Theory of Stochastic Processes
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ON SOME RANDOM INTEGRAL OPERATORS GENERATED BY AN ARRATIA FLOW

We study some properties of a random integral operator in $L_2(\mathbb{R})$ whose kernel is generated by a stationary point process related to an Arratia flow. To prove that this random operator is not bounded we estimate the rate of growth of the maximal amount of clusters in Arratia flow on intervals of unit length.

1. Introduction

Let $\{x(u,s),\ u\in\mathbb{R},\ s\in[0;t]\}$ be an Arratia flow [1] on the interval $[0;t]$, $t>0$, and $\{y(u,s),\ u\in\mathbb{R},\ s\in[0;t]\}$ be its conjugated Arratia flow [9]. Define a random operator T_t in $L_2(\mathbb{R})$ which describes the shift of functions along $x(\cdot,t):\mathbb{R}\to\mathbb{R}$, i. e.

$$(T_tf)(\cdot)=f(x(\cdot,t)),\qquad f\in L_2(\mathbb{R}).$$

It is proved in [6] that the image under T_t of any boundedly supported function is a zero function with positive probability. However, for any $f\neq 0$ the function $T_t(f*p)$, where $p(u)=\frac{1}{\sqrt{2\pi}}e^{-\frac{u^2}{2}}$, is not zero almost surely. Moreover, by the formula of change of variables for an Arratia flow [11], the following equality holds

$$\|T_t(f*p)\|^2_{L_2(\mathbb{R})}=\int_{\mathbb{R}}\int_{\mathbb{R}}f(u)f(v)\sum_{\theta\in\Theta_t}\Delta y(\theta,t)p(u-\theta)p(v-\theta)dudv, \tag{1}$$

where Θ_t is a set of all points of discontinuity of the function $y(\cdot,t)\colon\mathbb{R}\to\mathbb{R}$, and $\Delta y(\theta,t)=y(\theta+,t)-y(\theta-,t)$.

The right-hand side of (1) is a quadratic form of a random integral operator K_t in $L_2(\mathbb{R})$ with the kernel

$$k_t(u,v)=\sum_{\theta\in\Theta_t}\Delta y(\theta,t)p(u-\theta)p(v-\theta). \tag{2}$$

In this paper we prove that K_t is a strong random operator [15] in $L_2(\mathbb{R})$, and is not a bounded one. However, for an orthogonal projector $Q_{a,b},\ a<b$, from $L_2(\mathbb{R})$ onto $L_2([a;b])$ random operators $K_tQ_{a,b},\ Q_{a,b}K_t$ are bounded. Moreover, with probability one $Q_{a,b}K_tQ_{a,b}$ is nuclear. In the article we estimate the rate of convergence of $\|Q_{-n;n}K_tQ_{-n;n}\|$ to infinity when $n\to\infty$.

2. Random integral operators and point processes

For a fixed $t>0$ let us define the set

$$\Theta_t=x(\mathbb{R},t).$$

Since $\{y(u,r),\ u\in\mathbb{R},\ r\in[0;t]\}$ is an Arratia flow on the interval $[0;t]$ in the backward time, and its trajectories don't cross trajectories of $\{x(u,r),\ u\in\mathbb{R},\ r\in[0;t]\}$, the set Θ_t consists of all points of discontinuity of the function $y(\cdot,t)\colon\mathbb{R}\to\mathbb{R}$.

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Thus, the stationarity of $\{x(u,r),\ u\in\mathbb{R},\ r\in[0;t]\}$ with respect to the spatial variable implies the stationarity of the point process Θ_t. Let us build a random measure on $\mathcal{B}(\mathbb{R})$ in the following way

$$\eta_t(B)=\sum_{\theta\in\Theta_t}\Delta y(\theta,t)\delta_\theta(B),\qquad B\in\mathcal{B}(\mathbb{R}).$$

Since Θ_t is stationary, η_t is a stationary random measure.

Lemma 2.1. *For any bounded set $B\in\mathcal{B}(\mathbb{R})$*

$$E\eta_t(B)=\lambda(B),$$

where λ is the Lebesgue measure on $\mathbb{R}$.

Proof. By the definition of η_t

$$E\eta_t(B)=E\sum_{\theta\in\Theta_t\cap B}\Delta y(\theta,t).$$

Using the formula of change of variables for an Arratia flow [11] one can check equalities

$$E\sum_{\theta\in\Theta_t\cap B}\Delta y(\theta,t)=E\int_{\mathbb{R}}1\!\!\mathrm{I}_B(x(u,t))du=E\|T_t1\!\!\mathrm{I}_B\|^2_{L_2(\mathbb{R})}.$$

It is proved in [2] that for any function $f\in L_2(\mathbb{R})$

$$E\|T_tf\|^2_{L_2(\mathbb{R})}=\|f\|^2_{L_2(\mathbb{R})}.$$

Hence,

$$E\|T_t1\!\!\mathrm{I}_B\|^2_{L_2(\mathbb{R})}=\lambda(B),$$

which proves the lemma. $\square$

Corollary 2.1. *An analog of Campbell's formula for a point process* [12] *holds*

$$E\sum_{\theta\in\Theta_t}\Delta y(\theta,t)h(\theta)=\int_{\mathbb{R}}h(u)du \tag{3}$$

for any non-negative function $h\in L_1(\mathbb{R})$.

Let us consider a random integral operator K_t in $L_2(\mathbb{R})$ mentioned in the introduction.

Theorem 2.1. *For any $t>0$ K_t is a strong random operator in $L_2(\mathbb{R})$.*

Proof. Let us show that for any $f\in L_2(\mathbb{R})$

$$E\int_{\mathbb{R}}(K_tf(u))^2du<+\infty.$$

Denote by $p_a(u)=\frac{1}{\sqrt{2\pi a}}e^{-\frac{u^2}{2a}},\ a>0$. Hence, the following relations hold

$$E\int_{\mathbb{R}}(K_tf(u))^2du=E\int_{\mathbb{R}}\int_{\mathbb{R}}p_2(u-v)(f*p)(u)(f*p)(v)\eta_t(du)\eta_t(dv)\le$$

$$\le\int_{\mathbb{R}}\int_{\mathbb{R}}|f(r_1)|\cdot|f(r_2)|E\int_{\mathbb{R}}\int_{\mathbb{R}}p_2(u-v)p(u-r_1)p(v-r_2)\eta_t(du)\eta_t(dv)dr_1dr_2.$$

One can check that

$$E\int_{\mathbb{R}}\int_{\mathbb{R}}p_2(u-v)p(u-r_1)p(v-r_2)\eta_t(du)\eta_t(dv)=$$

$$=E\sum_{k_1,k_2\in\mathbb{Z}}\sum_{\substack{\theta_1\in\Theta_t\cap[k_1;k_1+1)\\ \theta_2\in\Theta_t\cap[k_2;k_2+1)}}p_2(\theta_1-\theta_2)p(\theta_1-r_1)p(\theta_2-r_2)\Delta y(\theta_1,t)\Delta y(\theta_2,t)\le$$

$$\leq \sum_{k_1,k_2\in\mathbb{Z}} \max_{\substack{u\in[k_1;\,k_1+1)\\ v\in[k_2;\,k_2+1)}} p_2(u-v)p(u-r_1)p(v-r_2)\cdot E \sum_{\substack{\theta_1\in\Theta_t\cap[k_1;\,k_1+1)\\ \theta_2\in\Theta_t\cap[k_2;\,k_2+1)}} \Delta y(\theta_1,t)\Delta y(\theta_2,t).$$

Let $\{w(a,r),\ r\geq 0\}$ be a Wiener process, $w(a,0)=a$. Define

$$\tau=\inf\{\ s\geq 0\mid w(a,s)=0\ \},$$

and consider a new process $\widetilde{w}(a,r)=w(a,r\wedge\tau),\ r\geq 0$. Then

$$E \sum_{\substack{\theta_1\in\Theta_t\cap[k_1;\,k_1+1)\\ \theta_2\in\Theta_t\cap[k_2;\,k_2+1)}} \Delta y(\theta_1,t)\Delta y(\theta_2,t) \leq \left(E\Big(y(k_1+1,t)-y(k_1,t)\Big)^2\right)^{\frac{1}{2}}\times$$

$$\times\left(E\Big(y(k_2+1,t)-y(k_2,t)\Big)^2\right)^{\frac{1}{2}}=E\widetilde{w}(1,2t)^2.$$

Consequently,

$$E\int_{\mathbb{R}}\int_{\mathbb{R}} p_2(u-v)p(u-r_1)p(v-r_2)\eta_t(du)\eta_t(dv)\leq$$

$$\leq C_t\sum_{k_1,k_2\in\mathbb{Z}} \max_{\substack{u\in[k_1;\,k_1+1)\\ v\in[k_2;\,k_2+1)}} p_2(u-v)p(u-r_1)p(v-r_2), \tag{4}$$

where

$$C_t=E\widetilde{w}(1,2t)^2<+\infty.$$

It suffices to show that expression in the right-hand side (4) is a kernel, which generates a bounded integral operator in $L_2(\mathbb{R})$. Let us notice that the following inequality is true

$$\max_{\substack{u\in[k_1;\,k_1+1)\\ v\in[k_2;\,k_2+1)}} p_2(u-v)p(u-r_1)p(v-r_2)\leq[p(r_1-k_1)+p(r_1-k_1-1)]\times$$

$$\times[p(r_2-k_2)+p(r_2-k_2-1)]\times[2p_2(k_1-k_2)+p_2(k_1-k_2-1)+p_2(k_1+1-k_2)]\,. \tag{5}$$

For any $a,b\in\mathbb{R}$ there exists positive constant C such that

$$\sum_{k\in\mathbb{Z}} p(a-k)p(k-b)\leq Cp_2(a-b).$$

Thus, with some constants $\widetilde{C}_i>0,\ i=\overline{1,2}$, the following relations hold

$$\sum_{k_1,k_2\in\mathbb{Z}} p(a-k_1)p(b-k_2)p_2(k_1-k_2)=$$

$$=\widetilde{C}_1\sum_{k_1,k_2\in\mathbb{Z}} p(a-k_1)p(b-k_2)\int_{\mathbb{R}} p(k_1-r)p(r-k_2)dr=$$

$$=\widetilde{C}_1\int_{\mathbb{R}}\sum_{k_1\in\mathbb{Z}} p(a-k_1)p(k_1-r)\times\sum_{k_2\in\mathbb{Z}} p(b-k_2)p(k_2-r)dr\leq$$

$$\leq\widetilde{C}_2\int_{\mathbb{R}} p_2(a-r)p_2(b-r)dr=\widetilde{C}_2 p_4(a-b).$$

Hence, by (5), the right-hand side (4) is bounded by the kernel $p_4(r_1-r_2)$ with some constant. This kernel generates a bounded operator in $L_2(\mathbb{R})$. Consequently, there exists $\widetilde{C}_t>0$ such that for any function $f\in L_2(\mathbb{R})$

$$E\int_{\mathbb{R}}(K_t f(u))^2\,du\leq\widetilde{C}_t\int_{\mathbb{R}}\int_{\mathbb{R}}|f(r_1)|\cdot|f(r_2)|\cdot p_4(r_1-r_2)dr_2dr_2<+\infty,$$

which proves the statement. □

Lemma 2.2. *K_t is not a bounded random operator in $L_2(\mathbb{R})$.*

Proof. It suffices to show that

$$\sup_{n\in\mathbb{N}} \|K_t 1\!\!1_{[n;n+1]}\|_{L_2(\mathbb{R})} = +\infty \text{ a. s.}$$

One can check that there exists $b>0$ such that the following relations are true

$$\|K_t 1\!\!1_{[n;n+1]}\|^2_{L_2(\mathbb{R})} \geq \int_{\mathbb{R}} \left(\sum_{\theta\in\Theta_t\cap[n;n+1]} \Delta y(\theta,t) \int_n^{n+1} p(u-\theta)du p(v-\theta) \right)^2 dv \geq$$

$$\geq b \sum_{\theta\in\Theta_t\cap[n;n+1]} (\Delta y(\theta,t))^2 . \tag{6}$$

Let us consider random variables

$$\zeta_n = \sum_{\theta\in\Theta_t\cap[n;n+1]} (\Delta y(\theta,t))^2 , \qquad n\in\mathbb{N}.$$

One can check that

$$\sup_{n\in\mathbb{N}} \zeta_n = +\infty \quad \text{ a. s.} \tag{7}$$

Really, since the trajectories of $\{x(u,r),\ u\in\mathbb{R},\ r\in[0;t]\}$ and $\{y(u,r),\ u\in\mathbb{R},\ r\in[0;t]\}$ don't cross, for any $m\in\mathbb{N}$ and uniform partition $\{u_j,\ j=\overline{1,2M+2}\}$ of $[-M;M]$

$$\mathbb{P}\{\zeta_1\geq M\} = \mathbb{P}\left\{ \sum_{\theta\in\Theta_t\cap[1;2]} (\Delta y(\theta,t))^2 \geq M \right\} \geq$$
$$\geq \mathbb{P}\{\, 1\leq x(u_1,t) = x(u_2,t) < x(u_3,t) = x(u_4,t) < \ldots$$
$$\ldots < x(u_{2M+1},t) = x(u_{2M+2},t) \leq 2\} > 0.$$

Thus, $\operatorname{essup}\zeta_1 = +\infty$, which, by stationarity and mixing property [10] of the sequence $\{\zeta_n\}_{n\in\mathbb{N}}$, proves (7). Hence,

$$\sup_{n\in\mathbb{N}} \|K_t 1\!\!1_{[n;n+1]}\|^2_{L_2(\mathbb{R})} \geq b \sup_{n\in\mathbb{N}} \zeta_n = +\infty \qquad \text{a. s.}$$

□

For a fixed $a<b$ consider an orthogonal projector $Q_{a,b}$ of $L_2(\mathbb{R})$ onto $L_2([a;b])$, which we identify with the subspace of $L_2(\mathbb{R})$ of functions supported on $[a;b]$.

Lemma 2.3. *Random operators $Q_{a,b}K_t$ and $K_tQ_{a,b}$ are bounded in $L_2(\mathbb{R})$.*

Proof. One can check, by Hölder inequality, that for any $f,g\in L_2(\mathbb{R})$

$$(K_tQ_{a,b}f,g)_{L_2(\mathbb{R})} \leq (b-a)^{\frac{1}{2}} \|f\|_{L_2(\mathbb{R})}\|g\|_{L_2(\mathbb{R})} \sum_{\theta\in\Theta_t} \Delta y(\theta,t) \max_{u\in[a;b]} p(u-\theta). \tag{8}$$

Let us notice that the same upper estimation of $(Q_{a,b}K_tf,g)_{L_2(\mathbb{R})}$ is true.

Due to (3) the following equality holds

$$E\sum_{\theta\in\Theta_t} \Delta y(\theta,t) \max_{u\in[a;b]} p(u-\theta) = \int_{\mathbb{R}} \max_{u\in[a;b]} p(u-v)dv < +\infty.$$

Lemma is proved. □

Lemma 2.4. *For any $a<b$ with probability one the random operator $Q_{a,b}K_tQ_{a,b}$ is nuclear in $L_2(\mathbb{R})$.*

Proof. For a fixed $\theta \in \Theta_t$ denote by B_θ an integral operator in $L_2(\mathbb{R})$ with the kernel

$$h_\theta(u,v) = p(u-\theta)p(v-\theta)1\!\!1_{[a;b]}(u)1\!\!1_{[a;b]}(v).$$

The nuclear norm of B_θ equals to $\|p(\cdot-\theta)1\!\!1_{[a;b]}(\cdot)\|^2_{L_2(\mathbb{R})}$. Since

$$Q_{a,b}K_tQ_{a,b} = \sum_{\theta\in\Theta_t} \Delta y(\theta,t)B_\theta,$$

the nuclear norm of $Q_{a,b}K_tQ_{a,b}$ is the value

$$\sum_{\theta\in\Theta_t} \Delta y(\theta,t)\|p(\cdot-\theta)1\!\!1_{[a;b]}(\cdot)\|^2_{L_2(\mathbb{R})}. \tag{9}$$

By (3), the following relations are true

$$E\sum_{\theta\in\Theta_t} \Delta y(\theta,t)\|p(\cdot-\theta)1\!\!1_{[a;b]}(\cdot)\|^2_{L_2(\mathbb{R})} = \int_{\mathbb{R}} \|p(\cdot-u)1\!\!1_{[a;b]}(\cdot)\|^2_{L_2(\mathbb{R})}du =$$

$$= \int_{\mathbb{R}}\int_a^b p^2(u-v)dudv < +\infty.$$

Consequently, the value (9) is finite with probability one, which proves the lemma. □

Since K_t is not a bounded random operator, the operator norm of $Q_{-n,n}K_tQ_{-n,n}$ tends to infinity when $n\to\infty$. Let us notice, that the rate of growth of $\|Q_{-n,n}\widetilde{K}Q_{-n,n}\|$ was investigated in [7], where $\widetilde{K}$ is a random integral operator in $L_2(\mathbb{R})$ with the kernel generated by a stationary point process Θ

$$k(u,v) = \sum_{\theta\in\Theta} p(u-\theta)p(v-\theta).$$

It was proved in [7] that

$$\|Q_{-n,n}\widetilde{K}Q_{-n,n}\|^2 \ge \max_{k=\overline{0,n}} |\Theta\cap[k;k+1]|. \tag{10}$$

In the next section, in the case of the point process Θ_t, we estimate the rate of growth not only of the value in the right-hand side (10), but also of

$$\max_{k=\overline{0,n}} \sum_{\theta\in\Theta_t\cap[k;k+1]} (\Delta y(\theta,t))^2.$$

Using this one can estimate the rate of growth of $\|Q_{-n,n}K_tQ_{-n,n}\|$ to infinity, because

$$\|Q_{-n,n}K_tQ_{-n,n}\|^2 \ge \max_{k=\overline{0,n}} \sum_{\theta\in\Theta_t\cap[k;k+1]} (\Delta y(\theta,t))^2. \tag{11}$$

3. Rate of growth of $\|Q_{-n,n}K_tQ_{-n,n}\|$ to infinity

For any $n\in\mathbb{N}\cup\{0\}$ let us consider a random variable

$$\xi_n = |x([n;n+1],t)|,$$

where $|A|$ is a cardinality of the set A. Since Arratia flow is stationary with respect to a spatial variable, $\{\xi_n\}_{n\in\mathbb{N}\cup\{0\}}$ is a stationary sequence.

Theorem 3.1. *There exist positive C_0 and R such that for any $C\ge C_0$*

$$\frac{1}{C^4}\ln\mathbb{P}\{\xi_0\ge C\} \ge -Rt. \tag{12}$$

Proof. Consider an integer $C > 0$. Denote by $\{u_k,\ k = \overline{0, C+1}\}$ the uniform partition of $[0;1]$. Thus,

$$\mathbb{P}\{\xi_0 \geq C\} = \mathbb{P}\{|x([0;1],t)| \geq C\} \geq$$

$$\geq \mathbb{P}\{x(u_0,t) < x(u_1,t) < \ldots < x(u_{C+1},t)\}. \tag{13}$$

Consider a Wiener process $\vec{w}(\vec{u},\cdot)$ in $\mathbb{R}^{C+2}$ which starts from the point

$$\vec{u} = (u_0, u_1, \ldots, u_{C+1}).$$

Denote by

$$\Delta_{C+2} = \left\{\vec{v} \in \mathbb{R}^{C+2} |\ v_0 \leq v_1 \leq \ldots \leq v_{C+1}\right\}.$$

Let us notice that

$$\mathbb{P}\{x(u_0,t) < x(u_1,t) < \ldots < x(u_{C+1},t)\} =$$

$$= \mathbb{P}\{\text{ for any } \ s \leq t \ \ \vec{w}(\vec{u},s) \notin \partial\Delta_{C+2}\}. \tag{14}$$

Since $\frac{1}{\sqrt{2}(C+1)}$ is a distance from the point $\vec{u}$ to any

$$H_i = \{\vec{v} \in \mathbb{R}^{C+2} |\ v_i = v_{i+1}\}, \quad i = \overline{0, C},$$

the probability in the right-hand side of (14) is not less than

$$\mathbb{P}\left\{ \text{ for any } \ s \leq t \ \ \|\vec{w}(\vec{0},s)\|_{\mathbb{R}^{C+2}} < \frac{1}{\sqrt{2}(C+1)} \right\},$$

where $\vec{w}(\vec{0},\cdot)$ is a Wiener process in $\mathbb{R}^{C+2}$, $\vec{w}(\vec{0},0) = \vec{0}$. Hence,

$$\mathbb{P}\{x(u_0,t) < x(u_1,t) < \ldots < x(u_{C+1},t)\} \geq$$

$$\geq \mathbb{P}\left\{ \sup_{s\in[0;t]} \|\vec{w}(\vec{0},s)\|_{\mathbb{R}^{C+2}} \leq \frac{1}{2(C+1)} \right\},$$

and

$$\mathbb{P}\left\{ \sup_{s\in[0;t]} \|\vec{w}(\vec{0},s)\|_{\mathbb{R}^{C+2}} \leq \frac{1}{2(C+1)} \right\} \geq$$

$$\geq \mathbb{P}\left\{ \text{ for any } \ j = \overline{0, C+1} \ \sup_{s\in[0;t]} |w_j(0,s)| \leq \frac{1}{2(C+2)^{\frac{3}{2}}} \right\},$$

where $\{w_j(0,\cdot)\}_{j=\overline{0,C+1}}$ are independent Wiener processes in $\mathbb{R}$, $w_j(0,0) = 0$. Consequently,

$$\mathbb{P}\left\{ \sup_{s\in[0;t]} \|\vec{w}(\vec{0},s)\|_{\mathbb{R}^{C+2}} \leq \frac{1}{2(C+1)} \right\} \geq$$

$$\geq \left(\mathbb{P}\left\{ \sup_{s\in[0;t]} |w_0(0,s)| \leq \frac{1}{2(C+2)^{\frac{3}{2}}} \right\} \right)^{C+2}.$$

Due to [13] ((3), p.261),

$$\mathbb{P}\left\{ \sup_{s\in[0;t]} |w_0(0,s)| \leq \frac{1}{2(C+2)^{\frac{3}{2}}} \right\} \sim \frac{4}{\pi} e^{-\frac{\pi^2}{8}4t(C+2)^3}, \ C \to \infty.$$

Thus, there exists $C_0 > 0$ such that for any $C \geq C_0$

$$\mathbb{P}\left\{ \sup_{s\in[0;t]} |w_0(0,s)| \leq \frac{1}{2(C+2)^{\frac{3}{2}}} \right\} \geq \frac{2}{\pi} e^{-\frac{\pi^2}{2}t(C+2)^3}.$$

Hence, there exists $R > 0$ such that for arbitrary $C \ge C_0$

$$\frac{1}{C^4} \ln \mathbb{P}\{|x([0;1],t)| \ge C\} \ge \frac{C+2}{C^4}\left(\ln\frac{2}{\pi} + \ln e^{-\frac{\pi^2}{2}t(C+2)^3}\right) \ge -Rt,$$

which proves the theorem. □

Using Theorem 3.1 one can prove that with probability one $\max_{k=\overline{0,n}} \xi_n$ tends to infinity when $n \to \infty$.

Theorem 3.2. *For any $\beta \in (0; \frac{1}{4})$ almost surely*

$$\frac{\max\limits_{k=\overline{0,n}} \xi_k}{(\ln n)^{\frac{1}{4}-\beta}} \to +\infty,\ n \to \infty. \tag{15}$$

Proof. For an integer $n \ge 3$ define $N_n = \left[\frac{n}{[\sqrt{8\ln n}]}\right]$. For any $j = \overline{0, N_n}$ denote by

$$k_j^n = j \cdot [\sqrt{8\ln n}].$$

To prove (15) it suffices to show that with probability one

$$\frac{\max\limits_{j=\overline{0,N_n}} \xi_{k_j^n}}{(\ln n)^{\frac{1}{4}-\beta}} \to +\infty,\ n \to \infty. \tag{16}$$

Due to Borel-Cantelli lemma, to prove (16) it is enough to check a convergence of the series

$$\sum_{n=3}^{\infty} \mathbb{P}\left\{\max_{j=\overline{0,N_n}} \xi_{k_j^n} \le C(\ln n)^{\frac{1}{4}-\beta}\right\} \tag{17}$$

for any $C > 0$. Denote by

$$\alpha(h) = \sup\{|\mathbb{P}(B \cap D) - \mathbb{P}(B)\mathbb{P}(D)|,$$
$$B \in \mathcal{F}_{-\infty}^{u},\ D \in \mathcal{F}_{u+h}^{+\infty},\ u \in \mathbb{R}\},$$

where $\mathcal{F}_u^v = \sigma\{x(w,\cdot),\ w \in [u;v]\}$, $h > 0$. It was proved in [10] (Lemma 4.2) that for any $h > 0$

$$\alpha(h) \le 2\sqrt{\frac{2}{\pi}} \int_h^{+\infty} e^{-\frac{v^2}{2}} dv. \tag{18}$$

The process $\{x(u,t),\ u \in \mathbb{R}\}$ is stationary. Thus, for any $j = \overline{0, N_n}$

$$\mathbb{P}\left\{\xi_{k_j^n} \le C(\ln n)^{\frac{1}{4}-\beta}\right\} = \mathbb{P}\left\{\xi_0 \le C(\ln n)^{\frac{1}{4}-\beta}\right\},$$

which implies the following relations

$$\mathbb{P}\left\{\max_{j=\overline{0,N_n}} \xi_{k_j^n} \le C(\ln n)^{\frac{1}{4}-\beta}\right\} \le$$

$$\le \alpha([\sqrt{8\ln n}]) \sum_{j=0}^{N_n-1} \mathbb{P}\left\{\xi_0 \le C(\ln n)^{\frac{1}{4}-\beta}\right\}^j + \mathbb{P}\left\{\xi_0 \le C(\ln n)^{\frac{1}{4}-\beta}\right\}^{N_n}.$$

To show a convergence of the series (17) it suffices to check that series

$$\sum_{n=3}^{\infty} \alpha([\sqrt{8\ln n}]) \sum_{j=0}^{N_n-1} \mathbb{P}\left\{\xi_0 \le C(\ln n)^{\frac{1}{4}-\beta}\right\}^j, \tag{19}$$

$$\sum_{n=3}^{\infty} \mathbb{P}\left\{\xi_0 \le C(\ln n)^{\frac{1}{4}-\beta}\right\}^{N_n} \tag{20}$$

converge. For any $h>0$, by [14] (Lemma 12.9, p.349), the inequality is true

$$\frac{1}{\sqrt{2\pi}}\int_h^{+\infty} e^{-\frac{v^2}{2}}dv \le \frac{1}{h\sqrt{2\pi}}e^{-\frac{h^2}{2}}. \tag{21}$$

Due to (21), the following relations hold

$$\sum_{n=3}^{\infty}\alpha([\sqrt{8\ln n}])\sum_{j=0}^{N_n-1}\mathbb{P}\left\{\xi_0\le C(\ln n)^{\frac14-\beta}\right\}^j \le$$

$$\le \sum_{n=3}^{\infty}\alpha([\sqrt{8\ln n}])N_n \le 4\sum_{n=3}^{\infty}\frac{n}{[\sqrt{8\ln n}]}\cdot\frac{1}{[\sqrt{8\ln n}]\sqrt{2\pi}}e^{-\frac{[\sqrt{8\ln n}]^2}{2}} < +\infty.$$

Hence, the series (19) converges.

By Theorem 3.1, for any $n\ge n_1$ the following inequalities hold

$$\frac{n}{\sqrt{8\ln n}}\mathbb{P}\left\{\xi_0 > C\left(\ln n\right)^{\frac14-\beta}\right\} \ge$$

$$\ge \frac{n}{\sqrt{8\ln n}}e^{-\frac{R\pi^2}{2}t\left(C(\ln n)^{\frac14-\beta}\right)^4} \ge \frac{n^{\frac12}}{(8\ln n)^{\frac12}}. \tag{22}$$

Consequently, by (22), the series (20) converges. □

In the next we estimate a rate of growth $\|Q_{-n,n}K_tQ_{-n,n}\|$ to infinity, when $n\to\infty$. To do this, by (11), we consider a random variable

$$\zeta_k = \sum_{\theta\in\Theta_t\cap[k;k+1]}(\Delta y(\theta,t))^2$$

for $k\in\mathbb{N}\cup\{0\}$, and prove that with probability one $\max_{k=\overline{0,n}}\zeta_k$ tends to infinity when $n\to\infty$.

Theorem 3.3. *With probability one*

$$\frac{\ln\ln n}{\ln n}\cdot\max_{k=\overline{0,n}}\zeta_k\to+\infty,\ n\to\infty. \tag{23}$$

Proof. For a natural number $n\ge 3$ consider values N_n and k_j^n, $j=\overline{0,N_n}$, which are defined in the proof of Theorem 3.2. Let us show that almost surely

$$\frac{\ln\ln n}{\ln n}\cdot\max_{j=\overline{0,N_n}}\zeta_{k_j^n}\to+\infty,\ n\to\infty. \tag{24}$$

By Borel-Cantelli lemma, it is sufficient to prove that

$$\sum_{n=3}^{\infty}\mathbb{P}\left\{\max_{j=\overline{0,N_n}}\zeta_{k_j^n}\le C\frac{\ln n}{\ln\ln n}\right\}<+\infty \tag{25}$$

for any $C>0$. Since the process $\{y(u,t),\ u\in\mathbb{R}\}$ is stationary,

$$\mathbb{P}\left\{\zeta_{k_j^n}\le C\frac{\ln n}{\ln\ln n}\right\}=\mathbb{P}\left\{\zeta_0\le C\frac{\ln n}{\ln\ln n}\right\}.$$

As in the proof of Theorem 3.2, the following inequality holds

$$\mathbb{P}\left\{\max_{j=\overline{0,N_n}}\zeta_{k_j^n}\le C\frac{\ln n}{\ln\ln n}\right\}\le$$

$$\le\alpha([\sqrt{8\ln n}])\sum_{j=0}^{N_n-1}\mathbb{P}\left\{\zeta_0\le C\frac{\ln n}{\ln\ln n}\right\}^j+\mathbb{P}\left\{\zeta_0\le C\frac{\ln n}{\ln\ln n}\right\}^{N_n}.$$

Due to (18), the series $\sum_{n=3}^{\infty} \alpha([\sqrt{8\ln n}])N_n$ converges. Thus,

$$\sum_{n=3}^{\infty} \alpha([\sqrt{8\ln n}]) \sum_{j=0}^{N_n-1} \mathbb{P}\left\{\zeta_0 \le C\frac{\ln n}{\ln\ln n}\right\}^j < +\infty.$$

Let us check that

$$\sum_{n=3}^{\infty} \mathbb{P}\left\{\zeta_0 \le C\frac{\ln n}{\ln\ln n}\right\}^{N_n} < +\infty. \tag{26}$$

To do this we will use the following theorem.

Theorem 3.4. *For any $t > 0$ there exists $a_t > 0$ such that for any $C \ge t$*

$$\mathbb{P}\{\zeta_0 \ge C\} \ge a_t \frac{1}{\sqrt{C}} e^{-\frac{C}{2t}}.$$

Proof. Since the trajectories of conjugated flows

$$\{y(u,r),\ u \in \mathbb{R},\ r \in [0;t]\} \text{ and } \{x(u,r),\ u \in \mathbb{R},\ r \in [0;t]\}$$

don't cross, for any $C \ge 0$

$$\mathbb{P}\{\zeta_0 \ge C\} = \mathbb{P}\left\{\sum_{\theta\in\Theta_t\cap[0;1]} (\Delta y(\theta,t))^2 \ge C\right\} \ge$$

$$\ge \mathbb{P}\left\{x(0,t) = x(\sqrt{C},t),\ x(0;t) \in [0;1]\right\}. \tag{27}$$

Let $\{w_1(s),\ s \ge 0\}$, $\{w_2(s),\ s \ge 0\}$ be independent Wiener processes, and $w_1(0) = w_2(0) = 0$. Denote by

$$\tau = \inf\left\{r \ge 0 :\ w_1(r) = \sqrt{C} + w_2(r)\right\}.$$

Then, to estimate (27) one can notice

$$\mathbb{P}\left\{x(0,t) = x(\sqrt{C},t),\ x(0;t) \in [0;1]\right\} = \mathbb{P}\{\tau \le t,\ w_1(t) \in [0;1]\} \ge$$

$$\ge \mathbb{P}\left\{\sqrt{C} + w_2(t) \le 0,\ w_1(t) \in [0;1]\right\} = \int_0^1 \frac{1}{\sqrt{2\pi t}} e^{-\frac{u^2}{2t}} du \cdot \int_{\sqrt{C}}^{+\infty} \frac{1}{\sqrt{2\pi t}} e^{-\frac{u^2}{2t}} du.$$

For any $h > 0$, due to [14] (Lemma 12.9, p.349), the inequality is true

$$\frac{1}{\sqrt{2\pi}} \int_h^{+\infty} e^{-\frac{v^2}{2}} dv \ge \frac{h}{h^2+1} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{h^2}{2}}.$$

Hence, for all $C \ge t$ the following estimation holds

$$\int_0^1 \frac{1}{\sqrt{2\pi t}} e^{-\frac{u^2}{2t}} du \cdot \int_{\sqrt{C}}^{+\infty} \frac{1}{\sqrt{2\pi t}} e^{-\frac{u^2}{2t}} du \ge a_t \frac{1}{\sqrt{C}} e^{-\frac{C}{2t}},$$

where

$$a_t = \frac{1}{8\pi} \int_0^1 e^{-\frac{u^2}{2t}} du.$$

The theorem is proved.

□

Due to Theorem 3.4 relations hold

$$N_n\mathbb{P}\left\{\zeta_0 \geq C\frac{\ln n}{\ln\ln n}\right\} = \frac{n}{[\sqrt{8\ln n}]}\mathbb{P}\left\{\zeta_0 \geq C\frac{\ln n}{\ln\ln n}\right\} \geq$$

$$\geq \frac{a_t}{\sqrt{8C}} \cdot \frac{n\sqrt{\ln\ln n}}{\ln n} e^{-\frac{C\ln n}{2t\ln\ln n}}. \tag{28}$$

Hence, by (28), (26) is true. Consequently, for any $C > 0$ the series (25) converges, which proves (24). Thus, (23) holds. □

Due to Theorem 3.4 and inequality (11) the following estimation holds

$$\frac{\ln\ln n}{\ln n} \cdot \left\|Q_{[-n;n]}K_t Q_{[-n;n]}\right\|^2 \to \infty, \quad n \to \infty \qquad \text{a. s.} \tag{29}$$

Let us notice that in the case of a random operator $\widetilde{K}_t$, which is generated by a stationary Poisson point process with intensity one, the same estimation of $\|Q_{[-n;n]}\widetilde{K}_t Q_{[-n;n]}\|^2$ as in (29) was proved in [7]. The rate of growth of this value is different in the case $\widetilde{K}_t$ generated by Θ_t. Really, by Theorem 3.2, for any $\beta \in (0; \frac{1}{4})$

$$\frac{1}{(\ln n)^{\frac{1}{4}-\beta}} \cdot \|Q_{[-n;n]}\widetilde{K}_t Q_{[-n;n]}\|^2 \to \infty, \quad n \to \infty \qquad \text{a. s.}$$

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