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GAUSSIAN APPROXIMATION FOR RESIDUALS OF STATIONARY AUTOREGRESSIVE PROCESS IN HÖLDER NORM

The paper treats the hölderian approximation for partial sums process of stationary autoregressive residuals (AR(p), $p \geq 1$). We consider the polygonal smoothed process of these partial sums and we prove the Hölder convergence of this sequence of processes to the Brownian motion for any order $\alpha < \frac{1}{2}$. A statistical application of this convergence to detect epidemic change and simulation results are also presented.

1. Introduction

The study of the asymptotic behavior of stochastic processes is an important topic in probability theory, widely used in applied statistics. In fact, in parametric and non-parametric inference, many statistical applications (estimation, testing hypothesis) which are usually based on continuous functionals of paths of processes as partial sums process, empirical process and quantile process are solved using the weak convergence of stochastic processes. The asymptotic behavior of these processes, in particular partial sums process, was established. The well know invariance principle which is introduced by Donsker-Prokhorov states that, for a sequence of independent identically distributed random variables $(X_i)_{i\geq 1}$ with $\mathbb{E}X_1 = 0$ and $\mathbb{E}X_i^2 = 1$, the random polygonal lines

$$\widehat{W}_n(t) = \frac{1}{\sigma\sqrt{n}}(\sum_{i=0}^{[nt]} X_i + (nt - [nt])X_{[nt]+1}), \quad t \in [0, 1],$$

converges in distribution to some Brownian motion in $C[0, 1]$. Since the paths of the processes used in statistics as partial sums process, empirical process, quantile process and the limit process as Brownian motion, Brownian bridge have a better regularity than the bare continuity (e.g. hölderian), it is then legitimate to study these limit theorems in Banach space $H_\alpha^0[0, 1]$. Indeed, this hölderian topology provides more continuous functionals of the paths, thus, more statistical applications. The first result in this direction goes back to Lamperti [14], which extends the Donsker-Prohorov invariance principle to Hölder spaces. For more results in this area, see those of Račkauskas and Suquet [20, 21] and Hamadouche [9, 10].

Among processes also used in statistical applications, we find linear processes, autoregressive processes, the partial sums process of residuals of autoregressive processes.

The theory of autoregressive models or more generally, time series theory is very studied and has applications in different areas such as econometrics, medicine and biology. For a detailed review, we refer to Gourieroux and Monfort [8], Brockwell and Davis [4], Csörgő and Horváth [6], etc.

The weak convergence of partial sums of the residuals in regression models were established by MacNeil [15, 16]. MacNeil and Jandhyala [12, 17] studied the properties of the residuals process for non-linear regression respectively for linear regression model and its applications to change points detection. Further, Kulperger [13] extended the results of

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MacNeil to autoregressive model, while Bai [1] was interested to stationary ARMA(p,q) models. These works are first established in the framework of the classical function space $C[0,1]$ or the Skorokhod space $D[0,1]$. Since the weak Hölder convergence offers more continuous functionals than $C[0,1]$ for statistical applications, some extensions have been made in H_α^0. Račkauskas [19] developed Hölder convergence of the partial sums of regression residuals, Markevičiūtė, Račkauskas and Suquet [18], studied some hölderians functional theorems for the first order nearly non-stationary autoregressive process and their applications in epidemic change. In the same direction, Račkauskas and Rastené [22] derived the convergence, in Hölder spaces, of polygonal line processes of partial sums of residuals of the AR(1) model. These theoretical results found statistical applications for solving many problems as epidemic change detection in the mean, variance, etc.
Our current contribution aims to study in hölderian functional framework, the asymptotic behavior of polygonal line processes built on partial sums of residuals of a stationary AR(p) model, for $p \geq 1$. The paper is organized as follows. Section 2 is devoted to hölderian functional framework. Definitions and some properties on autoregressive processes are presented in section 3. Section 4 treats our main result which is the hölderian convergence of partial sums of residual of autoregressive process to the Brownian motion. In section 5, we present a statistical application for testing epidemic change in means of the innovations of a stationary AR(p) and to illustrate our results, some numerical simulations are also described in section 6.

2. Hölderian functional framework

2.1. **Definitions.** We define the Hölder space $H_\alpha[0,1]$ $(0 < \alpha \leq 1)$, as the space of functions f vanishing at 0 such that

$$\|f\|_\alpha = \sup_{0<|t-s|\leq 1} \frac{|f(t)-f(s)|}{|t-s|^\alpha} < +\infty.$$

We denote by $\omega_\alpha(f,\delta)$ the Hölder modulus of continuity of the function f:

$$\omega_\alpha(f,\delta) = \sup_{0<|t-s|\leq \delta} \frac{|f(t)-f(s)|}{|t-s|^\alpha}.$$

Define the subspace $H_\alpha^0[0,1]$ of $H_\alpha[0,1]$ by

$$f \in H_\alpha^0[0,1] \text{ if and only if} f \in H_\alpha[0,1] \text{ and } \lim_{\delta\to 0} \omega_\alpha(f,\delta) = 0.$$

$(H_\alpha, \|.\|_\alpha)$ is a non-separable Banach space. For $0 < \beta < \alpha$, $(H_\alpha, \|.\|_\beta)$ is separable and H_α is topologically embedded in H_β. $(H_\alpha^0, \|.\|_\alpha)$ is a separable closed subspace of $(H_\alpha, \|.\|_\alpha)$ (cf. Ciesielski [5]).
The separability of the space $H_\alpha^0[0,1]$ allows us to prove tightness instead of relative compacity of a sequence of distributions of random processes (cf. Billingsley [2]).

2.2. **Weak convergence and tightness in H_α^0.** We consider stochastic process with Hölderian paths as random element of the functional space H_α. As the canonical injection of H_α^0 in H_α is continuous, weak convergence in H_α^0 implies weak convergence in H_α. The study of weak convergence of random elements of H_α^0 is based on the following result.

Proposition 1 (Hamadouche [9]). *The weak convergence in H_α^0 of a sequence of processes $(\xi_n, n \geq 1)$ is equivalent to the tightness in H_α^0 of the sequence of distributions $P_n = P\xi_n^{-1}$ of random elements ξ_n and the convergence of the finite-dimensional distributions of ξ_n.*

So to prove the weak convergence in H_α^0, we need tightness. Among sufficient conditions of tightness, we have the well known result.

Theorem 1 (Lamperti [14]). *Let $(\xi_n)_{n\geq 1}$ be a sequence of processes vanishing at 0 and suppose there are $\delta > 0$, $\gamma > 0$ and $c > 0$ such that*

$$\mathbb{E}\left|\xi_n(t) - \xi_n(s)\right|^{\gamma} \leq c \ \left|t - s\right|^{1+\delta}.$$

Then the sequence $(\xi_n)_{n\geq 1}$ is tight in H_α^0 for $0 < \alpha < \frac{\delta}{\gamma}$.

3. Autoregressive processes and residuals

We consider the AR(p) model, $(p \geq 1)$

$$\left\{ \begin{array}{l} X_i = \rho_1 X_{i-1} + \rho_2 X_{i-2} + ... + \rho_p X_{i-p} + \varepsilon_i, \quad i = 1, ..., n, \ n \geq 1; \\ X_0 = X_{-1} = ... = X_{1-p} = 0, \end{array} \right. \tag{1}$$

where $\{X_i\}_{i=1}^n$ is a set of observations, $\varepsilon_1, ..., \varepsilon_n$ are independent identically distributed random variables with mean zero and finite variance $\sigma^2 = \mathbb{E}\varepsilon_1^2$ and $(\rho_1, ..., \rho_p)$, $\rho_p \neq 0$ are the parameters of model.
Similarly, we define the residuals $(\hat{\varepsilon}_i, i = 1, 2, ..., n)$ by

$$\begin{aligned} \hat{\varepsilon}_i &= X_i - \sum_{j=1}^{p} \hat{\rho}_j X_{i-j} \\ &= \varepsilon_i - \sum_{j=1}^{p} (\hat{\rho}_j - \rho_j) X_{i-j}, \qquad i = 1, 2, ..., n, \end{aligned} \tag{2}$$

where $\hat{\rho} = (\hat{\rho}_1, ..., \hat{\rho}_p)^{'}$ is the least squares estimator of $\rho = (\rho_1, ..., \rho_p)^{'}$ given by

$$\hat{\rho} = (X^{'}X)^{-1}X^{'}Y, \tag{3}$$

with $Y = (X_{p+1}, ..., X_n)^{'} = X\rho + Z$, $Z = (\varepsilon_{p+1}, ..., \varepsilon_n)^{'}$ and X is $(n-p) \times p$ matrix given by

$$X = \begin{pmatrix} X_p & X_{p-1} & \cdots & X_1 \\ X_{p+1} & X_p & \cdots & X_2 \\ \vdots & \vdots & \ddots & \vdots \\ X_{n-1} & X_{n-2} & \cdots & X_{n-p} \end{pmatrix}.$$

In order to get the autoregressive process AR (p) models more synthetic, we give some properties of the backshift operator and operators derived therefrom. We use these operators to manipulate AR (p) models and to determine the conditions under which these processes admit a stationary and causal representation.
We define the backshift operator $\mathbf{B}$ with action is defined on the space random variables as a linear function which operates on the time index of a series and shifts time back one time unit to form a new series i.e

$$\mathbf{B}X_i = X_{i-1}.$$

This operator has the following properties (cf. Bluduc [3], Gourieroux and Monfort [8] and Brockwell and Davis [4]).
- The backshift operator $\mathbf{B}$ is invertible. Its inverse $\mathbf{B}^{-1} = \mathbf{F}$ is defined by

$$\mathbf{B}^{-1}X_i = \mathbf{F}X_i = X_{i+1} \ \ (\mathbf{F} \text{ is called forward operator}).$$

- If $\mathbf{B}$ is composed j times with itself, we obtain,

$$\mathbf{B}o\mathbf{B}o...o\mathbf{B} = \mathbf{B}^j,$$

$$\text{such as } \forall \ j \ \in \mathbb{N}, \ \ \mathbf{B}^j X_i = X_{i-j}. \tag{4}$$

More generally, by combining linearly these different powers, we built a new operator

$$a(\mathbf{B}) = 1 + a\mathbf{B} + ... + a_p\mathbf{B}^p. \tag{5}$$

Under some conditions, the operator $a(\mathbf{B})$ may be reversed. Indeed, by the following proposition.

Proposition 2 (Gourieroux and Monfort [8]). *Let $a(\mathbf{B})$ the polynomial defined by (5). If the norms of the roots of $a(\mathbf{B}) = 0$ are not on the unit circle, then $a(\mathbf{B})$ has a unique inverse $b(\mathbf{B})$ such that $a(\mathbf{B}).b(\mathbf{B}) = 1$ and we have*

$$b(\mathbf{B}) = \sum_{m=-\infty}^{+\infty} b_m \mathbf{B}^m,$$

such that $\sum_{m=-\infty}^{+\infty} |b_m| < \infty$ with $b_0 = 1$.
On the other hand, if the roots of $a(\mathbf{B}) = 0$ are outside the unit disk, we have

$$b(\mathbf{B}) = \sum_{m=0}^{+\infty} b_m \mathbf{B}^m,$$

such that $\sum_{m=0}^{+\infty} |b_m| < \infty$ with $b_0 = 1$.

Then, we deduce that $a(\mathbf{B})^{-1}$ is well defined if the norms of the roots of $a(\mathbf{B})$ are not on the unit circle and for some sequence (b_m) such that $\sum_{m=-\infty}^{+\infty} |b_m| < \infty$,

$$\begin{aligned}
\|a(\mathbf{B})^{-1}\| = \|b(\mathbf{B})\| &= \|\sum_{m=-\infty}^{+\infty} b_m \mathbf{B}^m\| \\
&\leq \sum_{m=-\infty}^{+\infty} |b_m| \|\mathbf{B}^m\| \\
&\leq \sum_{m=-\infty}^{+\infty} |b_m| < \infty,
\end{aligned}$$

i.e, there exists a constant C such that

$$\|b(\mathbf{B})\| \leq \mathrm{C}. \tag{6}$$

However, setting $a_0 = 1$, $a_j = -\rho_j$ for $j = 1, ..., p$ and using the polynomial $a(\mathbf{B}) = 1 + a\mathbf{B} + ... + a_p\mathbf{B}^p$, we can write the process defined in (1) as follows

$$\begin{aligned}
\varepsilon_i &= X_i + a\mathbf{B}X_i + ... + a_p\mathbf{B}^p X_i \\
&= a(\mathbf{B})X_i, \qquad i =, 1, 2, ..., n,
\end{aligned}$$

or

$$X_i = a(\mathbf{B})^{-1}\varepsilon_i, \qquad i = 1, 2, ..., n, \tag{7}$$

if the norms of the roots of $a(\mathbf{B}) = 0$ are not on the unit circle. The following theorem gives necessary and sufficient conditions for the stationarity and causality of an AR(p) process.

Theorem 2 (Brockwell and Davis [4]). *The autoregressive process (X_i) is causal and stationary if and only if the polynomial $a(z) = \sum_{m=0}^{p} a_m z^m$ with $a_0 = 1$ satisfies*

$$a(z) \neq 0 \text{ for all } z \in \mathbb{C} \text{ such that } |z| \leq 1. \tag{8}$$

From this theorem, it follows that if (8) holds, the process (X_i) is stationary. From Proposition 2, the AR(p) process defined in (1) has a moving average $MA(\infty)$ representation.

$$\begin{aligned} X_i &= \frac{1}{a(\mathbf{B})}\varepsilon_i = b(\mathbf{B})\varepsilon_i \\ &= \sum_{m=0}^{\infty} b_m \mathbf{B}^m \varepsilon_i \\ &= \sum_{m=0}^{\infty} b_m \varepsilon_{i-m}, \qquad i = 1, 2, ..., n, \end{aligned} \tag{9}$$

with $\sum\limits_{m=0}^{+\infty} |b_m| < \infty$ and $b_0 = 1$, i.e., X_i is expressed as a convergent infinite sum of past terms of innovations ε_i.

4. Hölderian convergence of partial sums process of residuals of AR(p) process

We investigate the polygonal line processes $\widehat{V}_n(t)$ obtained by linear interpolation between the points $(\frac{k}{n}, \sum_{i=1}^k \hat{\varepsilon}_i)$.

$$\begin{aligned} \widehat{V}_n(t) &= \sum_{i=1}^{[nt]} \hat{\varepsilon}_i + (nt - [nt])\hat{\varepsilon}_{[nt]+1}, \quad t \in [0,1], \quad n = 1, 2, ... \\ &= \sum_{i=1}^{[nt]} \varepsilon_i + (nt - [nt])\varepsilon_{[nt]+1} - \sum_{j=1}^{p} (\hat{\rho}_j - \rho_j)[\sum_{i=1}^{[nt]} X_{i-j} + (nt - [nt])X_{[nt]+1-j}] \\ &= W_n(t) - \sum_{j=1}^{p} (\hat{\rho}_j - \rho_j)\theta_n^j, \end{aligned}$$

where

$$W_n(t) = \sum_{i=1}^{[nt]} \varepsilon_i + (nt - [nt])\varepsilon_{[nt]+1} \tag{10}$$

and

$$\theta_n^j = \sum_{i=1}^{[nt]} X_{i-j} + (nt - [nt])X_{[nt]+1-j}.$$

Theorem 3. *Let $0 < \alpha < 1/2$. Suppose that $\lim\limits_{t\to+\infty} t^{1/(1/2-\alpha)} P(|\varepsilon_1| > t) = 0$ and all the roots of equation $a(z) = \sum\limits_{m=0}^{p} a_m z^m = 0$, with $a_0 = 1$ are outside the unit disk, then the process $\sigma^{-1} n^{-1/2} \widehat{V}_n$ converges weakly to the Brownian motion in H_α^0.*

The proof of this theorem is based on the following results.

Proposition 3 (Rackauskas and Suquet [20])**.** *The process $n^{-1/2} W_n$ defined by* (10) *converges weakly to the Wiener process W in H_α^0 for all $\alpha < 1/2$ if and only if*

$$\lim_{t\to+\infty} t^{1/(1/2-\alpha)} P(|\varepsilon_1| > t) = 0.$$

Lemma 1 (Horváth [11])**.** *If the condition* (8) *holds, then, as $n \to \infty$ we have*

$$n^{1/2}(\hat{\rho} - \rho) = O_p(1).$$

Proof of Theorem 3. By Lamperti's invariance principle, $\sigma^{-1}n^{-1/2}W_n$ converges weakly to the Brownian motion in H^0_α, $\forall\alpha < \frac{1}{2}$, then it is sufficient, to prove that

$$\|\sigma^{-1}n^{-1/2}\widehat{V}_n - \sigma^{-1}n^{-1/2}W_n\|_\alpha \xrightarrow[n\to+\infty]{P} 0.$$

We have

$$\begin{aligned}\|\sigma^{-1}n^{-1/2}\widehat{V}_n - \sigma^{-1}n^{-1/2}W_n\|_\alpha &= \|\sigma^{-1}n^{-1/2}\sum_{j=1}^{p}(\hat{\rho}_j - \rho_j)\theta_n^j\|_\alpha \\ &\leq \sigma^{-1}\sum_{j=1}^{p}|n^{1/2}(\hat{\rho}_j - \rho_j)|\|n^{-1}\theta_n^j\|_\alpha. \qquad (11)\end{aligned}$$

If the condition (8) holds, we get

$$\|\theta_n^j\|_\alpha \leq \|b(\mathbf{B})\|\|W_n\|_\alpha.$$

In fact

$$\|\theta_n^j\|_\alpha = \max_{1\leq s<k\leq n}|(k-s)/n|^{-\alpha}|\sum_{i=1}^{k}X_{i-j} - \sum_{i=1}^{s}X_{i-j}|,$$

by (9), we have

$$X_i = \sum_{m=0}^{+\infty}b_m\varepsilon_{i-m}, \text{ with } \sum_{m=0}^{+\infty}|b_m| < \infty \text{ and } b_0 = 1$$

i.e

$$X_{i-j} = \sum_{m=0}^{+\infty}b_m\varepsilon_{i-m-j}, \text{ with } \sum_{m=0}^{+\infty}|b_m| < \infty \text{ and } b_0 = 1.$$

From (4), we deduce that

$$X_{i-j} = \sum_{m=0}^{+\infty}b_m\varepsilon_{i-m-j} = \sum_{m=0}^{+\infty}b_m\mathbf{B}^m\varepsilon_{i-j} = b(\mathbf{B})\varepsilon_{i-j},$$

thus, we get

$$\begin{aligned}\|\theta_n^j\|_\alpha &= \max_{1\leq s<k\leq n}|(k-s)/n|^{-\alpha}|\sum_{i=1}^{k}b(\mathbf{B})\varepsilon_{i-j} - \sum_{i=1}^{s}b(\mathbf{B})\varepsilon_{i-j}| \\ &= \max_{1\leq s<k\leq n}|(k-s)/n|^{-\alpha}|b(\mathbf{B})(\sum_{i=1}^{k}\varepsilon_{i-j} - \sum_{i=1}^{s}\varepsilon_{i-j})| \\ &\leq \|b(\mathbf{B})\|\max_{1\leq s<k\leq n}|(k-s)/n|^{-\alpha}|\sum_{i=1}^{k}\varepsilon_{i-j} - \sum_{i=1}^{s}\varepsilon_{i-j}| \\ &= \|b(\mathbf{B})\|\max_{1\leq s<k\leq n}|(k-s)/n|^{-\alpha}|W_n(\frac{k-j}{n}) - W_n(\frac{s-j}{n})| \\ &= \|b(\mathbf{B})\|\|W_n\|_\alpha. \qquad (12)\end{aligned}$$

Hence, it follows that

$$\|\sigma^{-1}n^{-1/2}\widehat{V}_n - \sigma^{-1}n^{-1/2}W_n\|_\alpha \leq \sigma^{-1}\|b(\mathbf{B})\|\|n^{-1}W_n\|_\alpha\sum_{j=1}^{p}|n^{1/2}(\hat{\rho}_j - \rho_j)|.$$

We have by (6)

$$\|b(\mathbf{B})\| \leq \text{C, with } \sum_{m=0}^{+\infty}|b_m| < \infty.$$

In addition, we can check that

$$n^{-1}W_n \xrightarrow{P} 0,$$

indeed,

$$n^{-1}W_n = n^{-1/2}n^{-1/2}W_n,$$

and from Proposition 3,

$$n^{-1/2}W_n \underset{n\to+\infty}{\xrightarrow{D}} W, \quad in \ \ H^0_\alpha. \tag{13}$$

Using Slutsky's theorem, it follows

$$n^{-1}W_n \xrightarrow{P} 0, \quad in \ \ H^0_\alpha. \tag{14}$$

On the other hand, by Lemma 1, we have

$$n^{1/2}(\hat{\rho} - \rho) = O_p(1).$$

Thus, we get

$$n^{1/2} \underset{1\le j\le p}{max} |\hat{\rho}_j - \rho_j| = O_p(1),$$

this implies

$$n^{1/2}\sum_{j=1}^{p} |\hat{\rho}_j - \rho_j| = O_p(1). \tag{15}$$

Finally, from (6), (14) and (15), we deduce that

$$\|\sigma^{-1}n^{-1/2}\widehat{V}_n - \sigma^{-1}n^{-1/2}W_n\|_\alpha \underset{n\to+\infty}{\xrightarrow{P}} 0.$$

this implies

$$\sigma^{-1}n^{-1/2}\widehat{V}_n \underset{n\to+\infty}{\xrightarrow{D}} W,$$

in H^0_α for all α such that $0 < \alpha < 1/2$. This achieves the proof of Theorem 3. □

5. Applications for epidemic change

In this section, we investigate some epidemic change in the innovations of the p-order autoregressive process.

We consider the same model as in section 2,

$$\begin{aligned} X_i &= \rho_1 X_{i-1} + \rho_2 X_{i-2} + ... + \rho_p X_{i-p} + \varepsilon_i \\ &= \sum_{j=1}^{p} \rho_j X_{i-j} + \varepsilon_i, \quad i = 1, 2, ..., n, \ , n \ge 1 \end{aligned}$$

where $\{X_i\}_{i=1}^n$ is a set of observations and the innovations $\varepsilon_1, ..., \varepsilon_n$ are independent identically distributed random variables with mean zero and finite variance $\sigma^2 = \mathbb{E}\varepsilon_1^2$.

We want to test the standard null hypothesis:
$(H_0) : \mathbb{E}\varepsilon_1 = \mathbb{E}\varepsilon_2 = ... = \mathbb{E}\varepsilon_n = \mu_0 = 0,$
against the epidemic alternative:
(H_A) : there exist $1 < k^* < m^* < n$ and some constant $\mu \ne \mu_0$ such that

$$\mathbb{E}\varepsilon_i = \mu \mathbf{1}_{I_n}(i), 1 \le i \le n,$$

where $I_n = k^* + 1, ..., k^* + l^*$ is the epidemic interval, $\mathbf{1}_{I_n}$ denotes its indicator function and $l^* = m^* - k^*$ the length of the epidemic. We assume that l^* go to infinity with n and $\frac{l^*}{n}$ tends to zero when n go to infinity.

To detect a short epidemic change in the mean of innovations of the p-order autoregressive process, we build the α-Hölderian uniform increments statistics $\widehat{T}(n,\alpha)$, $0<\alpha<1/2$ based on the residuals $\hat{\varepsilon}_i$.

$$\widehat{T}(n,\alpha)=\max_{1\le l\le n} l^{-\alpha}\max_{1\le k\le n-l}|\sum_{i=k+1}^{k+l}\hat{\varepsilon}_i-\frac{l}{n}\sum_{i=1}^{n}\hat{\varepsilon}_i|. \tag{16}$$

Let

$$T(\alpha)=\sup_{0\le h\le 1} h^{-\alpha}\sup_{0\le t\le 1-h}|B(t+h)-B(t)|, \tag{17}$$

where $B(t)=W(t)-tW(1)$, $t\in[0,1]$ is the Brownian bridge associated to Winner process W.

5.1. Convergence of $\widehat{T}(n,\alpha)$.

Proposition 4. *Let $0<\alpha<1/2$. Assume that $\lim_{t\to+\infty} t^{1/(1/2-\alpha)}P(|\varepsilon_1|>t)=0$ and all roots of equation $a(z)=\sum_{m=0}^{p}a_mz^m=0$, with $a_0=1$ are outside the unit disk, then under H_0 we have*

$$\sigma^{-1}n^{-1/2}\widehat{T}(n,\alpha)\xrightarrow[n\to+\infty]{D}T(\alpha). \tag{18}$$

The proof of this convergence is based on the following results.

Lemma 2 (Račkauskas, Suquet [21])**.** *Let $(\eta_n)_{n\ge1}$ be a tight sequence of random elements in the separable Banach space B and g_n, g be continuous functionals $B\to\mathbb{R}$. Assume that g_n converges pointwise to g on B and $(g_n)_n$ is equicontinuous. Then*

$$g_n(\eta_n)=g(\eta_n)+o_P(1).$$

Lemma 3 (Račkauskas, Suquet [21])**.** *Let $(B,\|.\|)$ be a normed space and $q:B\to\mathbb{R}$ such that*
(a) q is subadditive: $q(x+y)\le q(x)+q(y)$, $x,y\in B$,
(b) q is symmetric: $q(-x)=q(x)$, $x\in B$,
(c) For some constant C, $q(x)\le C\parallel x\parallel$, $x\in B$.
Then q satisfies the Lipschitz condition

$$|q(x+y)-q(x)|\le C\parallel y\parallel,\quad x,y\in B. \tag{19}$$

If $\mathcal{F}$ is any set of functionals q fulfilling (a), (b) end (c) with the same constant C, then (a), (b) and (c) are inherited by $g(x)=\sup\{q(x),\ q\in\mathcal{F}\}$, which therefore satisfies (19).

Proof. Proof of Proposition 4 Consider the functionals g_n, g, defined on $H_\alpha[0,1]$ by

$$g_n(x):=\max_{1\le i<j\le n} I(x,\frac{i}{n},\frac{j}{n}),\quad g(x)=:\sup_{0<s<t<1} I(x,s,t), \tag{20}$$

where

$$I(x,s,t)=\frac{|x(t)-x(s)-(t-s)x(1)|}{|t-s|^\alpha},\quad 0<t-s<1. \tag{21}$$

We have

$$n^\alpha\widehat{T}(n,\alpha)=n^\alpha\max_{1\le l\le n} l^{-\alpha}\max_{1\le k\le n-l}|S(k+l)-S(k)-\frac{l}{n}S(n)|,$$

with $S(t)=\sum_{i\le t}\hat{\varepsilon}_i$.

Clearly the functional $q = I(x,s,t)$ satisfies conditions (a) and (b) of Lemma 3. For condition (c) we have

$$\begin{aligned} q(x) = I(x,s,t) &= \frac{|x(t)-x(s)-(t-s)x(1)|}{|t-s|^\alpha} \\ &\le \frac{|x(t)-x(s)|}{|t-s|^\alpha} + \frac{1}{|t-s|^{\alpha-1}}|x(1)| \\ &\le 2\|x\|_\alpha. \end{aligned}$$

Then $\forall\ 0 \le s < t \le 1$, the functionals $q(x)$ satisfy the conditions (a), (b) and (c) of Lemma 3 with the same constant $C = 2$. It follows by Lemma 2 that

$$g_n = \max_{1\le i<j\le n} q(x,i,j,n) \text{ and } g = \sup_{0<s<t<1} q(x,s,t)$$

are Lipschitz on H_α^0 with the same constant C.

Observe that

$$\widehat{T}(n,\alpha) = g_n(\widehat{V}_n) \text{ and } T(\alpha) = g(W), \tag{22}$$

where $\widehat{V}_n(t)$, $t \in [0,1]$ is a polygonal line process built on residuals $\hat{\varepsilon}_i$

$$\widehat{V}_n(t) = \sum_{i=1}^{[nt]} \hat{\varepsilon}_i + (nt - [nt])\hat{\varepsilon}_{[nt]+1}, \quad t \in [0,1].$$

From Theorem 3, we have

$$\sigma^{-1}n^{-1/2}\widehat{V}_n \xrightarrow[n\to+\infty]{D} W. \tag{23}$$

Applying Lemma 2 with $\eta_n = \sigma^{-1}n^{-1/2}\widehat{V}_n$, we get

$$g_n(\sigma^{-1}n^{-1/2}\widehat{V}_n) = g(\sigma^{-1}n^{-1/2}\widehat{V}_n) + o_P(1). \tag{24}$$

Finally, the convergence of $\sigma^{-1}n^{-1/2}\widehat{T}(n,\alpha)$ to $T(\alpha)$ follows by (22), (23), (24) and continuous mapping theorem. □

5.2. Consistence of $\widehat{T}(n,\alpha)$.

Proposition 5. *Let $0 < \alpha < 1/2$. Under H_A and (8), we assume that*

$$\mathop{lim}_{t\to+\infty} t^{1/(1/2-\alpha)} P(|\varepsilon_1| > t) = 0,$$

and

$$\lim_{n\to+\infty} \sigma^{-1} n^{-1/2+\alpha}(l^*)^{(1-\alpha)} = +\infty.$$

Then

$$\sigma^{-1}n^{-1/2+\alpha}\widehat{T}(n,\alpha) \xrightarrow[n\to\infty]{P} +\infty. \tag{25}$$

Proof. We have

$$\widehat{T}(n,\alpha) = \max_{1\le l\le n} l^{-\alpha} \max_{1\le k\le n-l} \Big| \sum_{i=k}^{k+l} \hat{\varepsilon}_i - \frac{l}{n} \sum_{i=1}^{n} \hat{\varepsilon}_i \Big|.$$

Under H_A, we have

$$\begin{aligned}
\sum_{i=1}^{n}\hat{\varepsilon}_i &= \sum_{i=1}^{k-1}(\varepsilon_i - \sum_{j=1}^{p}(\hat{\rho}_j-\rho_j)b(\mathbf{B})\varepsilon_{i-j}) + \sum_{i=k}^{k+l}(\varepsilon_i - \mu - \sum_{j=1}^{p}(\hat{\rho}_j-\rho_j)b(\mathbf{B})(\varepsilon_{i-j}+\mu)) \\
&+ \sum_{i=k+l+1}^{n}(\varepsilon_i - \sum_{j=1}^{p}(\hat{\rho}_j-\rho_j)b(\mathbf{B})\varepsilon_{i-j}) \\
&= \sum_{i=1}^{k-1}(\varepsilon_i - \sum_{j=1}^{p}(\hat{\rho}_j-\rho_j)b(\mathbf{B})\varepsilon_{i-j}) + \sum_{i=k}^{k+l}(\varepsilon_i - \sum_{j=1}^{p}(\hat{\rho}_j-\rho_j)b(\mathbf{B})\varepsilon_{i-j}) \\
&+ \sum_{i=k+l+1}^{n}(\varepsilon_i - \sum_{j=1}^{p}(\hat{\rho}_j-\rho_j)b(\mathbf{B})\varepsilon_{i-j}) + l\mu(1-b(\mathbf{B})\sum_{j=1}^{p}(\hat{\rho}_j-\rho_j)) \\
&= \sum_{i=1}^{n}(\varepsilon_i - \sum_{j=1}^{p}(\hat{\rho}_j-\rho_j)b(\mathbf{B})\varepsilon_{i-j}) + l\mu(1-b(\mathbf{B})\sum_{j=1}^{p}(\hat{\rho}_j-\rho_j)),
\end{aligned}$$

and

$$\begin{aligned}
\sum_{i=k}^{k+l}\hat{\varepsilon}_i &= \sum_{i=k}^{k+l}(\varepsilon_i + \mu - \sum_{j=1}^{p}(\hat{\rho}_j-\rho_j)b(\mathbf{B})(\varepsilon_{i-j}+\mu)) \\
&= \sum_{i=k}^{k+l}(\varepsilon_i - \sum_{j=1}^{p}(\hat{\rho}_j-\rho_j)b(\mathbf{B})\varepsilon_{i-j}) + l\mu(1-b(\mathbf{B})\sum_{j=1}^{p}(\hat{\rho}_j-\rho_j)).
\end{aligned}$$

So, we obtain

$$\begin{aligned}
\widehat{T}(n,\alpha) &= \max_{1\le l\le n} l^{-\alpha}\max_{1\le k\le n-l}|\sum_{i=k}^{k+l}\hat{\varepsilon}_i - \frac{l}{n}\sum_{i=1}^{n}\hat{\varepsilon}_i| \\
&= \max_{1\le l\le n} l^{-\alpha}\max_{1\le k\le n-l}|\sum_{i=k}^{k+l}(\varepsilon_i - \sum_{j=1}^{p}(\hat{\rho}_j-\rho_j)b(\mathbf{B})\varepsilon_{i-j}) \\
&\quad + l\mu(1-b(\mathbf{B})\sum_{j=1}^{p}(\hat{\rho}_j-\rho_j)) \\
&\quad - \frac{l}{n}(\sum_{i=1}^{n}(\varepsilon_i - \sum_{j=1}^{p}(\hat{\rho}_j-\rho_j)b(\mathbf{B})\varepsilon_{i-j}) + l\mu(1-b(\mathbf{B})\sum_{j=1}^{p}(\hat{\rho}_j-\rho_j))| \\
&= \max_{1\le l\le n} l^{-\alpha}\max_{1\le k\le n-l}|l\mu(1-\frac{l}{n})(1-b(\mathbf{B})\sum_{j=1}^{p}(\hat{\rho}_j-\rho_j)) + \sum_{i=k}^{k+l}\hat{\varepsilon}_i - \frac{l}{n}\sum_{i=1}^{n}\hat{\varepsilon}_i| \\
&\ge \frac{1}{(l^*)^\alpha}(|l^*\mu(1-\frac{l^*}{n})(1-b(\mathbf{B})\sum_{j=1}^{p}(\hat{\rho}_j-\rho_j))| - |\sum_{I_n}\hat{\varepsilon}_i - \frac{l^*}{n}\sum_{i=1}^{n}\hat{\varepsilon}_i|).
\end{aligned}$$

Hence

$$\begin{aligned}
\sigma^{-1}n^{-1/2+\alpha}\widehat{T}(n,\alpha) \ge \sigma^{-1}n^{-1/2+\alpha}\frac{1}{(l^*)^\alpha}(|l^*\mu(1-\frac{l^*}{n})(1-b(\mathbf{B})\sum_{j=1}^{p}(\hat{\rho}_j-\rho_j))| \\
- |\sum_{I_n}\hat{\varepsilon}_i - \frac{l^*}{n}\sum_{i=1}^{n}\hat{\varepsilon}_i|).
\end{aligned}$$

By Theorem 3, we have

$$\sigma^{-1}n^{-1/2+\alpha}(\sum_{I_n}\hat{\varepsilon}_i - \frac{l^*}{n}\sum_{i=1}^{n}\hat{\varepsilon}_i) = O_p(1).$$

From (6), $||b(\mathbf{B})|| < \infty$ and by Lemme 1, $n^{1/2}\sum_{j=1}^{p}|\hat{\rho}_j - \rho_j| = O_p(1)$.
Thus, we get

$$\sigma^{-1}n^{-1/2+\alpha}\hat{T}(n,\alpha) \xrightarrow[n\to\infty]{P} +\infty \text{ when } \lim_{n\to+\infty}\sigma^{-1}n^{-1/2+\alpha}(l^*)^{(1-\alpha)} = +\infty.$$

□

6. Numerical results

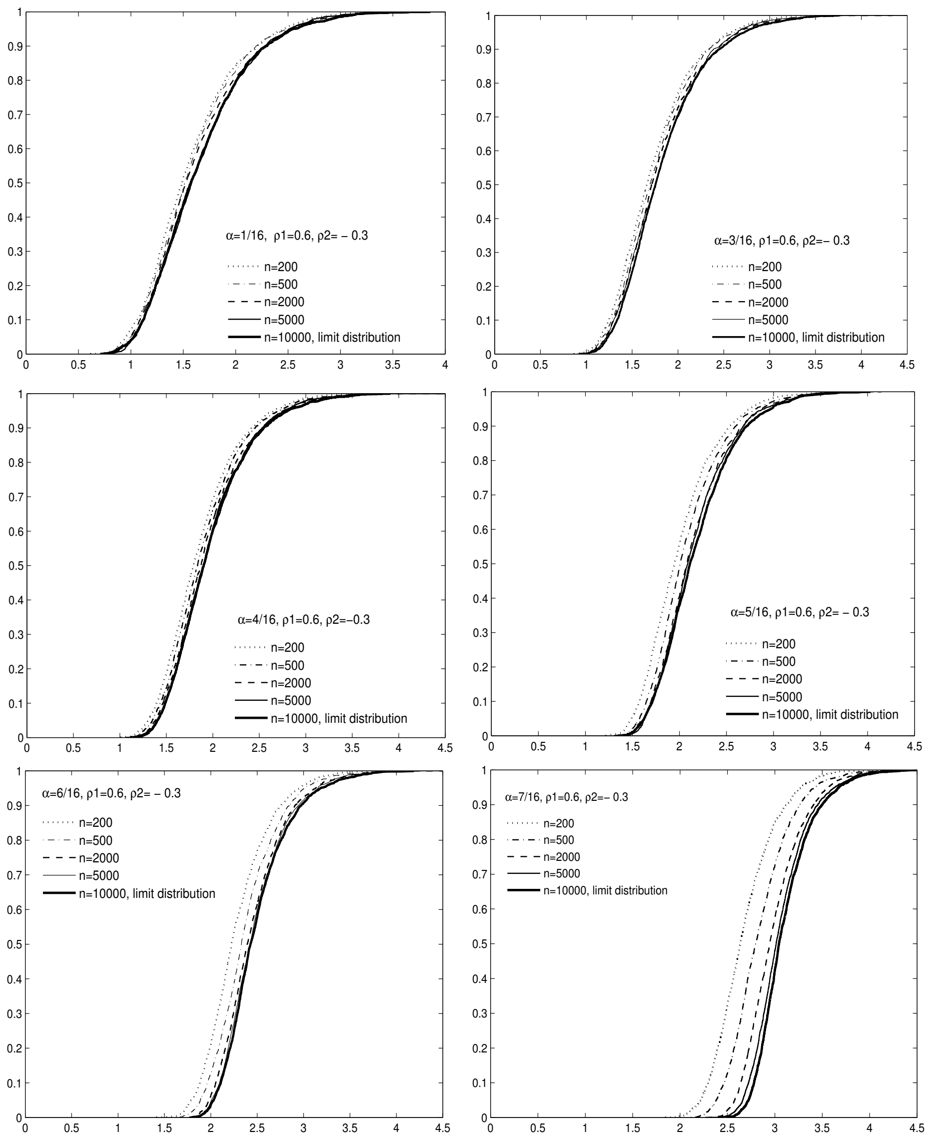

Figure 1. Empirical distribution function of $\|\sigma^{-1}n^{-1/2}\widehat{V}_n\|_\alpha$ for $\sigma = 1$ and different $\alpha = 1/16; 3/16; 1/4; 5/16; 6/16; 7/16$

In this section, we illustrate our theoretical results discussed above with some simulation results. First, to understand the behavior of the statistical test $\|\sigma^{-1}n^{-1/2}\widehat{V}_n\|_\alpha$, we investigate asymptotic approximations of the empirical cumulative distribution of this process for $\sigma = 1$ and different values of α. We present tabulated critical values of the limit random variable $T(\alpha)$. We analyze the performance of the test statistic $\|\sigma^{-1}n^{-1/2}\widehat{V}_n\|_\alpha$, under H_0 as well as under the alternative H_A. Since the epidemic change operates in the mean of ε_i, therefore, to study the behavior of $\|\sigma^{-1}n^{-1/2}\widehat{V}_n\|_\alpha$, we simulate 2000 realizations of this random variable constructed from residuals of an $AR(2)$ model with parameters $\rho_1 = 0.6$, $\rho_2 = -0.3$ for sample size $n = 200; 500; 2000; 5000; 10000$ respectively.

From Figure 1, we clearly see the convergence of the sampling distribution to the limiting distribution as n increases. Also, we can see that the distribution function of $\|n^{-1/2}\widehat{V}_n\|_\alpha$ depend on α, in fact, we note that when α is closer to 1/2 the convergence becomes slower.

To find an approximation of critical values for associated significance level c, we have chosen several α values and generated 5000 random values of the limit statistic $T(\alpha)$ and we took empirical quantiles of level $1 - c$ to approximate the unknown critical values for significance level c. Brownian bridge in each replication of $T(\alpha)$ is approximated by the partial sum process

$$\xi_n(t) = \frac{1}{\sqrt{n}}\left(\sum_{i=0}^{[nt]} y_i - t\sum_{i=0}^{n} y_i\right), \quad t \in [0,1], \quad \xi_n(0) = 0. \tag{26}$$

Here y_j, $j = 1, ..., n$, are independent standard normal random variables and $n = 5000$. The results are given in Table 1.

	$c = 0.1$	$c = 0.05$	$c = 0.025$	$c = 0.01$	$c = 0.005$
$T(\alpha = 1/16)$	1.6931	1.8188	1.9386	2.1099	2.1952
$T(\alpha = 2/16)$	1.7869	1.9188	2.0500	2.1979	2.3431
$T(\alpha = 3/16)$	1.9662	2.1057	2.2170	2.3715	2.4909
$T(\alpha = 4/16)$	2.1365	2.2729	2.4020	2.5768	2.6752
$T(\alpha = 5/16)$	2.3906	2.5401	2.6938	2.8470	2.9464
$T(\alpha = 6/16)$	2.7552	2.9093	3.0398	3.2174	3.3522
$T(\alpha = 7/16)$	3.3787	3.5290	3.6633	3.8114	3.9361

TABLE 1. Approximation of critical values of the limit statistic $T(\alpha)$

	$\sigma^{-1}n^{-\frac{1}{2}}\widehat{T}(n,\alpha)$		
n	$\alpha = 1/16$	$\alpha = 4/16$	$\alpha = 7/16$
$(cv = 1.8188)$	$(cv = 2.2745)$	$(cv = 3.5290)$	
10	0.7500	0.9893	1.3601
30	1.0444	1.3598	1.9824
50	1.1138	1.4677	2.1947
100	1.2014	1.5520	2.4250
500	1.2704	1.6712	2.7644
1000	1.2805	1.6712	2.8509
10000	1.3135	1.7635	2.9150

TABLE 2. Values of $\sigma^{-1}n^{-\frac{1}{2}}\widehat{T}(n,\alpha)$ under H_0 for level $c = 0.05$ and for different values of n and α

Table 2 gives the values of the statistic $\sigma^{-1}n^{-\frac{1}{2}}\widehat{T}(n,\alpha)$ for several values of n and α. cv is the critical value given by Table 1 for level $c = 0.05$. The critical region is defined by

$$W = \{(x_1, x_2, ..., x_n)\ /\sigma^{-1}n^{-\frac{1}{2}}\widehat{T}(n,\alpha) > cv\}.$$

So, if we calculate for different value of α, μ and $l^* = m^* - k^*$ for level $c = 0.05$, 1000 realizations of the statistic $\sigma^{-1}n^{-\frac{1}{2}}\widehat{T}(n,\alpha)$, with size $n = 500$ when (H_A) holds, we obtain the values presented in Table 3.

		$\sigma^{-1}n^{-\frac{1}{2}}\widehat{T}(n,\alpha)$		
l^*	μ	$\alpha = 1/16$ ($cv = 1.8188$)	$\alpha = 4/16$ ($cv = 2.2745$)	$\alpha = 7/16$ ($cv = 3.5290$)
	1	1.2810	1.7201	2.8128
	3	1.3705	2.0994	4.6851
5	4	1.4312	2.5127	-
	8	1.7494	-	-
	0.5	1.2879	1.6962	2.8014
	1.5	1.3869	2.0011	3.6522
10	2	1.4829	2.2891	-
	5	1.8752	-	-
20	0.5	1.3500	1.7889	2.8950
	1	1.5271	2.1736	3.6724
	1.5	1.6938	2.6777	-
	1.8	1.8219	-	-
50	0.5	1.6357	2.1694	3.3225
	0.6	1.7452	2.3679	3.5906
	0.7	1.8757	-	-

TABLE 3. Values of $\sigma^{-1}n^{-\frac{1}{2}}\widehat{T}(n,\alpha)$ under H_A for level $c = 0.05$

From Table 3, we observe that the statistic $\sigma^{-1}n^{-\frac{1}{2}}\widehat{T}(n,\alpha)$ detects short epidemic and when l^* increases and α goes to 1/2, it detects very quickly this epidemic.

In the other hand, to evaluate the performance of the statistic $\sigma^{-1}n^{-\frac{1}{2}}\widehat{T}(n,\alpha)$, we will estimate its power for different values of l^*, k^*, α and μ. To this end, we have simulated when (H_1) holds, 1000 realizations of this statistic of size n and we have computed the reject proportion of (H_0) (number of rejections of H_0/number of values). We define then the basic parameter set ($n = 1000$, $l^*/n = 0.05$, $k^*/n = 0.5$, $\mu = 0.8$), modifying the separate parameters, we obtain the test power presented in Table 4.

From Table 4, we see that the test power for all parameter values, is the lowest, when $\alpha = 0$ but it increases with α (i.e α is closer to 1/2). We also note that the test becomes more powerful when the number of observations n and the duration of the epidemic increase. However, the test power is not affected by the position of the epidemic and it does appear that it detects significantly even small changes in the mean as $\mu - \mu_0 = 0.6$.

For a more detailed illustration, we present the so called size-power curves on a correct size-adjusted (not nominal size) basis (see Davidson and MacKinnon [7]). For every parameters set (n, α, l^* et μ), we have computed replications of $\sigma^{-1}n^{-\frac{1}{2}}\widehat{T}(n,\alpha)$ for a number $R = 1000$ under (H_0) and a number $R = 3000$ under the alternative (H_A) and the corresponding p-value estimators noted $\hat{p}_j$

$$\hat{p}_j = \frac{1}{N}\sum_{k=1}^{N}\mathbf{1}\{L_k > T_j\},$$

Modified parameters	$\alpha = 1/16$	$\alpha = 4/16$	$\alpha = 7/16$
$\mu = 0.6$	0.274	0.543	0.682
$\mu = 0.8$	0.514	0.865	0.947
$\mu = 1$	0.808	0.996	1.000
$n = 500$	0.199	0.434	0.624
$n = 1000$	0.514	0.865	0.947
$n = 2000$	0.968	1.000	1.000
$l^*/n = 0.02$	0.096	0.128	0.389
$l^*/n = 0.05$	0.514	0.865	0.947
$l^*/n = 0.1$	0.996	0.998	1.000
$k^*/n = 0.3$	0.492	0.890	0.952
$k^*/n = 0.5$	0.514	0.865	0.947
$k^*/n = 0.7$	0.493	0.880	0.955

TABLE 4. Empirical power for the significance level $c = 0.05$

with L_k, $k = 1, ..., N$ ($N = 5000$), are the simulated values of the limit statistic $T(\alpha)$ and T_j, $j = 1, ..., R$, are the values of the statistic $\sigma^{-1} n^{-\frac{1}{2}} \widehat{T}(n, \alpha)$. After that, we display, on y-axis, the empirical distribution function for p-values under (H_A) (which is the empirical power function) and on x-axis, the empirical distribution function for p-values under (H_0) (instead of the nominal size c).

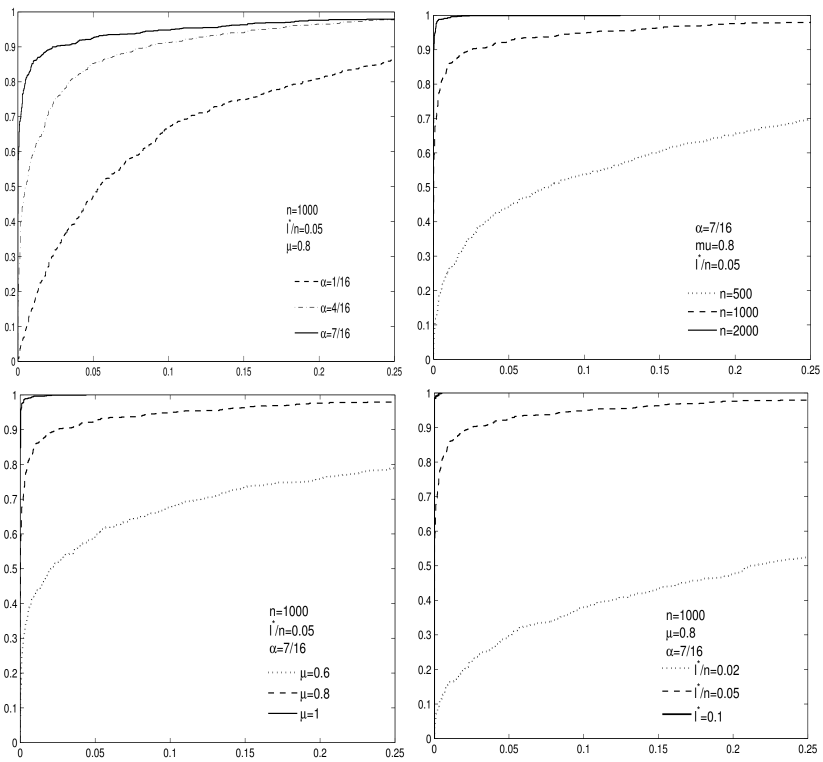

FIGURE 2. The adjusted size-power curve plots

We clearly see from Figure 2, how for true size values from $[0, 0.25]$, the test power increases quickly when α increases. Also, it appears that the test power increases with the length of epidemic l^* and the sample size n (with $\frac{l^*}{n}$ constant).

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