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**ON CONVERGENCE RATE FOR ERLANG–SEVASTYANOV TYPE
MODELS WITH INFINITELY MANY SERVERS**

IN MEMORY AND TO THE 90TH ANNIVERSARY OF A.D. SOLOVYEV

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Polynomial convergence rate to stationarity is shown for extended Erlang – Sevastyanov’s model with variable intensities of service and arrivals.

1. INTRODUCTION

Consider a service system or a process with countably many servers and one incoming flow of “events” or customers “of the same type” such that the incoming flow has intensity $\lambda(X)$, which depends on the number n of the customers in the system, and some variables $(x^0, x^1, \dots, x^n) \in R^{n+1}$; we will use notations $X = (n, x^0, x^1, \dots, x^n)$ and $x = (x^0, x^1, \dots, x^n)$, where $n \in Z_+$. Then, a bit non-rigorously, existence of intensity $\lambda(X_t)$ means that

$$P(\text{one new customer arrives on } (t, t + \Delta) \mid X_t) = \lambda(X_t)\Delta + o(\Delta),$$

as $\Delta \downarrow 0$; actually, for *continuous* intensities this definition is strict but continuity will not be assumed. Here $X_t = (n_t; x_t^0; x_t^1, \dots, x_t^{n_t})$ where n_t signifies the number of customers in the system at time t , each x_t^i , $1 \leq i \leq n_t$, stands for the elapsed service time of the corresponding customer at time t , while x_t^0 signifies the time from the most recent arrival. The latter makes sense mainly for states where $n_t = 0$; however, for simplicity of presentation we keep the same notations for all states; just for $n \geq 1$, x^0 must coincide with one of the other x^i values (namely, with the minimal one). Every customer with the elapsed time of service x^i is being served by the corresponding server also with some intensity $h(X^i)$; respectively, the vector $(x_t^1, \dots, x_t^{n_t})$ consists of all such elapsed times of all different customers currently present in the system; the meaning of x_t^0 was explained earlier.

In such a state space the process X_t is definitely Markov; however, for discontinuous intensities $\lambda(\cdot)$ a justification of existence of the process is needed, which issue was discussed, e.g., in [5] and [26]. In particular, in [5] it has been proved that the process defined in such a way is well-defined and is strong Markov, which will be useful in the sequel.

The problem addressed in this paper is convergence rate to the stationary regime under appropriate conditions. Recently such convergence rate was studied for the case where λ may only depend on n – the number of customers on service – but not on other continuous variables. Here we address a more general case. The price for this generality is that the explicit formulae for the stationary distribution due to Fortet and Sevastyanov are not valid any more (but, of course, modelling is available).

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Recall for completeness that Erlang's formulae for the stationary regime in the case of exponentially distributed service time (the system $M/M/\infty$) are known since the article [7] ("Erlang formulae **B**" in the sequel literature),

$$(1) \quad p_n = p_0 \frac{\lambda^n}{\mu^n}, \quad p_0 = 1 - \frac{\lambda}{\mu}.$$

where μ^{-1} is expectation of the service time, under the assumption of the convergence of the series $(\sum_n p_n)$, i.e., for $\lambda < \mu$. This was extended under appropriate assumptions (see, e.g., [11, Ch.4, §4-5]) to the case where $\lambda = \lambda_n$ may depend on the current number of customers n currently in the system,

$$(2) \quad p_n = P(n \text{ customers in the system}) = p_0 \frac{\prod_{i=0}^{n-1} \lambda_i}{\mu^n},$$

where

$$p_0 = \left(\sum_{n=0}^{\infty} \frac{\prod_{i=0}^{n-1} \lambda_i}{\mu^n} \right)^{-1},$$

and $\prod_{i=1}^0 \lambda_i \equiv 1$, and to a more general situation where *intensity* μ may also depend on n (in which case μ^n in (2) should be replaced by $\prod_{i=1}^n \mu_i$). Similar formulae also hold true for finitely many servers in the system under the condition that the customers arriving while all servers are busy become lost.

Fortet [8] for a finite number of servers case gave the density of a stationary distribution for a general – *non-exponential* – service time distribution possessing a density $g(x) = G'(x)$ where G is the distribution function of service time with a finite mean value $\mu^{-1} := \int_0^{\infty} t dG(t) \equiv \int_0^{\infty} (1 - G(t)) dt$:

$$(3) \quad p(n; x) = p(n; x^1, \dots, x^n) = p_0 \prod_{i=0}^{n-1} \lambda_i (1 - G(x^i)), \quad n \geq 1,$$

where p_0 is the normalizing constant; in the case of infinite number of servers it is given by the same formula as in (2). Sevastyanov [17] extended and strengthened this result to the case without assumption on the existence of a density g and for the first time has proved *convergence in total variation* of the non-stationary system to its stationary regime using his version of the ergodic theorem for Markov processes with general state spaces. This was a breakthrough and by this reason the author's view is that the name "Erlang–Sevastyanov systems" is appropriate here. In [17] the number of servers was assumed finite and waiting in a queue unavailable, as in [8]. Further, there were several extensions of this result to the case of infinitely many servers [13, 15, 19, 21]. In all these cases starting from [8], in particular situations with a constant intensity μ the explicit formula (2) holds true, although, the process which equals the number of customers in the system at time t is not Markov.

The paper [21] was based on the MSc project by the author on the topic suggested by Professor A.D. Solov'yev. The ultimate goal of that time – to estimate rates of convergence for Erlang–Sevastyanov systems – was partially realised only recently [23] where a polynomial convergence for such systems with finitely or infinitely many servers has been established under certain assumptions on the intensities; a bit earlier in [12] exponential rate was proved under more severe than in [23] assumptions. The problem of evaluating convergence rates without assuming anything about intensities apparently remains open. At the same time, neither in [17], nor in [21] existence of intensities was assumed. Note that there are also general results about exponential or polynomial convergence for Markov processes and for *regeneration* processes to stationarity under the assumption of appropriate exponential or polynomial recurrence along with a "local

mixing” or regeneration [1, 2, 9, 10, 16, 20], et al. So far, none of these general results cover directly the particular Erlang–Sevastyanov type systems and their convergence rates and especially the setting with a “more general” dependence of $\lambda(\cdot)$ of the “whole state” of the process. In any case, the results proposed below do not follow from earlier results, including those from [23]. On the other hand, close results for single-server systems were recently studied in [22, 25, 26], and some ideas from the latter articles will be used in the sequel. A general dependence of intensity of service $h(\cdot)$ on all coordinates of the process (assuming that the description of this process makes it Markov) is allowed. Initially, the result of this paper was announced in [24]. The literature in the references is by all means not complete; more references can be found in [23].

The paper consists of Introduction, Main result, Proof of main result.

2. ERLANG – SEVASTYANOV TYPE SYSTEM: MAIN RESULT

Let us introduce the state space \mathcal{X} : it is a union of countably many subsets,

$$\mathcal{X} = \bigcup_{n=0}^{\infty} \{(n, x^0, \dots, x^n), x^0, \dots, x^n \geq 0\};$$

denote also $\mathcal{X}_n := \{(n, x^0, \dots, x^n), x^0, \dots, x^n \geq 0\}$. To any $n \geq 0$ there correspond $n + 1$ non-negative coordinates (x^1, \dots, x^n) and x^0 , which signify, respectively, the elapsed times of service of all existing n customers (x^1, \dots, x^n) and the time from the last arrival (x^0), including for $n = 0$. It is convenient to assume that a newly arrived customer with number $n + 1$ gets a coordinate $x^i = 0$ for any $i = 1, \dots, n + 1$ with equal probabilities. For state $X = (n, x^0, \dots, x^n)$, denote $x = (x^0, \dots, x^n)$, and $n(X) = n(x) = n$ (i.e., we do not distinguish $n(X)$ and $n(x)$). To establish convergence rate, we will use a characteristic of the service time distribution called intensity of service,

$$h(t) := \frac{g(t)}{1 - G(t)}, \quad t \geq 0, \quad g(t) = G'(t).$$

If the intensity function is constant, it means an exponential distribution of the service time. With a convention $\sum_{j=1}^0 = 0$, denote

$$V_{m,a}(X) := \left(\sum_{j=1}^{n(X)} (1 + x^j)^m \right)^a, \quad \Lambda := \sup_{X \in \mathcal{X}} (\lambda(X)/(n(X) \vee 1)),$$

$$\underline{\lambda}_0 := \inf_{Y \in \mathcal{X}_0} \lambda(Y), \quad \bar{\lambda}_n := \sup_{X \in \mathcal{X}_n} \lambda(X), \quad n \geq 0.$$

Note that $V_{m,a}$ as a function does not depend on x^0 and that this function is symmetric with respect to (x^1, \dots, x^n) . Notation $a \vee b$ stands for $\max(a, b)$, and $a \wedge b = \min(a, b)$. The process X_t is assumed right-continuous.

Theorem 1. *Let there exist $C_0, D > 0$, $m > 1$, $a > 1$ and $\ell > 0$ such that*

$$(4) \quad \frac{C_0}{1+t} \leq h(t) \leq D, \quad t \geq 0,$$

$$(5) \quad 0 < \underline{\lambda}_0 \leq \Lambda < \infty,$$

and

$$(6) \quad C_0 > \left(a + \frac{\ell + 1}{m} \right) \left(m + \Lambda 2^{a + \frac{\ell + 1}{m}} \right).$$

Then for any $0 < k < \ell$, if C_0 is large enough, then there exist constants $C, C_1 > 0$ such that for every $X_0 \in \mathcal{X}$ and $t \geq 0$,

$$(7) \quad \|\mu_t^{X_0} - \mu\|_{TV} \leq \frac{C(V_{m,a+\ell/m}(X_0) \vee 1 + C_1)}{t^{k+1}} \wedge 2,$$

where μ_t^X is the distribution of X_t with the initial value X , μ is the unique stationary measure of the process, and $\|\cdot\|_{TV}$ is the total variation distance.

Remark 1. The condition (6) used here is a bit more precise than in [23]; probably it may allow some further generalizations. For any $k > 0$, the bound (7) with *some* m and a is valid for any C_0 which is large enough. Recall that strong Markov property holds true for our system, see [5]. Uniqueness of stationary distribution as well as its existence is a part of the statement of the Theorem.

Remark 2. The constant C, C_1 in (7) are not just finite but do admit some effective estimate, which will be seen in the proof. In the earlier versions including [23] similar constants in the main estimate – e.g., in the main inequality in [23] – also admit certain effective estimates in principle. Nonetheless, in the present paper we show step by step how such estimates may be achieved. There is a hope that better and more precise bounds may be obtained in the future. More that that, there is a hypothesis that the condition (5) may be relaxed to a condition similar to (4),

$$(8) \quad \frac{C'_0}{1+t} \leq \lambda(0, t) \leq \Lambda < \infty, \quad t \geq 0,$$

with some C'_0 . Yet, this seems more technical and we do not pursue this goal here.

3. PROOF OF THEOREM 1

0. We will be using notation $*$ for an arbitrary value of any coordinate of the process: for example, $(1, *, 0) = (X = (1, x^0, 0) \in \mathcal{X} : n(X) = 1, x^0 \geq 0)$, and $\mathcal{X}_0 = \{(0, *)\}$. The starting idea is to construct a Lyapunov function and to apply coupling method. The coupling algorithm to be used here will be a “two-step” one, which makes it, perhaps, just a little bit unusual. Eventually, we are going to show that roughly speaking the state $(1, 0, 0)$ may be considered as a “generalised regeneration”, with some uniformly bounded from above polynomial moments for the distributions of the length between the generalised regeneration periods. (The word “generalised” here means that it is necessary to change probability space to make it work.) Due to the strong Markov property, this will suffice for the proof.

Denote

$$\tau_0 := \inf(t \geq 0 : X_t \in \mathcal{X}_0), \quad \tau_{01} := \inf(t > \tau_0 : X_t = (1, 0, 0)),$$

and

$$\tau_{010} := \inf(t > \tau_{01} : X_t = (0, *)).$$

The *two-step coupling* idea is to consider two versions of the process, one of them stationary (existence of which is yet to be established), and to couple *firstly the first* components of the two processes waiting when they both are simultaneously in the set \mathcal{X}_0 , and secondly coupling their remaining components at their transition from $(0, *)$ to $(1, 0, 0)$.

1. To estimate τ_0 , we construct a Lyapunov function only for the variables (n, x^1, \dots, x^n) , i.e., ignoring the component x^0 . We want to show that the process

$$X_t = (n(X_t), X_t^0, X_t^1, \dots, X_t^n)$$

with probability one hits the set $\{X : n(X) = 0\} \equiv \mathcal{X}_0$, being also positive recurrent uniformly in the variable x^0 . (Note that given the assumptions this is reasonable, since only $\lambda_0(\cdot)$ depends on x^0 essentially, and $\lambda_0(\cdot)$ itself is bounded away from zero and from infinity.) So, informally speaking, we may arrange *coupling* of two versions of the process – the original one (X) and the stationary one (say, Y) – on their joint jump from $n = 0$ to $n = 1$ (or, more precisely, from the set $\mathcal{X}_0 \times \mathcal{X}_0$ to state $(1, 0, 0, 1, 0, 0) \in \mathcal{X}_1 \times \mathcal{X}_1$).

Note, however, that existence of a stationary measure itself is yet to be shown; this will be addressed at step 9 of this proof which consists of 10 major steps (the preliminary zero one is not counted): the first part is fairly close to the calculus from [23], while the second one is a “simple coupling” on the passage of the system “from $n = 0$ to $n = 1$ ”. For the convenience of the reader and by the suggestion of the referee we keep most of the calculus trying to minimize references which would make it necessary to read another paper ([23]), even though it increases the volume to some extent. The constants $C, C',$ etc. in the calculus may change from line to line; different constants on one line will be denoted by C, C', C'' .

For $X = (n, x^0, x^1, \dots, x^n)$ with $n \geq 1$ and for $1 \leq j \leq n$ denote

$$X^{j,-} := (n-1, x^0, x^1, \dots, x^{j-1}, x^{j+1}, \dots, x^n)$$

(i.e., the component x^j has been dropped, which means one service completed and which automatically decreases the value of n by one unit), and for $1 \leq j \leq n+1$,

$$X^{j,+} := (n+1, 0, x^1, \dots, x^{j-1}, 0, x^j, \dots, x^n)$$

(i.e., a new component 0 has been added between the old x^{j-1} and x^j , which means a new arrival and which automatically increases the value of n by one unit and the component x^0 jumps to zero). For $n = 0$ we have $X = (0, x^0)$ with some $x^0 \geq 0$, and then ($j = 1$)

$$X^{1,+} := (1, 0, 0).$$

The element $X^{j,-}$ is not defined for $n(X) = 0$.

Now, the *extended generator* of our *Markov process* (see, e.g., [5]) reads,

$$(9) \quad \begin{aligned} Lg(X) = Lg(n, x) = & \lambda(X) \frac{1}{n(x)+1} \sum_{j=1}^{n(x)+1} [g(X^{j,+}) - g(X)] \\ & + 1(n(x) > 0) \sum_{j=1}^{n(x)} h(x^j) [g(X^{j,-}) - g(X)] + \sum_{i=0}^{n(x)} \frac{\partial}{\partial x^i} g(n, x). \end{aligned}$$

It would be a “honest” generator of the Markov process – see [6] – under the assumption of continuity of all intensities λ and h . A corresponding *Dynkin’s formula* – which is, actually, the definition of extended generator (except that we do not state the exact domain of this operator, but only some sub-domain, which is sufficient for our goals) – has a form,

$$(10) \quad \mathbb{E}_{X_0} g(X_t) = g(X_0) + \int_0^t \mathbb{E}_{X_0} Lg(X_s) ds,$$

for any function g from the class of bounded C_b^1 functions (i.e., with bounded continuous derivatives in any x^i , $0 \leq i \leq n(X)$). The *proof* of this formula for L given in (2) follows from the “complete expectation” formula (by analogy with the complete probability formula); some details in a slightly different setting may be found in [26]; see also [5]. Equivalently, the equation (10) may be stated as follows: the difference

$$(11) \quad g(X_t) - g(X_0) - \int_0^t Lg(X_s) ds$$

is a martingale (see, e.g., [5]). (We note that because of the jumps the accurate writing should have used $Lg(X_{s-})$ under the integral here, but since integration is performed with respect to the Lebesgue measure, it is equivalent to $Lg(X_s)$.)

Using just bounded functions is often not sufficient. It follows further that for *locally bounded* C^1 functions (i.e., with one continuous derivative in all components (x^0, \dots, x^n))

the same expression (11) is a *local martingale* (i.e. a martingale stopped by some appropriate “localizing sequence” of stopping times, see, e.g., [14]). For the sequel, note that another form of Dynkin’s formula for bounded functions $f(t, X)$ from the class C_b^1 in all x^i and in t reads (f_s denotes $\partial f/\partial s$),

$$\mathbb{E}_{X_0} f(t, X_t) = f(0, X_0) + \int_0^t \mathbb{E}_{X_0} (f_s(s, X_s) + Lf(s, X_s)) ds.$$

Equivalently, it can be stated that for f from the specified class of functions the difference

$$f(t, X_t) - f(0, X_0) - \int_0^t (f_s(s, X_s) + Lf(s, X_s)) ds$$

is a martingale. Similarly to the case of functions $g(X)$, it follows that for *locally bounded* C^1 functions $f(t, X)$ with locally bounded first derivatives with respect to t and all x^i the latter difference is a *local martingale*.

2. Let us show that $V_{m,a}$ may serve as a Lyapunov function. For $X_t \notin \mathcal{X}_0$ and the constants m and a satisfying a weakened version of the standing assumption (6),

$$(12) \quad C_0 > a(m + \Lambda 2^a),$$

and with $n = n(X_t)$ and with M_t being some local martingale we have (cf. with [23, Lemma 2]),

$$\begin{aligned} dV_{m,a}(X_t) &= \lambda_n(X_t) \left(\left(1 + \sum_{j=1}^n (1 + X_t^j)^m \right)^a - \left(\sum_{j=1}^n (1 + X_t^j)^m \right)^a \right) dt \\ &\quad - \sum_{i=1}^n h(X_t^i) \left(- \left(\sum_{1 \leq j \leq n, j \neq i} (1 + X_t^j)^m \right)^a + \left(\sum_{j=1}^n (1 + X_t^j)^m \right)^a \right) dt \\ &\quad + \sum_{i=1}^n \left(\left(\sum_{j=1}^n (1 + X_t^j + dt)^m \right)^a - \left(\sum_{j=1}^n (1 + X_t^j)^m \right)^a \right) + dM_t \\ (13) \quad &\equiv (I_1 - I_2 + I_3)dt + dM_t. \end{aligned}$$

Due to the assumption (4) we get,

$$I_2 \geq C_0 \sum_{i=1}^n (1 + X_t^i)^{-1} (1 + X_t^i)^m V_{m,a-1}(X_t) = C_0 V_{m-1,1}(X_t) V_{m,a-1}(X_t).$$

Hence, we can see that, at least, the “main term” $-I_2$ is negative for $X_t \notin \mathcal{X}_0$.

Our next task is to show that I_1 and I_3 are dominated by I_2 . Then it would imply that the stationary measure (see step 9 for its existence) integrates some polynomial. This, in turn, would allow to extend our Lyapunov function so as to include some multiplier that depends on time (see step 2 below). The latter would provide for some $k > 0$ a crucial bound $E_x \tau_0^{k+1} < \infty$ along with its quantitative version (see the estimate (21) below). Finally, the similar inequality for τ_{01} would imply coupling between the original process and its stationary version (but not an immediate regeneration at τ_0 , unlike in [23]), which would mean a certain rate of convergence to the stationary regime.

We estimate (cf. with [23, Lemma 3]) with $n = n(X)$,

$$I_1 = \lambda(X_t) \left(\left(1 + \sum_{j=1}^n (1 + X_t^j)^m \right)^a - \left(\sum_{j=1}^n (1 + X_t^j)^m \right)^a \right) \leq \bar{\lambda}_n a 2^a V_{m,a-1}(X_t).$$

Due to the inequality $n \leq V_{m-1,1}(X_t)$ we find,

$$I_1 \leq \bar{\lambda}_n a 2^a V_{m,a-1}(X_t) \leq \Lambda n a 2^a V_{m,a-1}(X_t) \leq \Lambda a 2^a V_{m-1,1}(X_t) V_{m,a-1}(X_t).$$

Further, we have,

$$I_3 = \sum_{i=1}^n \left(a \left(\sum_{j=1}^n (1 + X_t^j)^m \right)^{a-1} m \sum_{j=1}^n (1 + X_t^j)^{m-1} \right) = am V_{m-1,1}(X_t) V_{m,a-1}(X_t).$$

Notice that both the estimate for I_1 and the expression for I_3 are comparable with the estimate for I_2 . Overall,

$$I_1 - I_2 + I_3 \leq -(C_0 - \Lambda a 2^a - ma) V_{m-1,1}(X_t) V_{m,a-1}(X_t) < 0,$$

for $X_t \notin \mathcal{X}_0$. So, we get,

$$(14) \quad \mathbb{E}_X V_{m,a}(X_{t \wedge \tau_0}) + (C_0 - a(m + \Lambda 2^a)) \mathbb{E}_X \int_0^{t \wedge \tau_0} V_{m-1,1}(X_s) V_{m,a-1}(X_s) ds \leq V_{m,a}(X).$$

A formal justification of (14) uses a localising sequence from the definition of a local martingale. In our case, let us denote $T_R := \inf(t \geq 0 : n(X_t) + \max_{0 \leq i \leq n(X_t)} x_t^i \geq R)$ for any $R > 0$. Then, all term in the version of Dynkin's formula

$$(15) \quad \mathbb{E}_X V_{m,a}(X_{t \wedge \tau_0 \wedge T_R}) = \mathbb{E}_X \int_0^{t \wedge \tau_0 \wedge T_R} LV_{m,a}(X_s) ds + V_{m,a}(X),$$

are bounded and the identity (15) itself follows, e.g., from the ‘‘complete expectation’’ formula mentioned earlier. In other words, the process

$$M_{t \wedge \tau_0 \wedge T_R} := V_{m,a}(X_{t \wedge \tau_0 \wedge T_R}) - \int_0^{t \wedge \tau_0 \wedge T_R} LV_{m,a}(X_s) ds - V_{m,a}(X)$$

is a stopped martingale in t . Due to the bounds on all terms of $LV(X_s)$, we get

$$(16) \quad \mathbb{E}_X V_{m,a}(X_{t \wedge \tau_0 \wedge T_R}) + (C_0 - a(m + \Lambda 2^a)) \mathbb{E}_X \int_0^{t \wedge \tau_0 \wedge T_R} V_{m-1,1}(X_s) V_{m,a-1}(X_s) ds \leq V_{m,a}(X),$$

and now (14) follows from (16) by Fatou's lemma as $R \rightarrow \infty$, as required, since $T_R \rightarrow \infty$ (because the first component of the process X has jumps ± 1 , other components increase between their exclusively negative jumps with rate $+1$, and because intensity of arrivals satisfy the condition $\bar{\lambda}_n \leq \Lambda n$, $n \geq 1$, so that neither the first nor any other component of the process may achieve infinity over a finite time with a positive probability, cf. [11]) and since at any t the trajectories of the process are continuous almost surely (the latter due to the existence of intensities of both arrivals and service). In the sequel in similar places we will drop this standard part related to localising sequence. As $t \rightarrow \infty$, due to (12) and by Fatou's lemma we obtain,

$$(17) \quad \mathbb{E}_X V_{m,a}(X_{\tau_0}) + (C_0 - a(m + \Lambda 2^a)) \mathbb{E}_X \int_0^{\tau_0} V_{m-1,1}(X_s) V_{m,a-1}(X_s) ds \leq V_{m,a}(X).$$

Further, in particular, it follows that $\mathbb{E}_X \tau_0 < \infty$ for any X . In the sequel we shall see that also $\mathbb{E}_X \tau_{01} < \infty$, which does signify that the process X is positive recurrent (see the details in steps 2–3 below). According to the Harris–Khasminsky principle – fully justified by steps 2–3 and 9 below – there is an invariant measure μ (we show in

the sequel that it is unique) and $V_{m-1,1}(X)V_{m,a-1}(X)$ is integrable with respect to this measure μ . Due to the elementary inequality (see [23])

$$V_{m,1}(X)^{(m-1)/m} \leq V_{m-1,1}(X),$$

(also recall that $V_{m,a}(X_{\tau_0}) = 1$ and $V_{m,a}(X)V_{m,b}(X) = V_{m,a+b}(X)$), we obtain,

$$\mathbb{E}_X V_{m,a}(X_{\tau_0}) + (C_0 - a(m + \Lambda)) \mathbb{E}_X \int_0^{\tau_0} V_{m,a-1/m}(X_s) ds \leq V_{m,a}(X).$$

In particular, for any t ,

$$(18) \quad \mathbb{E}_X V_{m,a}(X_{\tau_0}) \vee \mathbb{E}_X V_{m,a}(X_{t \wedge \tau_0}) \leq V_{m,a}(X).$$

3. Let $V_{m,a,k}(t, X) := (1+t)^k V_{m,a}(X)$, $k < \ell$. Similarly to the above, we have due to Dynkin's formula,

$$(19) \quad \begin{aligned} dV_{m,a,k}(t, X_t) &= (1+t)^k [I_1 - I_2 + I_3] dt + d\tilde{M}_t + k(1+t)^{k-1} V_{m,a}(X_t) dt \\ &\leq -(1+t)^k (C_0 - a(m + \Lambda)) V_{m,a-1/m}(X_t) dt + k(1+t)^{k-1} V_{m,a}(X_t) dt + d\tilde{M}_t. \end{aligned}$$

Now the task is to ensure that the negative part in the right hand side of the last expression prevails all other terms. We will be using the inequality established in the step 2 above. The second term may be split into two parts,

$$\begin{aligned} I &:= k(1+t)^{k-1} V_{m,a}(X_t) \\ &\equiv I \times 1(k(1+t)^{k-1} V_{m,a}(X_t) \leq \epsilon(1+t)^k V_{m,a-1/m}(X_t)) \\ &\quad + I \times 1(k(1+t)^{k-1} V_{m,a}(X_t) > \epsilon(1+t)^k V_{m,a-1/m}(X_t)). \end{aligned}$$

The first part of this term here with ' $\leq \epsilon$ ', is dominated by the main negative expression if $\epsilon < C_0 - a(m + \Lambda 2^a)$.

Let us estimate the second part of this term. We have, for any $\ell > k$,

$$\begin{aligned} &I \times 1(k(1+t)^{k-1} V_{m,a}(X_t) > \epsilon(1+t)^k V_{m,a-1/m}(X_t)) \\ &\leq I \times \frac{(k V_{m,a}(X_t))^\ell}{(\epsilon(1+t) V_{m,a-1/m}(X_t))^\ell} = I \times \frac{k^\ell}{(\epsilon(1+t))^\ell} V_{m,\ell/m}(X_t). \end{aligned}$$

Therefore, the second part of the second term does not exceed

$$k(1+t)^{k-1} \times \frac{k^\ell}{(\epsilon(1+t))^\ell} V_{m,a+\ell/m}(X_t).$$

Denote $a' := a + \ell/m$, and recall that, in any case,

$$(20) \quad C_0 > a'(m + \Lambda 2^{a'}).$$

Now, let us collect all terms and their bounds, integrate and take expectation,

$$\begin{aligned} &\mathbb{E}_X V_{m,a,k}(t \wedge \tau_0, X_{t \wedge \tau_0}) + (C - a(m + \Lambda) - \epsilon) \mathbb{E}_X \int_0^{t \wedge \tau_0} (1+s)^k V_{m,a-1/m}(X_s) ds \\ &\leq V_{m,a}(X) + C' \int_0^\infty \mathbb{E}_X 1(s \leq t \wedge \tau_0) (1+s)^{k-1-\ell} V_{m,a+\ell/m}(X_s) ds. \end{aligned}$$

Here by virtue of (18) and due to the condition (20),

$$\mathbb{E}_X 1(s \leq t \wedge \tau_0) V_{m,a+\ell/m}(X_s) \leq V_{m,a+\ell/m}(X).$$

Thus,

$$\int_0^{\infty} \mathbb{E}_X 1(s \leq t \wedge \tau_0) (1+s)^{k-1-\ell} V_{m,a+\ell/m}(X_s) ds \leq C V_{m,a+\ell/m}(X),$$

where the constant C clearly admits an effective estimate. Further, due to Fatou's lemma, for $k < \ell$ this implies,

$$\mathbb{E}_X V_{m,a,k}(\tau_0, X_{\tau_0}) + C' \mathbb{E}_X \int_0^{\tau_0} (1+s)^k V_{m,a-1/m}(X_s) ds \leq V_{m,a}(X) + C V_{m,a+\ell/m}(X),$$

again with two new constants C, C' . Since $V_{m,a-1/m}(X_s) \geq 1$ for $s < \tau_0$ (and since $\tau_0 = 0$ for $X_0 \in \mathcal{X}_0$), we obtain

$$\mathbb{E}_X \tau_0^{k+1} \leq C V_{m,a}(X) + C V_{m,a+\ell/m}(X).$$

or, with one more new constant C ,

$$(21) \quad \mathbb{E}_X \tau_0^{k+1} \leq C V_{m,a+\ell/m}(X).$$

Here also the constant C does allow an effective estimate. Emphasize that so far we still used a bit relaxed version (20) (namely, $C_0 > a'(m + \Lambda 2^{a'})$ with $a' = a + \ell/m$) of our standing assumption (6).

4. For the hitting time τ_{01} defined earlier it follows straight away due to (21) and from the assumption (5) that

$$(22) \quad \mathbb{E}_{X_{\tau_0}} (\tau_{01} - \tau_0)^{k+1} \leq \int_0^{\infty} s^{k+1} \underline{\lambda}_0 \exp(-\underline{\lambda}_0 s) ds = \underline{\lambda}_0^{-(k+1)} \Gamma(k+2) < \infty,$$

and, hence, since $V_{m,a+\ell/m}(X) \geq 1$ for $n(X) > 0$, we obtain

$$(23) \quad \mathbb{E}_X \tau_{01}^{k+1} = \mathbb{E}_X \mathbb{E}_{X_{\tau_0}} (\tau_0 + (\tau_{01} - \tau_0))^{k+1} \leq C V_{m,a+\ell/m}(X) \quad (n(X) > 0).$$

Recall for completeness that, according to (22), for $n(X) = 0$ we have $\mathbb{E}_X \tau_{01}^{k+1} \leq C = \underline{\lambda}_0^{-(k+1)} \Gamma(k+2) < \infty$. Thus, (23) can be written in a more general form for *any* initial value X with some new C as follows,

$$(24) \quad \mathbb{E}_X \tau_{01}^{k+1} = \mathbb{E}_X \mathbb{E}_{X_{\tau_0}} (\tau_0 + (\tau_{01} - \tau_0))^{k+1} \leq C V_{m,a+\ell/m}(X) \vee 1.$$

Moreover, since $V_{m,a}(X) \equiv 0$ for $X \in \mathcal{X}_0$, and thus, $1(\tau_0 \leq s < \tau_{01}) V_{m,a}(X_s) \equiv 0$, we also get under the full assumption (6) with “ $\ell + 1$ ”, that

$$(25) \quad \begin{aligned} \mathbb{E}_X \int_0^{\tau_{01}} V_{m,a+\ell/m}(X_s) ds &= \mathbb{E}_X \left(\int_0^{\tau_0} + \int_{\tau_0}^{\tau_{01}} \right) V_{m,a+\ell/m}(X_s) ds \\ &= \mathbb{E}_X \int_0^{\tau_0} V_{m,a+\ell/m}(X_s) ds \leq C V_{m,a+(\ell+1)/m}(X) \vee 1. \end{aligned}$$

5. Sufficiency of the bound (23). For some readers it could be a “common knowledge” that the inequality (23) suffices for the proof of the statement of the Theorem because the state $(1, 0, 0)$ may be regarded as a regeneration state (yet, note that the sojourn time of the process at this state as well as at all other states is zero). However, for the author it was not so easy to provide a proper reference which would justify exactly what is stated including the bound (7). So, for the sake of completeness and for the convenience of the

reader we offer a version of coupling below. The reader who knows a right reference may skip the remaining paragraphs.

6. Now, for the hitting time

$$\tau_{010} := \inf(t > \tau_{01} : X_t = (0, *)),$$

which will play its role in the sequel, we can use the same estimate (21) from the previous step, which gives the following bound:

$$(26) \quad \mathbb{E}_{X_{\tau_{01}}} (\tau_{010} - \tau_{01})^{k+1} \leq CV_{m,a+\ell/m}((1, 0, 0)) = C.$$

As an immediate corollary we also get from (26) and (23) that

$$(27) \quad \mathbb{E}_X \tau_{010}^{k+1} \leq CV_{m,a+(\ell+1)/m}(X) \vee 1.$$

Naturally, all constants C here (and earlier) may depend on k .

7. “**Hitting the set $\mathcal{X}_0 \times \mathcal{X}_0$** ”. Similarly to (21), it may be justified *for the couple of independent processes* (X_t, Y_t) with possibly different initial values that

$$\mathbb{E}_{X_0, Y_0} \bar{\tau}_{00}^{k+1} < \infty,$$

with a bound,

$$(28) \quad \mathbb{E}_{X_0, Y_0} \bar{\tau}_{00}^{k+1} \leq C (V_{m,a+\ell/m}(X_0) + V_{m,a+\ell/m}(Y_0)),$$

with any $k < \ell$, where

$$\bar{\tau}_{00} := \inf(t \geq 0 : (X_t, Y_t) \in \mathcal{X}_0 \times \mathcal{X}_0).$$

The inequality (28) can be established similarly to the lines of proving (38–39) in [23]. So, let us consider a couple of processes X_t and Y_t on two independent probability spaces where the process Y_t is a stationary version of X_t (with a stationary initial distribution: the existence of this distribution will be *independently* justified in step 9 in the sequel). Consider new Lyapunov functions for the two component process:

$$\bar{V}_{m,a}(X, Y) := V_{m,a}(X) + V_{m,a}(Y),$$

and

$$\bar{V}_{k,m,a}(t, X, Y) := V_{k,m,a}(t, X) + V_{k,m,a}(t, Y).$$

It follows from Dynkin’s formulae for the couple (X_t, Y_t) and for the function $\bar{V}_{m,a,k}(t, X_t, Y_t)$ with the restrictions on the parameters given in the assumptions of the Theorem that for $(X_t, Y_t) \notin \mathcal{X}_0 \times \mathcal{X}_0$ we have,

$$\begin{aligned} & d\bar{V}_{m,a,k}(t, X_t, Y_t) \\ &= (1+t)^k [I_1^X - I_2^X + I_3^X + I_1^Y - I_2^Y + I_3^Y] dt + k(1+t)^{k-1} \bar{V}_{m,a}(X_t, Y_t) dt + d\bar{M}_t, \end{aligned}$$

with some local martingale \bar{M}_t and with natural notations

$$\begin{aligned} I_1^X &= \lambda(X_t) \left(\left(1 + \sum_{j=1}^n (1 + X_t^j)^m \right)^a - \left(\sum_{j=1}^n (1 + X_t^j)^m \right)^a \right), \\ I_2^X &= \sum_{i=1}^n h(X_t^i) \left(- \left(\sum_{1 \leq j \leq n, j \neq i} (1 + X_t^j)^m \right)^a + \left(\sum_{j=1}^n (1 + X_t^j)^m \right)^a \right), \\ I_3^X &= \sum_{i=1}^n \left(a \left(\sum_{j=1}^n (1 + X_t^j)^m \right)^{a-1} m \sum_{j=1}^n (1 + X_t^j)^{m-1} \right) \end{aligned}$$

(former I_1, I_2 , and I_3), and

$$\begin{aligned} I_1^Y &= \lambda(Y_t) \left(\left(1 + \sum_{j=1}^n (1 + Y_t^j)^m \right)^a - \left(\sum_{j=1}^n (1 + Y_t^j)^m \right)^a \right), \\ I_2^Y &= \sum_{i=1}^n h(Y_t^i) \left(- \left(\sum_{1 \leq j \leq n, j \neq i} (1 + Y_t^j)^m \right)^a + \left(\sum_{j=1}^n (1 + Y_t^j)^m \right)^a \right), \\ I_3^Y &= \sum_{i=1}^n \left(a \left(\sum_{j=1}^n (1 + Y_t^j)^m \right)^{a-1} m \sum_{j=1}^n (1 + Y_t^j)^{m-1} \right). \end{aligned}$$

For any t where both $X_t \notin \mathcal{X}_0$ and $Y_t \notin \mathcal{X}_0$, the terms I_2^X and I_2^Y dominate their counterparts I_1^X, I_3^X and I_1^Y, I_3^Y , respectively. Let us inspect the case where one of the processes belongs to \mathcal{X}_0 , say, $n(Y_t) = 0$. In this case $X_t \notin \mathcal{X}_0$, but the term $I_2^Y = 0$, and we have,

$$I_1^Y = \lambda(Y_t) = \lambda(0, y_t) \leq \bar{\lambda}_0, \quad I_3^Y = 0$$

(the latter because $\sum_{i=1}^0 = 0$). Hence, $n(Y_t) = 0$ implies

$$I_1^Y - I_2^Y + I_3^Y = \lambda(0, y_t) \leq \bar{\lambda}_0.$$

So, the total sum of non-negative terms admits the bound (cf. with (19)),

$$I_1^X + I_3^X + I_1^Y + I_3^Y \leq (1+t)^k [(a(m+\Lambda)V_{m,a-1/m}(X_t) + \bar{\lambda}_0] + k(1+t)^{k-1} V_{m,a}(X_t),$$

while the modulus of the only negative term (I_2^X) equals

$$C_0(1+t)^k V_{m,a-1/m}(X_t).$$

In the case $n(X_t) = 0$ instead, the situation is, of course, symmetric for X and Y and the same conclusions hold. So, the earlier considerations for one process X_t in the steps 2–3 based on the inequality

$$C_0 > a(m+\Lambda)$$

remain valid and lead to the desired estimate (28) if we only check that

$$C_0 > a(m+\Lambda) + \bar{\lambda}_0.$$

However, the latter bound does follow from the standing assumption (6):

$$C_0 > \left(a + \frac{\ell+1}{m} \right) \left(m + \Lambda 2^{a+\frac{\ell+1}{m}} \right),$$

since $a > 1$ and, hence, $C_0 > am + 2a\Lambda > am + a\Lambda + \bar{\lambda}_0$.

8. Coupling at the “ $\mathcal{X}_0 \mapsto \mathcal{X}_1$ ” passage. Further, from any state $(0, *, 0, *) \in \mathcal{X}_0 \times \mathcal{X}_0$, coupling is achieved with a positive (bounded away from zero) probability over a unit time interval on the passage to state $(1, 0, 0, 1, 0, 0) \in \mathcal{X}_1 \times \mathcal{X}_1$ due to the assumption

$$0 < \underline{\lambda}_0 \leq \bar{\lambda}_0 < \infty.$$

Indeed, denote

$$\bar{\tau}_{0011} := \inf(t \geq 0 : (X_t, Y_t) = (1, 0, 0, 1, 0, 0) \text{ after visiting } (0, *, 0, *)),$$

$$\bar{\tau}_{001} := \inf(t \geq 0 : (X_t, Y_t) \notin \mathcal{X}_0 \times \mathcal{X}_0 \text{ after visiting } (0, *, 0, *)).$$

It may happen that $\bar{\tau}_{001} = \bar{\tau}_{0011}$, although, in general, $\bar{\tau}_{001} \leq \bar{\tau}_{0011}$. The moment $\bar{\tau}_{0011}$ may be regarded as a moment of coupling, i.e., the moment where the two versions of the process meet up, after which due to the strong Markov property the processes may

be considered as equal, of course, after a corresponding change of the probability space. (Clearly, without such a change the moment $\bar{\tau}_{001}$ can never occur, i.e., it can equal infinity.) Indeed, after such a change, we may assume that each of the two processes perform jumps up from state $(0, *)$ according to two *independent* flows of “events”, one with intensity $\underline{\lambda}_0$ and the other due to the “remainder” flow with intensity $\lambda_0(\cdot) - \underline{\lambda}_0$. The point is that the flow with intensity $\underline{\lambda}_0$ may be regarded as *the same* for the two processes and, hence, on this new probability space both processes (more precisely, their equivalents) jump up to state $(1, 0, 0)$ simultaneously with probability at least $p_{01} := (1 - \exp(-\underline{\lambda}_0)) \times \exp(-2(\bar{\lambda}_0 - \underline{\lambda}_0))$ on a unit interval of time.

Note that similarly to the bound (26) and due to (28), for the stopping time

$$\bar{\tau}_{00100} := \inf(t > \bar{\tau}_{001} : (X_t, Y_t) \in \mathcal{X}_0 \times \mathcal{X}_0)$$

the following holds true with some finite constant $C > 0$:

$$(29) \quad \mathbb{E}_{X_{\bar{\tau}_{001}}, Y_{\bar{\tau}_{001}}} (\bar{\tau}_{00100} - \bar{\tau}_{001})^{k+1} \leq C \sup_{X, Y \in (\mathcal{X}_0 \times (1, 0, 0)) \cup ((1, 0, 0) \times \mathcal{X}_0)} \bar{V}_{m, a+\ell/m}(X, Y) = C.$$

Also, similarly to (22) and due to exactly the same calculus,

$$(30) \quad \sup_{X, Y \in \mathcal{X}_0} \mathbb{E}_{X, Y} \bar{\tau}_{001}^{k+1} \leq C = \underline{\lambda}_0^{-(k+1)} \Gamma(k+2).$$

Let us show that after the declared change of probability space, the moment $\bar{\tau}_{001}$ satisfies the bound,

$$(31) \quad \mathbb{E}_{X_0, Y_0} \bar{\tau}_{001}^{k+1} < \infty,$$

and that moreover, for **any** (this is the difference between (30) and (32) below) $X_0 = (n^1, x)$, $Y_0 = (n^2, y)$ and for $k < \ell$ we have,

$$(32) \quad \mathbb{E}_{X_0, Y_0} \bar{\tau}_{001}^{k+1} \leq C \bar{V}_{m, a+\ell/m}(X_0, Y_0) \vee 1,$$

with a new $C > 0$. The crucial bound (32) here is due to (28) and to the fact that coupling on the passage from $\mathcal{X}_0 \times \mathcal{X}_0$ to state $(1, 0, 0, 1, 0, 0)$ occurs with a positive probability (p_{01}) over a unit time (see above), with the help of one more geometric like series, as will be shown below. Indeed, denote $\bar{\tau}_{001} =: \bar{\tau}_{001}(1)$, $\bar{\tau}_{00100} =: \bar{\tau}_{00100}(1)$, $\bar{\tau}_{00100}(0) =: \bar{\tau}_{00}$, and by induction,

$$\begin{aligned} \bar{\tau}_{00100}(i) &:= \inf(t > \bar{\tau}_{001}(i) : (X_t, Y_t) = (0, *, 0, *)), \\ \bar{\tau}_{001}(i+1) &:= \inf(t > \bar{\tau}_{00100}(i) : (X_t, Y_t) \notin \mathcal{X}_0 \times \mathcal{X}_0), \quad i \geq 1. \end{aligned}$$

At each moment $\bar{\tau}_{001}(i)$ coupling is possible with a positive probability p_{01} – see the definition above – bounded away from zero. One of these moments will be a “successful coupling”, i.e., will be equal to $\bar{\tau}_{0011}$. (The change of probability space is assumed as prescribed earlier). By virtue of the bounds (28), (29) and (30) and using the representation for any $i \geq 1$,

$$\bar{\tau}_{001}(i) = \bar{\tau}_{00} + (\bar{\tau}_{001} - \bar{\tau}_{00}) + \sum_{j=1}^{i-1} ((\bar{\tau}_{001}(j+1) - \bar{\tau}_{00100}(j)) + (\bar{\tau}_{00100}(j) - \bar{\tau}_{001}(j))),$$

we get by induction with some new $C > 0$ for any $i \geq 1$,

$$(33) \quad \mathbb{E}_{X, Y} \bar{\tau}_{001}^{k+1}(i) \leq C i^k (V_{m, a+\ell/m}(X) + V_{m, a+\ell/m}(Y)) \vee 1 = C i^k \bar{V}_{m, a+\ell/m}(X, Y) \vee 1.$$

From (33) we finally get the decisive estimate for the coupling moment $\bar{\tau}_{0011}$,

$$(34) \quad \begin{aligned} \mathbb{E}_{X,Y} \bar{\tau}_{0011}^{k+1} &\leq \sum_{i=1}^{\infty} \mathbb{E}_{X,Y} \bar{\tau}_{001}^{k+1}(i) (1-p_{01})^{i-1} \\ &\leq C \sum_{i=1}^{\infty} (1-p_{01})^{i-1} i^k \bar{V}_{m,a+\ell/m}(X,Y) \vee 1 \leq C \bar{V}_{m,a+\ell/m}(X,Y) \vee 1. \end{aligned}$$

9. Existence of some invariant measure μ follows from the fact that $(1, 0, 0)$ is a regeneration state and from (21) and (23) (suffices for one component), by the Harris–Khasminsky formula

$$(35) \quad \mu(A) = c \mathbb{E}_{(1,0,0)} \int_0^{\tau_{01}} 1(X_t \in A) dt$$

(recall that the stopping time τ_{01} was defined earlier in the step 0). Here c is the normalising constant. For the sequel notice that by virtue of (35) and (25),

$$(36) \quad \int V_{m,a+\ell/m}(x) \mu(dx) < \infty,$$

however, for our aims the value of this integral also has to be estimated. From (35) (and with the constant c from (35)), we have by virtue of (25),

$$\begin{aligned} \int V_{m,a+\ell/m}(X) \mu(dX) &= c \mathbb{E}_{(1,0,0)} \int_0^{\tau_{01}} V_{m,a+\ell/m}(X_s) ds \\ &\leq c \frac{1}{C_0 - a'(m+\Lambda)} V_{m,a+(\ell+1)/m}(1, 0, 0) = \frac{c}{C_0 - a'(m+\Lambda)}, \end{aligned}$$

with $a' = a + \ell/m$, because $V_{m,a+(\ell+1)/m}(1, 0, 0) = 1$. So, in order to obtain an upper bound for the integral in (36), it remains to estimate the constant c from above. From (35) with $A = \Omega$ we get,

$$c = \frac{1}{\mathbb{E}_{(1,0,0)} \tau_{01}} \leq \frac{1}{\mathbb{E}_{(1,0,0)} \tau_0 + \mathbb{E}_{(0,0)} \tau_{01}} \leq \frac{1}{\frac{1}{\bar{\lambda}_0} + \frac{1}{D}} \equiv \frac{\bar{\lambda}_0 D}{\bar{\lambda}_0 + D}.$$

This is firstly because the value of τ_0 is stochastically minorated by a similar hitting time for a pure death process with a constant intensity D (cf. with the assumption (4)) which mean value equals D^{-1} , so that

$$\mathbb{E}_{(1,0,0)} \tau_0 \geq D^{-1},$$

and secondly, because the value of τ_{01} is stochastically minorated by a similar hitting time for a pure birth process with a constant intensity $\bar{\lambda}_0$ which mean value equals $\bar{\lambda}_0^{-1}$, so that

$$\mathbb{E}_{(0,0)} \tau_{01} \geq \bar{\lambda}_0^{-1}.$$

Therefore, we obtain the following bound for the value of the integral in (36):

$$(37) \quad \begin{aligned} \int_S V_{m,a+\ell/m}(X) \mu(dX) &= c \mathbb{E}_{(1,0,0)} \int_0^{\tau_{01}} V_{m,a+\ell/m}(X_s) ds \leq \\ &\leq \frac{\bar{\lambda}_0 D}{(\bar{\lambda}_0 + D)(C_0 - (a + \ell/m)(m + \Lambda))} =: C_1. \end{aligned}$$

Note that here again the assumption (6) was essentially used while deriving (37), precisely with “ $\ell + 1$ ”, since (25) was applied for which the condition (6) is required.

Returning now to (32), we can see that it follows by integration that for the distribution μ (and with a new constant C),

$$(38) \quad \mathbb{E}_{X_0, \mu} \bar{\tau}_{0011}^{k+1} \leq C V_{m, a+\ell/m}(X_0) \vee 1,$$

under the standing assumption (6) of the Theorem. Indeed, by virtue of (34) and a little more precisely,

$$\begin{aligned} \mathbb{E}_{X, \mu} \bar{\tau}_{0011}^{k+1} &= \int \mathbb{E}_{X, Y} \bar{\tau}_{0011}^{k+1} \mu(dY) \leq C \int \bar{V}_{m, a+\ell/m}(X, Y) \vee 1 \mu(dY) \\ &= C V_{m, a+\ell/m}(X) \vee 1 + C \int V_{m, a+\ell/m}(Y) \vee 1 \mu(dY) \leq C(V_{m, a+\ell/m}(X) \vee 1 + C_1 + 1). \end{aligned}$$

Note that here both constants C and C_1 admit some effective estimates.

10. The final step of the proof is quite standard in coupling. Consider two independent versions X and Y of our Markov process, one starting at X_0 and another at the stationary distribution μ found earlier. Now, on some new probability space as described, for example, in [3], we estimate, uniformly in $A \in \mathcal{B}(\mathbb{R}^1)$,

$$\begin{aligned} |(\mu_t^{X_0} - \mu)(A)| &\leq |\mathbb{E}_{X_0, \mu}(1(X_t \in A) - 1(Y_t \in A))| 1(t \geq \bar{\tau}_{0011}) \\ &\quad + |\mathbb{E}_{X_0, \mu}(1(X_t \in A) - 1(Y_t \in A))| 1(t < \bar{\tau}_{0011}) \\ &\leq \mathbb{E}_{X_0, \mu} 1(t < \bar{\tau}_{0011}) = \mathbb{P}_{X_0, \mu}(t < \bar{\tau}_{0011}) \leq \frac{\mathbb{E}_{X_0, \mu} \bar{\tau}_{0011}^{k+1}}{t^{k+1}} \leq \frac{C(V_{m, a+\ell/m}(X_0) \vee 1 + \tilde{C}_1)}{t^{k+1}} \end{aligned}$$

(with $\tilde{C}_1 = C_1 + 1$). This shows the main statement of the Theorem; the version with “ $\dots \wedge 2$ ”, clearly, can be used for small values of t since the left hand side in (7) may never exceed two. Uniqueness of a stationary distribution μ automatically follows from this convergence. The Theorem 1 is proved.

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