representation has been obtained in [22]:

Theorem 2.1 ([22]). *Let $T = x + \nabla\varphi$ be the optimal transportation taking $g \cdot \gamma$ into γ , where g is a sufficiently regular probability density. Then the following representation holds:*

$$(2.2) \quad \begin{aligned} I(g) = & 2\text{Ent}g + 2 \int (\Delta\varphi - \log \det(I + D^2\varphi))gd\gamma + \int \|D^2\varphi\|^2gd\gamma \\ & + \int \sum_{i=1}^n \text{Tr} \left[(\text{Id} + D^2\varphi)^{-1} (D^2\varphi_{x_i}) \right]^2 gd\gamma. \end{aligned}$$

Remark 2.2. (Regularity of φ). The gradient of φ is well-defined almost everywhere because $\frac{|x|^2}{2} + \varphi$ is a convex function. Identity (2.2) ensures that φ belongs to an appropriate second-order Sobolev space (see [7] for details). The reader can always assume that g is bounded away from zero and locally smooth; this implies the local smoothness of φ (see [22] and [7]). In almost all our statements the minimal assumption about g is $I(g) < \infty$. This case follows easily from the case of a smooth potential by the standard approximation procedure.

It is important to mention that all the terms in the right-hand side of (2.2) are non-negative. This result is closely related to the so-called Gaussian stability inequalities, which have been recently investigated in a series of papers [18], [3], [14], [12]. Technically speaking, these are estimates of the type

$$\frac{1}{2}I(g) - \text{Ent}g \geq F(g \cdot \gamma),$$

where F is a non-negative functional on probability densities (measures). In our work we apply other well-known results deeply connected with (2.1): the Gaussian Talagrand transportation inequality

$$(2.3) \quad \text{Ent}g \geq \frac{1}{2}W_2^2(\gamma, g \cdot \gamma)$$

and the HWI inequality

$$(2.4) \quad \frac{1}{2}W_2^2(g \cdot \gamma, \gamma) + \text{Ent}g \leq \sqrt{I(g)}W_2(\gamma, g \cdot \gamma)$$

(see [23] and [1]).

The Talagrand inequality and the HWI inequality follow from the following identity which is widely used in transportation inequalities (see [1, Theorem 9.3.1]).

Theorem 2.3. *Let $T(x) = x + \nabla u(x)$ be the optimal transportation taking $g \cdot \gamma$ to $f \cdot \gamma$. Then*

$$(2.5) \quad \text{Ent} f = \text{Ent} g + \frac{1}{2} W_2^2(g \cdot \gamma, f \cdot \gamma) + \int \langle \nabla u, \nabla g \rangle d\gamma + \int (\Delta u - \log \det(I + D^2 u)) g d\gamma.$$

To prove (2.3) we set $g = 1$ and use that $\Delta u - \log \det(I + D^2 u)$ is non-negative. For (2.4) we set $f = 1$ and apply the Cauchy inequality.

In the proof of our main result we apply the following theorem from [14].

Theorem 2.4 ([14], Theorem 1). *Assume that $\nu = g \cdot \gamma$ satisfies the Poincaré inequality*

$$\int \left(f - \int f g d\gamma \right)^2 g d\gamma \leq C_P \int |\nabla f|^2 g d\gamma$$

and $\int x g d\gamma = 0$. Then the following inequality holds:

$$(2.6) \quad \frac{1}{2} I(g) - \text{Ent} g \geq \frac{1}{2} \frac{C_P \log C_P - C_P + 1}{(C_P - 1)^2} I(g).$$

Remark 2.5. Stability estimates of the same type, but with non-sharp constants can be derived from (2.2) and (2.4). Let ν satisfy the assumptions of Theorem 2.4. Then

$$I(g) \geq 2\text{Ent} g + \frac{1}{C_P} W_2^2(\gamma, g \cdot \gamma).$$

Indeed, the result follows immediately from (2.2) and the following computations (we use the Poincaré inequality and the change of variables formula)

$$\int \|D^2 \varphi\|^2 g d\gamma = \sum_{i=1}^n \int |\nabla \varphi_{x_i}|^2 g d\gamma \geq \frac{1}{C_P} \int \varphi_{x_i}^2 g d\gamma - \frac{1}{C_P} \left(\int \varphi_{x_i} g d\gamma \right)^2,$$

$$\int \varphi_{x_i} g d\gamma = \int (T_i - x_i) g d\gamma = \int x_i d\gamma - \int x_i g d\gamma = 0.$$

Applying (2.4) we get the following estimate for arbitrary $K \geq 1$:

$$\frac{1}{2} W_2^2(g \cdot \gamma, \gamma) + \text{Ent} g \leq \sqrt{I(g)} W_2(g \cdot \gamma, \gamma) \leq \frac{1}{2} \left(K W_2^2(g \cdot \gamma, \gamma) + \frac{1}{K} I(g) \right).$$

Hence

$$\text{Ent} g \leq \frac{1}{2K} I(g) + \frac{K-1}{2} W_2^2(g \cdot \gamma, \gamma) \leq \frac{1}{2K} I(g) + \frac{C_P(K-1)}{2} (I(g) - 2\text{Ent} g).$$

Equivalently, $\text{Ent} g \leq \frac{1}{2} I(g) \left(\frac{\frac{1}{K} + (K-1)C_P}{1 + (K-1)C_P} \right)$. Choosing the optimal value of K , which is $K = 1 + \frac{1}{\sqrt{C_P}}$, one gets

$$(2.7) \quad I(g) - 2\text{Ent} g \geq \frac{1}{(1 + \sqrt{C_P})^2} I(g).$$

Note that this is a result of the same type as in Theorem 2.4, but for large values of C_P the constant in the right-hand side of (2.6) is of order $\frac{\log C_P}{C_P}$, which is stronger than our result. We observe that the proof of (2.7) modulo (2.2) is easier than the proof of (2.6), but we do not know how to deduce (2.6) from (2.2).

We now prove another stability result (2.8) similar to (2.6). In particular, both estimates are sharp: the equalities hold for $g = \lambda^{\frac{n}{2}} e^{\frac{(1-\lambda)}{2}|x|^2}$. Note, however, that under the assumption $I(g) < \infty$ the right-hand side of (2.6) is always finite and dimension-free, which is not the case for (2.8). This fact has rather unexpected interesting consequences (see Remark 2.7).

Theorem 2.6. *Assume that $\nu = g \cdot \gamma$ satisfies the Poincaré inequality*

$$\int f^2 d\nu \leq C_P \int |\nabla f|^2 d\nu, \quad \int f d\nu = 0$$

and $C_P \leq 1$. Then

$$(2.8) \quad \frac{1}{2}I(g) - \text{Ent}g \geq \frac{n(C_P \log C_P - C_P + 1)}{2C_P}.$$

Remark 2.7. Applying (2.8) to the infinite-dimensional ($n = \infty$) Gaussian measure γ , we obtain the following result: if $g \cdot \gamma$ satisfies $I(g) < \infty$ and admits a finite Poincaré constant C_P (for the Cameron–Martin norm), then $C_P \geq 1$. We believe that the assumption $I(g) < \infty$ is unnecessary for this observation and can be relaxed.

For the proof, we write the right-hand side as $\frac{n\Delta(C_P^{-1}-1)}{2}$, where $\Delta(t) = t - \log(1+t)$. Let us apply (2.2). Let $\lambda_i, i \in \{1, \dots, n\}$ be the eigenvalues of $D^2\varphi$. Then

$$\frac{1}{2}\|D^2\varphi\|_{HS}^2 + \Delta\varphi - \log \det(I + D^2\varphi) = \sum_{i=1}^n \frac{1}{2}\lambda_i^2 + \lambda_i - \log(1 + \lambda_i) = \frac{1}{2} \sum_{i=1}^n \Delta((\lambda_i + 1)^2 - 1).$$

Note that

$$\Delta^*(s) := \sup_{t \geq -1} \{st - \Delta(t)\} = -s - \log(1 - s), \quad s \leq 1.$$

Thus,

$$\begin{aligned} \frac{1}{2}\|D^2\varphi\|_{HS}^2 + \Delta\varphi - \log \det(I + D^2\varphi) &\geq \frac{1}{2} \sum_{i=1}^n [s((\lambda_i + 1)^2 - 1) - (-s - \log(1 - s))] \\ &= \frac{1}{2}s \sum_{i=1}^n (\lambda_i + 1)^2 + \frac{1}{2}n \log(1 - s) = \frac{1}{2}s\|I + D^2\varphi\|_{HS}^2 + \frac{1}{2}n \log(1 - s). \end{aligned}$$

Applying the relations

$$\|I + D^2\varphi\|_{HS}^2 = \sum_{i=1}^n |\nabla(x_i + \varphi_{x_i})|^2,$$

$$\int [x_i + \varphi_{x_i}] g d\gamma = \int y_i d\gamma = 0,$$

we obtain for $s \geq 0$

$$\begin{aligned} \frac{1}{2}I(g) - \text{Ent}g &\geq \frac{s}{2C_P} \sum_{i=1}^n \int |x_i + \varphi_{x_i}|^2 g d\gamma + \frac{1}{2}n \log(1 - s) \\ &= \frac{s}{2C_P} \sum_{i=1}^n \int |y_i|^2 d\gamma + \frac{1}{2}n \log(1 - s) = \frac{n}{2} \left(\frac{s}{C_P} + \log(1 - s) \right). \end{aligned}$$

Taking $s = 1 - C_P$ we obtain the desired result.

Identity (2.2) implies another estimate obtained earlier in [3, Theorem 1.1].

Corollary 2.8. *There holds the inequality*

$$I(g) - 2\text{Ent}g \geq n\Delta\left(\frac{1}{n} \int \left| \frac{\nabla g}{g} - x \right|^2 g d\gamma - 1\right).$$

Proof. Rewrite (2.2) in the following way:

$$I(g) = 2\text{Ent}_\gamma g + \int \left(\|D^2\Phi\|^2 - n - \log \det(D^2\Phi)^2 \right) g d\gamma \\ + \sum_{i=1}^n \left\| (D^2\Phi)^{-\frac{1}{2}} D^2\Phi_{x_i} (D^2\Phi)^{-\frac{1}{2}} \right\|^2 g d\gamma.$$

By another result of [22, Section 5] we have

$$(2.9) \quad \int \left| \frac{\nabla g}{g} - x \right|^2 g d\gamma = \int \|D^2\Phi\|^2 g d\gamma + \sum_i \int \left\| (D^2\Phi)^{-\frac{1}{2}} D^2\Phi_{x_i} (D^2\Phi)^{-\frac{1}{2}} \right\|^2 g d\gamma.$$

These two identities imply that

$$\int \left| \frac{\nabla g}{g} \right|^2 g d\gamma = 2\text{Ent}_\gamma g + \int \left(\left| \frac{\nabla g}{g} - x \right|^2 - n - \log \det(D^2\Phi)^2 \right) g d\gamma.$$

Using Jensen's inequality and (2.9) we obtain

$$- \int \log \det(D^2\Phi)^2 g d\gamma \geq -n \log \int \frac{\|D^2\Phi\|^2}{n} g d\gamma \geq -n \log \int \frac{1}{n} \left| \frac{\nabla g}{g} - x \right|^2 g d\gamma.$$

Hence

$$\int \left| \frac{\nabla g}{g} \right|^2 g d\gamma \geq 2\text{Ent}_\gamma g + \int \left(\left| \frac{\nabla g}{g} - x \right|^2 - n \right) g d\gamma - n \log \int \frac{1}{n} \left| \frac{\nabla g}{g} - x \right|^2 g d\gamma,$$

which completes the proof. \square

Certain stability estimates can be obtained under (one-sided) uniform bounds on the Hessian of the logarithmic potential $-\log g$. The proof is based on the Caffarelli contraction theorem (see [21] and the references therein, some new developments for higher order derivatives can be found in [19]).

Proposition 2.9. *Assume that*

$$\int x g d\gamma = 0$$

and

$$\text{Id} - D^2 \log g \geq \varepsilon \cdot \text{Id}$$

for some constant $\varepsilon > 0$. Then there exists a universal constant c such that

$$\text{Ent} g \geq \left(\frac{1}{2} + c\sqrt{\varepsilon} \right) W_2^2(\gamma, g \cdot \gamma).$$

Proof. Let $S(x) = x + \nabla\psi$ be the optimal transportation taking γ to $g \cdot \gamma$. According to the Caffarelli contraction theorem

$$I + D^2\psi \leq \frac{1}{\sqrt{\varepsilon}}.$$

Hence $\Delta\psi - \log \det D^2(I + D^2\psi) \geq c\sqrt{\varepsilon} \|D^2\psi\|^2$. Then it follows from (2.5) that

$$\text{Ent} g \geq \frac{1}{2} W_2^2(\gamma, g \cdot \gamma) + c\sqrt{\varepsilon} \int \|D^2\psi\|^2 d\gamma.$$

By the Gaussian Poincaré inequality

$$\int \|D^2\psi\|^2 d\gamma = \sum_{i=1}^n \int |\nabla\psi_{x_i}|^2 d\gamma \geq \sum_{i=1}^n \int \psi_{x_i}^2 d\gamma = W^2(g \cdot \gamma, \gamma),$$

which completes the proof. \square

We end this subsection with an extension of Theorem 2.4 under the stronger assumption that $g \cdot \gamma$ satisfies the log-Sobolev inequality. Roughly speaking, the log-Sobolev deficit can be estimated from below by

$$\frac{1}{C_{LSI}} K_{a_\nu},$$

where C_{LSI} is the constant in the log-Sobolev inequality and K_{a_ν} is the minimum of the Kantorovich functional with a cost function c satisfying $c(x) \sim x^2 \log x^2$ for large values of x and $c(W_2(\nu, \gamma)) = 0$.

Theorem 2.10. *Assume that $\nu = g \cdot \gamma$ satisfies the logarithmic Sobolev inequality*

$$\int f^2 \log f^2 d\nu - \int f^2 d\nu \cdot \log \left(\int f^2 d\nu \right) \leq C_{LSI} \int |\nabla f|^2 d\nu.$$

Then

$$\frac{1}{2} I(g) - \text{Ent} g \geq \frac{1}{C_{LSI}} K_{a_\nu}(\nu, \gamma),$$

where $a_\nu = W_2(\nu, \gamma)$ and $K_a(\nu, \gamma)$ is the minimum of the Kantorovich functional corresponding to the cost function

$$c_a(x) = a^2 \left(1 - \frac{|x|^2}{a^2} + \frac{|x|^2}{a^2} \log \frac{|x|^2}{a^2} \right).$$

Proof. Let $T(x) = x + \nabla \varphi$ be the optimal transportation taking ν to γ . We apply formula (2.2) and estimate the integral of $\|D^2 \varphi\|_{\text{HS}}^2$ from below:

$$\begin{aligned} C_{LSI} \int \|D^2 \varphi\|_{\text{HS}}^2 d\nu &= \sum_{j=1}^n C_{LSI} \int |\nabla \varphi_{x_j}|^2 d\nu \geq \sum_{j=1}^n \int \varphi_{x_j}^2 \log \left(\frac{\varphi_{x_j}^2}{\int \varphi_{x_j}^2 d\nu} \right) d\nu \\ &= W_2^2(\nu, \gamma) \int \sum_{j=1}^n \alpha_j \frac{\varphi_{x_j}^2}{\int \varphi_{x_j}^2 d\nu} \log \left(\frac{\varphi_{x_j}^2}{\int \varphi_{x_j}^2 d\nu} \right) d\nu, \end{aligned}$$

where $\alpha_j = \frac{\int \varphi_{x_j}^2 d\nu}{W_2^2(\nu, \gamma)}$. The function $t \mapsto t \log t$ is convex for $t > 0$ and $\sum_{j=1}^n \alpha_j = 1$. Hence the above expression is not less than

$$\begin{aligned} W_2^2(\nu, \gamma) \int \frac{\sum_{j=1}^n \varphi_{x_j}^2}{W_2^2(\nu, \gamma)} \log \left(\frac{\sum_{j=1}^n \varphi_{x_j}^2}{W_2^2(\nu, \gamma)} \right) d\nu \\ = \int |\nabla \varphi|^2 \left[\log \left(\frac{|\nabla \varphi|^2}{W_2^2(\nu, \gamma)} \right) - 1 + \frac{W_2^2(\nu, \gamma)}{|\nabla \varphi|^2} \right] d\nu \geq K_{a_\nu}(\nu, \gamma), \end{aligned}$$

which completes the proof. \square

2.3. Stability for the Talagrand transportation inequality. The aim of the following proposition is to give a simplified proof of another result from [14, Theorem 5] with a more precise constant.

Lemma 2.11. *The function $\Delta(t) = t - \log(1+t)$, $t > -1$ has the following properties:*

- (1) $\Delta(t)$ is convex,
- (2) $\Delta(\sqrt{t})$ is concave on $[0, +\infty)$ and, in particular, subadditive,
- (3) $\Delta(t) \geq \Delta(|t|)$,
- (4) $\Delta(t) \geq (1 - \log 2) \min(t, t^2)$ on $[0, +\infty)$.

Proposition 2.12. *Assume that*

$$\int x g d\gamma = 0.$$

The deficit $\text{Ent}g - \frac{1}{2}W_2^2(g \cdot \gamma, \gamma)$ of the Talagrand transportation inequality satisfies the following estimate:

$$\begin{aligned} \text{Ent}g - \frac{1}{2}W_2^2(g \cdot \gamma, \gamma) &\geq \Delta\left(\frac{1}{2}n^{-1/2}W_{1,1}(g \cdot \gamma, \gamma)\right) \\ &\geq (1 - \log 2) \min\left\{\frac{1}{2}n^{-1/2}W_{1,1}(g \cdot \gamma, \gamma), \frac{1}{4}n^{-1}W_{1,1}^2(g \cdot \gamma, \gamma)\right\} \end{aligned}$$

where $W_{1,1}$ is the transportation cost corresponding to $c(x, y) = \sum_{i=1}^n |x_i - y_i|$.

Proof. Let $S(x) = x + \nabla\psi$ be the optimal transportation taking γ to $g \cdot \gamma$ and let λ_i be all eigenvalues of $D^2\varphi$. Applying (2.5) we obtain

$$\begin{aligned} \text{Ent}g - \frac{1}{2}W_2^2(g \cdot \gamma, \gamma) &= \int \sum_{i=1}^n \lambda_i - \log(1 + \lambda_i) d\gamma \\ &= \int \sum_{i=1}^n \Delta(\lambda_i) d\gamma \geq \int \sum_{i=1}^n \Delta(|\lambda_i|) d\gamma = \int \sum_{i=1}^n \Delta(\sqrt{\lambda_i^2}) d\gamma \geq \int \Delta\left(\left[\sum_{i=1}^n \lambda_i^2\right]^{1/2}\right) d\gamma \\ &= \int \Delta(\|D^2\psi\|_{\text{HS}}) d\gamma \geq \Delta\left(\int \|D^2\psi\|_{\text{HS}} d\gamma\right). \end{aligned}$$

Now we note that

$$\begin{aligned} \int \|D^2\psi\|_{\text{HS}} d\gamma &= \int \left(\sum_{i=1}^n |\nabla\psi_{x_i}|^2\right)^{1/2} d\gamma \geq n^{-1/2} \int \sum_{i=1}^n |\nabla\psi_{x_i}| d\gamma \\ &\geq \frac{1}{2}n^{-1/2} \int \sum_{i=1}^n |\psi_{x_i}| d\gamma \geq \frac{1}{2}n^{-1/2}W_{1,1}(g \cdot \gamma, \gamma), \end{aligned}$$

where we apply the equality

$$\int \psi_{x_i} d\gamma = \int (x_i + \psi_{x_i}) d\gamma = \int x_i g d\gamma = 0$$

and the L^1 -Poincaré (Cheeger) inequality for γ :

$$\int |f - \int f d\gamma| d\gamma \leq 2 \int |\nabla f| d\gamma,$$

which completes the proof. \square

3. A PRIORI ESTIMATES FOR THE KÄHLER–EINSTEIN EQUATION

A moment measure on \mathbb{R}^n is a probability measure ν on \mathbb{R}^n that is the image of another probability measure $\mu = e^{-\Phi} dx$ under the mapping $x \rightarrow \nabla\Phi(x)$, where Φ is a convex function. If ν admits a smooth density ϱ , then Φ solves the Kähler–Einstein equation

$$\varrho(\nabla\Phi) \det D^2\Phi = e^{-\Phi}.$$

It was shown in [11] that every measure ν with zero mean satisfying the condition $\nu(L) = 0$ for any subspace L of dimension less than n is a moment measure. The function Φ is uniquely determined up to a translation.

We will be interested in the following Gaussian analog of the Kähler–Einstein equation: given a probability measure

$$\varrho dx = g \cdot \gamma,$$

find φ such that $g \cdot \gamma$ is the image of the log-concave probability measure

$$\rho \cdot \gamma = e^{-\varphi} \cdot \gamma$$

under the mapping

$$T(x) = x + \nabla\varphi(x).$$

Clearly, there is a simple connection between this problem and the "Euclidean" moment measure problem. Namely, Φ and φ are related by the following formula:

$$\Phi(x) = \frac{|x|^2}{2} + \varphi(x) + \frac{n}{2} \log 2\pi.$$

However, the Gaussian modification of the moment measure problem is meaningful in any infinite-dimensional space equipped with a Gaussian measure.

Since Φ is unique up to a translation, it will be natural to impose the following requirement that determines φ uniquely.

Assumption II. The measure $\rho \cdot \gamma = e^{-\varphi} \cdot \gamma$ satisfies the condition

$$\int x_i e^{-\varphi} d\gamma = 0 \quad \forall i.$$

The existence and uniqueness of φ follows from the results of [11]. It follows from the main result of [27] that $e^{-\varphi}$ gives a minimum to the following functional:

$$\mathcal{F}_\gamma(\rho) = -\frac{1}{2} W_2^2(g \cdot \gamma, \rho \cdot \gamma) + \int \rho \log \rho d\gamma.$$

We wish to find a condition on g which guarantees that F is bounded from below by a dimension-free functional depending on g .

3.1. Information controls \mathcal{F} . Throughout this subsection $\rho \cdot \gamma = e^{-\varphi} d\gamma$ is the (unique) minimum point of \mathcal{F}_γ with zero mean.

Lemma 3.1. *Assume that the measure $\rho \cdot \gamma = e^{-\varphi} d\gamma$ satisfies the inequality*

$$\text{Ent} \rho \leq \frac{1-\delta}{2} I(\rho)$$

for some $0 < \delta < 1$. Then

$$W_2(\rho \cdot \gamma, g \cdot \gamma) \leq \frac{1 + \sqrt{1-\delta}}{\delta} W_2(g \cdot \gamma, \gamma).$$

Proof. We have

$$W_2^2(\rho \cdot \gamma, \gamma) \leq 2 \text{Ent}(\rho) \leq (1-\delta) I(\rho) = (1-\delta) W_2^2(\rho \cdot \gamma, g \cdot \gamma).$$

Hence by the triangle inequality

$$W_2(\rho \cdot \gamma, g \cdot \gamma) \leq W_2(\rho \cdot \gamma, \gamma) + W_2(g \cdot \gamma, \gamma) \leq \sqrt{1-\delta} W_2(\rho \cdot \gamma, g \cdot \gamma) + W_2(g \cdot \gamma, \gamma),$$

which completes the proof. \square

Theorem 3.2. *There exists a pair of universal constants C_1, C_2 such that*

$$C_P \leq \max\{C_1, \exp(C_2 I(g))\},$$

where C_P is the Poincaré constant of the measure $e^{-\varphi} \cdot \gamma$.

Proof. Let $x + \nabla\psi$ be the optimal transportation taking $g \cdot \gamma$ to $e^{-\varphi} \cdot \gamma$. It is well-known that

$$x + \nabla\psi = T^{-1} \quad g \cdot \gamma\text{-a.e.}$$

and

$$W_2^2(\rho \cdot \gamma, g \cdot \gamma) = \int |\nabla\varphi|^2 e^{-\varphi} d\gamma = \int |\nabla\psi|^2 g d\gamma.$$

First we note that $e^{-\varphi} \cdot \gamma$ is a log-concave measure, hence it has finite moments of all orders and a finite Poincaré constant $C_P < \infty$ (see [1, Theorem 4.6.3]).

Note that

$$I(\rho) = W_2(\rho \cdot \gamma, g \cdot \gamma) \leq W_2(\rho \cdot \gamma, \gamma) + W_2(g \cdot \gamma, \gamma).$$

The right-hand side of this inequality is finite, because $g \cdot \gamma$ and $\rho \cdot \gamma$ have finite second moments. Thus, $I(\rho) < \infty$. Moreover, approximating g by smooth densities with uniformly bounded second derivatives of $\log g$ we can assume without loss of generality that $\nabla \psi$ is globally Lipschitz (see Theorem 3.4).

It follows from the previous lemma and Theorem 2.4 that

$$W_2(\rho \cdot \gamma, g \cdot \gamma) \leq \frac{1 + \sqrt{1 - \delta}}{\delta} W_2(g \cdot \gamma, \gamma),$$

where $\delta = \frac{C_P \log C_P - C_P + 1}{(C_P - 1)^2}$. Applying (2.5) we obtain

$$\begin{aligned} \int (\Delta \psi - \log \det(\text{Id} + D^2 \psi)) g d\gamma + \int g \log g d\gamma + \frac{1}{2} \int |\nabla \varphi|^2 e^{-\varphi} d\gamma \\ = - \int \varphi e^{-\varphi} d\gamma - \int \langle \nabla \psi, \nabla g \rangle d\gamma. \end{aligned}$$

By the log-Sobolev inequality

$$- \int \varphi e^{-\varphi} d\gamma \leq \frac{1}{2} \int |\nabla \varphi|^2 e^{-\varphi} d\gamma.$$

Hence

$$\begin{aligned} \int (\Delta \psi - \log \det(\text{Id} + D^2 \psi)) g d\gamma &\leq \sqrt{\int \frac{|\nabla g|^2}{g} d\gamma} \cdot W_2(g \cdot \gamma, \rho \cdot \gamma) \\ (3.1) \quad &\leq \frac{1 + \sqrt{1 - \delta}}{\delta} W_2(g \cdot \gamma, \gamma) \sqrt{I(g)} \leq \frac{1 + \sqrt{1 - \delta}}{\delta} I(g). \end{aligned}$$

Let us estimate C_P . By the Brascamb–Lieb inequality (see [1])

$$\int f^2 e^{-\varphi} d\gamma - \left(\int f e^{-\varphi} d\gamma \right)^2 \leq \int \langle (\text{Id} + D^2 \varphi)^{-1} \nabla f, \nabla f \rangle e^{-\varphi} d\gamma.$$

Hence

$$\begin{aligned} \int f^2 e^{-\varphi} d\gamma - \left(\int f e^{-\varphi} d\gamma \right)^2 &\leq \int \|(\text{Id} + D^2 \varphi)^{-1}\|_{op} e^{-\varphi} d\gamma \cdot \|\nabla f\|_{L^\infty(e^{-\varphi} \cdot \gamma)}^2 \\ &= \int \|\text{Id} + D^2 \psi\|_{op} g d\gamma \cdot \|\nabla f\|_{L^\infty(e^{-\varphi} \cdot \gamma)}^2. \end{aligned}$$

Since $e^{-\varphi} \cdot \gamma$ is a log-concave measure, it follows from a result of E. Milman on equivalence of the isoperimetric and concentration inequalities ([26], [25, Theorem 1.5] or [1, Theorem 8.7.1]) that

$$C_P \leq c \int \|\text{Id} + D^2 \psi\|_{op} g d\gamma$$

for some universal constant c . It follows from (3.1) that

$$\int (\|D^2 \psi\|_{op} - \log \det(I + \|D^2 \psi\|_{op})) g d\gamma \leq \frac{1 + \sqrt{1 - \delta}}{\delta} I(g).$$

Applying the inequality $\log(1 + t) \leq 2^{-1} + 2^{-1}t$, we observe that

$$\int (\|D^2 \psi\|_{op} - \log \det(I + \|D^2 \psi\|_{op})) g d\gamma \geq -1/2 + 1/2 \int \|D^2 \psi\|_{op} g d\gamma.$$

Hence for some universal constant C we have

$$C_P \leq C \left(1 + \frac{1 + \sqrt{1 - \delta}}{\delta} I(g) \right).$$

It remains to note that for large values of C_P one has $\delta \sim \frac{\log C_P}{C_P}$. This immediately implies the announced bound. \square

Finally, Theorem 3.2, Lemma 3.1, and Theorem 2.4 imply our main result.

Theorem 3.3. *Assume that g is a probability density such that $I(g) < \infty$. Then there exists a constant $C > 0$ depending only on $I(g)$ such that*

$$\mathcal{F}_\gamma \geq -C$$

and

$$W_2^2(g \cdot \gamma, \rho \cdot \gamma) \leq C,$$

where $\rho \cdot \gamma$ is the minimum point of \mathcal{F}_γ such that $\rho \cdot \gamma$ has zero mean.

Yet another result can be obtained under the uniform bound for the Hessian of $-\log g$ by applying the same techniques as in the proof of the Caffarelli contraction theorem. We do not give the full proof here (see, for instance, [21]), but only explain the main idea.

Theorem 3.4. *Let $-D^2 \log g \leq c \cdot \text{Id}$, $c > -1$. Then $C_P \leq 1 + c$ and $\mathcal{F}_\gamma \geq -C$, $W_2^2(g \cdot \gamma, \rho \cdot \gamma) \leq C$, for some constant C depending on c .*

Sketch of the proof. According to the Brascamb–Lieb inequality

$$\int f^2 d\mu - \left(\int f d\mu \right)^2 \leq \int \langle (D^2 \varphi + \text{Id})^{-1} \nabla f, \nabla f \rangle d\mu, \quad \mu = \rho \cdot \gamma.$$

Hence it is sufficient to show that $(D^2 \varphi + \text{Id})^{-1} \leq (1 + c) \text{Id}$, or, equivalently, $D^2 \psi + \text{Id} \leq (1 + c) \text{Id}$, where ψ is the dual potential. This estimate can be obtained by the standard maximum principle and differentiation of the Monge–Ampère equation. The maximum principle is applied in the situation

$$1 + \psi_{ee} = \Psi_{ee}, \quad \Psi(x) = \frac{|x|^2}{2} + \psi(x) + c(n),$$

where e is a fixed unit vector. Note that Ψ satisfies the Monge–Ampère equation

$$\Phi(\nabla \Psi) - \log \det D^2 \Psi = \frac{x^2}{2} - \log g + c'(n),$$

where $\Phi = \frac{|x|^2}{2} + \varphi$. Let us differentiate twice the equation

$$(3.2) \quad \langle \nabla \Phi(\nabla \Psi), \nabla \Psi_{ee} \rangle + \langle D^2 \Phi(\nabla \Phi) \nabla \Psi_e, \nabla \Psi_e \rangle - \text{Tr}[(D^2 \Psi)^{-1} D^2 \Psi_{ee}] \\ + \text{Tr}[(D^2 \Psi)^{-1} D^2 \Psi_e]^2 = 1 - (\log g)_{ee}.$$

At every local maximum point x_0 of the function Ψ_{ee} one has $\nabla \Psi_{ee} = 0$, $D^2 \Psi_{ee} \leq 0$. Note, moreover, that

$$\langle D^2 \Phi(\nabla \Phi) \nabla \Psi_e, \nabla \Psi_e \rangle = \Psi_{ee}.$$

From (3.2) we obtain

$$1 + \psi_{ee} = \Psi_{ee} \leq 1 + c,$$

which completes the proof.

3.2. High power and exponential integrability. In this subsection we establish a priori bounds for the entropy- and information-type integrals

$$\int |\log \rho|^p \rho d\gamma$$

and

$$\int \left| \frac{\nabla \rho}{\rho} \right|^p \rho d\gamma.$$

Here again $\rho \cdot \gamma = e^{-\varphi} d\gamma$ is the (unique) minimum point of \mathcal{F}_γ with zero mean. Several results of this type have been obtained in [5] for the standard Monge–Kantorovich problem. The proofs of the theorems of this subsection follow the ideas from [5], but here they are simpler because we benefit from the specific properties of our problem.

Theorem 3.5. *Assume that $I(g) < \infty$. Then*

$$\int \varphi^2 e^{-\varphi} d\gamma = \int (\log \rho)^2 \rho dx < \infty.$$

Assume, in addition, that g satisfies the Poincaré inequality with a constant C . Then for every $p > 0$ there exists a number c depending on $p, I(g), C$ such that

$$\int |\nabla \varphi|^p e^{-\varphi} d\gamma = \int \left| \frac{\nabla \rho}{\rho} \right|^p \rho d\gamma \leq c$$

and

$$\int |\varphi|^p e^{-\varphi} d\gamma = \int |\log \rho|^p \rho d\gamma \leq c.$$

Proof. The assumptions of the theorem imply that μ satisfies the Poincaré inequality (see Theorem 3.2). Next we note that the γ -integrability of $|\nabla \varphi|^p e^{-\varphi}$ implies the γ -integrability of $|\varphi|^p e^{-\varphi} d\gamma$. This follows from the Poincaré inequality (see an explanation in [5, formula (1.3)]) and the γ -integrability of $\varphi e^{-\varphi}$, i.e., the existence of $\text{Ent}(e^{-\varphi})$.

Then our first claim follows immediately from the finiteness of

$$\int |\nabla \varphi|^2 e^{-\varphi} d\gamma = W_2^2(\rho \cdot \gamma, g \cdot \gamma).$$

We now proceed by induction and assume that the theorem is proved for $p = 2m$. Let us show how to prove the claim for $2m + 1$. By the Kantorovich duality identity (see [8])

$$\frac{|\nabla \varphi|^2}{2} + \varphi + \psi(x + \nabla \varphi) = 0.$$

Hence

$$\int |\nabla \varphi|^{2m+1} e^{-\varphi} d\gamma = - \int \varphi |\nabla \varphi|^{2m-1} e^{-\varphi} d\gamma - \int \psi(x + \nabla \varphi) |\nabla \varphi|^{2m-1} e^{-\varphi} d\gamma.$$

We estimate the right-hand side by

$$c(m) \left(\int |\nabla \varphi|^{2m} e^{-\varphi} d\gamma + \int |\varphi|^{2m} e^{-\varphi} d\gamma + \int |\psi(x + \nabla \varphi)|^{2m} e^{-\varphi} d\gamma \right).$$

The integrals

$$\int |\varphi|^{2m} e^{-\varphi} d\gamma, \quad \int |\nabla \varphi|^{2m} e^{-\varphi} d\gamma$$

are bounded by a constant depending on $C, m, I(g)$ by the inductive assumption. Since $g \cdot \gamma$ satisfies the Poincaré inequality, it remains to show the integrability of $|\nabla \psi|^{2m} g$. But the integral of this function against γ equals the integral of $|\nabla \varphi|^{2m} e^{-\varphi}$. So the claim is proved for $p = 2m + 1$. Repeating the arguments we prove the assertion for $p = 2m + 2$. \square

We close this section with a result on the exponential integrability of $|\nabla \varphi|^2$. We apply the infimum-convolution inequality, which is known to be another form of the transportation inequality (see [1]):

$$(3.3) \quad \int e^{-f} d\gamma \leq e^{\int f^* d\gamma},$$

where

$$f^*(y) = - \inf_x \left(f(x) + \frac{1}{2} |x - y|^2 \right).$$

The duality identity and (3.3) immediately imply that for every $0 \leq \delta \leq 1$ one has

$$\begin{aligned} \int e^{\frac{\delta}{2}|\nabla\varphi|^2} e^{-(1-\delta)\varphi} d\gamma &= \int e^{-\delta\psi(x+\nabla\varphi)} e^{-\varphi} d\gamma = \int e^{-\delta\psi} g d\gamma \\ &\leq \left(\int e^{-\psi} d\gamma \right)^\delta \left(\int g^{\frac{1}{1-\delta}} d\gamma \right)^{1-\delta} \leq e^{\delta \int \varphi d\gamma} \left(\int g^{\frac{1}{1-\delta}} d\gamma \right)^{1-\delta}. \end{aligned}$$

In particular, we obtain the following result (note that unlike all other results in this paper we do not assume that $I(g) < \infty$).

Theorem 3.6. *Assume that $g \leq C$ and $\varphi \in L^1(\gamma)$. Then*

$$\int \exp\left(\frac{1}{2}|\nabla\varphi|^2\right) d\gamma \leq C \exp\left(\int \varphi d\gamma\right).$$

Remark 3.7. The assumption of integrability of φ may seem quite innocent, but it is not. For instance, if ρ vanishes outside a compact set, then the integral of φ is infinite. On the other hand, if φ is defined γ -a.e., then by the Cheeger inequality

$$\int |\varphi - \text{med}| d\gamma \leq 2 \int |\nabla\varphi| d\gamma,$$

where med is the median of φ . Then it follows immediately from Theorem 3.6 that the integral of $\exp\left(\frac{1}{2}|\nabla\varphi|^2\right)$ is bounded by a constant depending on C, med .

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