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p−MAJORIZING QUADRATIC STOCHASTIC OPERATORS

In this paper, we introduce a new class of the so-called **p**−majorizing quadratic stochastic operators which is the generalization of the class of quadratic doubly stochastic operators. We provide a criterion for the regularity of **p**−majorizing quadratic stochastic operators acting on 2D simplex. Some relevant examples are also provided.

1. Introduction

The dynamics of nonlinear stochastic operators acting on the finite dimensional simplex remains to be difficult and complex. The main problem is to study the asymptotic behavior of nonlinear stochastic operators associated with stochastic hyper-matrices. The simplest nonlinear stochastic operator is a quadratic stochastic operator (in short QSO) associated with a cubic stochastic matrix. The dynamics of QSO is sufficiently rich and quite complicated (see [3, 10, 11]). The QSO has an incredible application in population genetics (see [8]). Namely, it describes a distribution of the next generation in the population system if the distribution of the current generation was given. The QSO is a primary source for investigations of evolution of population genetics. In the paper [1], a mathematical model of a transmission of human ABO blood groups was described as the QSO on 7-dimensional simplex and based on some numerical investigations, the future ABO blood group distribution of Malaysian people was predicted. A self-contained exposition of the recent achievements and open problems in the theory of QSO was given in the paper [3].

The classical Perron–Frobenius theorem states that a trajectory of a linear stochastic operator associated with a positive square stochastic matrix always converges to a unique fixed point. In general, an analogy of the Perron–Frobenius theorem does not hold for a quadratic stochastic operator associated with a positive cubic stochastic matrix. Namely, its trajectories may converge to different fixed points depending on initial points or may not converge at all. Therefore, unlike linear stochastic operators, the structure of a set of all fixed points of QSO might be as complex as possible (see [17, 18]). It is of independent interest to consider the Perron–Frobenius problem in the nonlinear setting.

In the paper [12], the new class of the so-called **p**−majorizing QSO was introduced and the regularity problem under some constraints was studied. In general, the **p**−majorizing QSO may not be regular (see [19]). In this paper, we would like to provide a criterion for the regularity of the **p**−majorizing QSO. The dynamics of any QSO on 1D simplex is more or less clear (see [8]). However, there are many QSO on 2D simplex which remain to be investigated (see [10]). Therefore, we study the dynamics of **p**−majorizing QSO on 2D simplex. A more complete study on the dynamics of **p**−majorizing QSO on the higher dimensional simplex will be explored in another paper.

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A vector majorization is a preorder of dispersion for vectors with the same length and same sum of components. The vector majorization can be viewed as a preorder of distance from a uniform vector. A preorder of distance from any fixed non-uniform vector of positive components, so-called $\mathbf{p}-$majorization, is a generalization of usual vector majorization. Several equivalent definitions of $\mathbf{p}-$majorization and related concepts are discussed in the paper [6]. Let us provide some necessary notions and notations related to $\mathbf{p}-$majorization. Throughout this paper, we write vectors in the row forms.

Let $\mathbf{p} = (p_1, p_2, p_3) \in \mathbb{R}^3$. We write $\mathbf{p} \geq 0$ (resp. $\mathbf{p} > 0$) whenever $p_i \geq 0$ (resp. $p_i > 0$) for all $i = 1, 2, 3$. Let $\|\mathbf{x}\|_1 = \sum_{i=1}^{3} |x_i|$ for any $\mathbf{x} \in \mathbb{R}^3$ and $\mathbb{S}^2 = \{\mathbf{x} \in \mathbb{R}^3 : \mathbf{x} \geq 0,\ \|\mathbf{x}\|_1 = 1\}$ be a standard simplex. An element of the simplex $\mathbb{S}^2$ is called *a stochastic vector*. We write $\mathbb{P} \geq 0$ (resp. $\mathbb{P} > 0$) for a matrix $\mathbb{P}$ whenever $p_{ij} \geq 0$ (resp. $p_{ij} > 0$) for all i, j.

Definition 1.1. Let $\mathbf{x}, \mathbf{y}$ and $\mathbf{p} > 0$ be stochastic vectors. We say that $\mathbf{x}$ is $\mathbf{p}-$majorized by $\mathbf{y}$ with respect to $\mathbf{p}$ (written $\mathbf{x} \prec_{\mathbf{p}} \mathbf{y}$) if one has

$$\sum_{i=1}^{3} |x_i - tp_i| \leq \sum_{i=1}^{3} |y_i - tp_i|, \quad \forall\ t \in \mathbb{R}. \tag{1}$$

Note that if $\mathbf{p} = \mathbf{c} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ then the $\mathbf{p}-$majorization with respect to $\mathbf{p}$ is nothing but usual majorization (see [6, 9]). In this case, we shall use usual notation $\prec$ for usual majorization.

A matrix is said to be *stochastic* (resp. *doubly stochastic*) if its rows (resp. its rows and columns) are stochastic vectors. We denote the set of all stochastic (resp. doubly stochastic) matrices by SM (resp. DSM). Let us introduce the following set of stochastic matrices for a positive stochastic vector $\mathbf{p} > 0$

$$\mathsf{SM}[\mathbf{p}] = \{\mathbb{P} \in \mathsf{SM} : \mathbf{p}\mathbb{P} = \mathbf{p}\}. \tag{2}$$

The set $\mathsf{SM}[\mathbf{p}]$ of all stochastic matrices having a common fixed distribution $\mathbf{p} > 0$ is a convex compact subset of the set of all stochastic matrices SM. It is worth mentioning that if $\mathbf{p} = \mathbf{c} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ then $\mathsf{SM}[\mathbf{c}]$ is nothing but a set of all doubly stochastic matrices, i.e., $\mathsf{SM}[\mathbf{c}] = \mathsf{DSM}$. The following result was proven in [6, 9].

Theorem 1.1 ([6, 9])**.** *The following statements are equivalent:*

(*i*) *One has* $\mathbf{x} \prec_{\mathbf{p}} \mathbf{y}$;

(*ii*) *There is a stochastic matrix* $\mathbb{P} \in \mathsf{SM}[\mathbf{p}]$ *such that* $\mathbf{x} = \mathbf{y}\mathbb{P}$;

(*iii*) *One has* $\sum_{i=1}^{3} p_i \varphi\left(\frac{x_i}{p_i}\right) \leq \sum_{i=1}^{3} p_i \varphi\left(\frac{y_i}{p_i}\right)$ *for all convex continuous functions* $\varphi : [0, +\infty) \to \mathbb{R}$.

We shall interchangeably use (*i*) and (*ii*) throughout this paper.

Let us recall the notion of strong ergodicity of a sequence of stochastic matrices (see [20]). Let $\{\mathbb{P}_n\}_{n=1}^{\infty} \subset \mathsf{SM}$ be a sequence of stochastic matrices. Let $\mathbb{P}^{[r,s]} \equiv \mathbb{P}_r\mathbb{P}_{r+1} \cdots \mathbb{P}_s$ be a finite forward product and $\mathbb{P}^{[r,\infty]} \equiv \mathbb{P}_r\mathbb{P}_{r+1} \cdots \mathbb{P}_n\mathbb{P}_{n+1} \cdots$ be an infinity forward product of stochastic matrices. A sequence $\{\mathbb{P}_n\}_{n=1}^{\infty}$ of stochastic matrices is said to be *strongly ergodic* if $\left(\mathbb{P}^{[r,s]}\right)_{ij} \to p_j^{[r]}$ as $s \to \infty$ for every i, j, r. A stochastic matrix with identical rows is called *stable*. Thus, the sequence $\{\mathbb{P}_n\}_{n=1}^{\infty}$ of stochastic matrices is strongly ergodic if and only if the infinity forward product $\mathbb{P}^{[r,\infty]}$ tends to a single stable matrix.

A stochastic matrix $\mathbb{P}$ is called *scrambling* if for any i, j there is k such that $p_{ik}p_{jk} > 0$. In other words, a stochastic matrix $\mathbb{P}$ is *scrambling* if and only if any two rows are not orthogonal. We denote the set of all scrambling stochastic matrices by SSM. The set of all scrambling stochastic matrices SSM is convex.

Let $\Omega(\{\mathbb{P}_n\}) = \left\{\mathbb{P} \in \mathsf{SM} : \lim_{k\to\infty} \mathbb{P}_{n_k} = \mathbb{P}\right\}$ be an omega limiting set of a sequence $\{\mathbb{P}_n\}_{n=1}^{\infty}$ of stochastic matrices. We need the following result throughout this paper.

Theorem 1.2 ([7]). *If $\Omega(\{\mathbb{P}_n\}) \subset \mathsf{SSM} \bigcap \mathsf{SM}[\mathbf{p}]$ then a sequence $\{\mathbb{P}_n\}_{n=1}^{\infty}$ of matrices is strongly ergodic.*

2. p−Majorizing Quadratic Stochastic Operator

Let $\mathcal{Q} = (q_{ijk})_{i,j,k=1}^3$ be a cubic stochastic matrix, i.e.,

$$\sum_{k=1}^{3} q_{ijk} = 1, \quad q_{ijk} = q_{jik}, \quad q_{ijk} \geq 0, \quad 1 \leq i,j,k \leq 3.$$

We define a quadratic stochastic operator (in short QSO) $\mathcal{Q} : \mathbb{S}^2 \to \mathbb{S}^2$ associated with a cubic stochastic matrix $\mathcal{Q} = (q_{ijk})_{i,j,k=1}^3$ as follows

$$(\mathcal{Q}(\mathbf{x}))_k = x_1^2 q_{11k} + x_2^2 q_{22k} + x_3^2 q_{33k} + 2x_1x_2q_{12k} + 2x_1x_3q_{13k} + 2x_2x_3q_{23k}, \tag{3}$$

for all $1 \leq k \leq 3$. Here, we are using the same notation for the cubic stochastic matrix as well as for the associated QSO in order to show some correlation.

We define the following stochastic vectors and square stochastic matrices associated with the cubic stochastic matrix $\mathcal{Q} = (q_{ijk})_{i,j,k=1}^3$

$$\mathbf{q}_{ij\bullet} = (q_{ij1}, q_{ij2}, q_{ij3}), \qquad 1 \leq i,j \leq 3, \tag{4}$$

$$\mathbb{Q}_i = (q_{ijk})_{j,k=1}^3, \qquad 1 \leq i \leq 3, \tag{5}$$

$$\mathbb{Q}_{\mathbf{x}} = \mathbb{Q}_1 x_1 + \mathbb{Q}_2 x_2 + \mathbb{Q}_3 x_3, \qquad \forall\, \mathbf{x} \in \mathbb{S}^2. \tag{6}$$

Remark 2.1. It is worth mentioning that $\mathbb{Q}_{\mathbf{e}_i} = \mathbb{Q}_i$ for any vertex $\mathbf{e}_i = (\delta_{i1}, \delta_{i2}, \delta_{i3})$ of the simplex $\mathbb{S}^2$ where δ_{ij} is Kronecker's symbol.

It is easy to check that the QSO has the following vector and matrix forms

$$\mathcal{Q}(\mathbf{x}) = x_1^2\mathbf{q}_{11\bullet} + x_2^2\mathbf{q}_{22\bullet} + x_3^2\mathbf{q}_{33\bullet} + 2x_1x_2\mathbf{q}_{12\bullet} + 2x_1x_3\mathbf{q}_{13\bullet} + 2x_2x_3\mathbf{q}_{23\bullet},$$

$$\mathcal{Q}(\mathbf{x}) = \mathbf{x}\mathbb{Q}_{\mathbf{x}} = x_1 \cdot \mathbf{x}\mathbb{Q}_1 + x_2 \cdot \mathbf{x}\mathbb{Q}_2 + x_3 \cdot \mathbf{x}\mathbb{Q}_3$$

where $\mathbf{x} \in \mathbb{S}^2$ and $\mathbb{Q}_{\mathbf{x}} = \mathbb{Q}_1 x_1 + \mathbb{Q}_2 x_2 + \mathbb{Q}_3 x_3$ is a square stochastic matrix.

Since $\mathbf{q}_{ij\bullet} = \mathbf{q}_{ji\bullet}$ for all $1 \leq i,j \leq 3$, we have the following relation

$$\mathbb{Q}_{\mathbf{x}} = \mathbb{Q}_1 x_1 + \mathbb{Q}_2 x_2 + \mathbb{Q}_3 x_3 = \begin{pmatrix} \mathbf{x}\mathbb{Q}_1 \\ \mathbf{x}\mathbb{Q}_2 \\ \mathbf{x}\mathbb{Q}_3 \end{pmatrix} \tag{7}$$

where $\mathbf{x}\mathbb{Q}_1, \mathbf{x}\mathbb{Q}_2, \mathbf{x}\mathbb{Q}_3$ are respectively the first, the second, the third row vectors of $\mathbb{Q}_{\mathbf{x}}$.

Definition 2.1. The QSO $\mathcal{Q} : \mathbb{S}^2 \to \mathbb{S}^2$ given by (3) is said to be $\mathbf{p}$−majorizing with respect to a stochastic vector $\mathbf{p} > 0$ if one has $\mathbb{Q}_1, \mathbb{Q}_2, \mathbb{Q}_3 \in \mathsf{SM}[\mathbf{p}]$, i.e., the square stochastic matrices $\mathbb{Q}_1, \mathbb{Q}_2, \mathbb{Q}_3$ have a common fixed point $\mathbf{p} > 0$.

Definition 2.2. The QSO $\mathcal{Q} : \mathbb{S}^2 \to \mathbb{S}^2$ given by (3) is said to be scrambling (resp. positive) if one has $\mathbb{Q}_1, \mathbb{Q}_2, \mathbb{Q}_3 \in \mathsf{SSM}$, (resp. $\mathbb{Q}_1, \mathbb{Q}_2, \mathbb{Q}_3 > 0$) i.e., the square stochastic matrices $\mathbb{Q}_1, \mathbb{Q}_2, \mathbb{Q}_3$ are scrambling (resp. positive).

Remark 2.2. It is worth mentioning that the QSO $\mathcal{Q} : \mathbb{S}^2 \to \mathbb{S}^2$ given by (3) is $\mathbf{c}$−majorizing with respect to $\mathbf{c} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ if and only if the square matrices $\mathbb{Q}_1, \mathbb{Q}_2, \mathbb{Q}_3$ are doubly stochastic and $\mathbb{Q}_1 + \mathbb{Q}_2 + \mathbb{Q}_3 = \mathbb{E}$ where $\mathbb{E} = (1)_{j,k=1}^3$ is a square matrix having entries only 1. In this case, the QSO is called doubly stochastic (see [2, 4]). The dynamics of such kind of operators were studied in the papers [5, 13, 14, 15, 16] and the results of this paper generalize all results of the papers [5, 13, 14, 15, 16].

3. Some Examples

In this section, we provide some examples for $\mathbf{p}-$majorizing QSO on 2D simplex.

Example 3.1. Let $\mathbf{p} \in \mathbb{S}^2$ and $\mathbf{p} > 0$. Without loss of generality, we may assume that $0 < p_3 \leq p_2 \leq p_1 < 1$. Let $s = p_2 + p_3$ and $t = \frac{p_3}{p_2}$. It is clear that $0 < s, t \leq 1$. Let us define the following set

$$\mathbb{S}_{\mathbf{p}} = \big\{\mathbf{x} \in \mathbb{S}^2 : \quad 0 < x_3 < st, \quad 0 \vee (s-t) < x_2 < s, \\ 0 \vee [p_1(1+t) - t] < x_1 < 1 \wedge [p_1(1+t)]\big\},$$

where $a \vee b = \max\{a, b\}$ and $a \wedge b = \min\{a, b\}$. The set $\mathbb{S}_{\mathbf{p}} \subset \mathbb{S}^2$ is nonempty.

For any vector $\mathbf{q}_1 \in \mathbb{S}_{\mathbf{p}}$, we define the following vectors

$$\mathbf{q}_3 = \frac{p_2 + p_3}{p_3}\mathbf{p} - \frac{p_2}{p_3}\mathbf{q}_1, \qquad \mathbf{q}_2 = (1 - \frac{p_3}{p_2})\mathbf{q}_1 + \frac{p_3}{p_2}\mathbf{q}_3.$$

It is easy to see that $\mathbf{q}_3, \mathbf{q}_2 \in \mathbb{S}^2$. By means of stochastic vectors $\mathbf{q_1}, \mathbf{q_2}, \mathbf{q_3}, \mathbf{p}$, we define the following square stochastic matrices

$$\mathbb{Q}_1 = \begin{pmatrix} \mathbf{p} \\ \mathbf{p} \\ \mathbf{p} \end{pmatrix}, \qquad \mathbb{Q}_2 = \begin{pmatrix} \mathbf{p} \\ \mathbf{q}_2 \\ \mathbf{q}_1 \end{pmatrix}, \qquad \mathbb{Q}_3 = \begin{pmatrix} \mathbf{p} \\ \mathbf{q}_1 \\ \mathbf{q}_3 \end{pmatrix}$$

where $\mathbf{p}$ is the first, the second, and the third row vectors of $\mathbb{Q}_1$ and so on. Due to the construction of stochastic vectors $\mathbf{q_1}, \mathbf{q_2}, \mathbf{q_3}$, the square stochastic matrices $\mathbb{Q}_1, \mathbb{Q}_2, \mathbb{Q}_3$ have a common fixed point $\mathbf{p}$, i.e, $\mathbb{Q}_1, \mathbb{Q}_2, \mathbb{Q}_3 \in \mathsf{SM}[\mathbf{p}]$. Consequently, by means of the square stochastic matrices $\mathbb{Q}_1, \mathbb{Q}_2, \mathbb{Q}_3$, we define QSO $\mathcal{Q}_{\mathbf{q_1},\mathbf{q_2},\mathbf{q_3}} : \mathbb{S}^2 \to \mathbb{S}^2$ as follows

$$\mathcal{Q}_{\mathbf{q_1},\mathbf{q_2},\mathbf{q_3}}(\mathbf{x}) = x_1^2\mathbf{p} + x_2^2\mathbf{q}_2 + x_3^2\mathbf{q}_3 + 2x_1x_2\mathbf{p} + 2x_1x_3\mathbf{p} + 2x_2x_3\mathbf{q}_1.$$

This operator is the $\mathbf{p}-$majorizing QSO.

Example 3.2. Let $\mathbf{p} \in \mathbb{S}^2$ and $\mathbf{p} > 0$. Without loss of generality, we may assume that $0 < p_3 \leq p_2 \leq p_1 < 1$. Let $\mathbf{e} = \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3 = (1, 1, 1)$. We first choose any stochastic vector $\mathbf{r}_3 \in \mathbb{S}^2$. Since $p_3 \leq p_2 \leq p_1$, we have that

$$0 \leq p_3(\mathbf{e} - \mathbf{r}_3) = p_3\mathbf{e} - p_3\mathbf{r}_3 \leq \mathbf{p} - p_3\mathbf{r}_3 \leq \mathbf{p} < \mathbf{e}.$$

Therefore, $\mathbf{r} = \frac{1}{1-p_3}\mathbf{p} - \frac{p_3}{1-p_3}\mathbf{r}_3 \in \mathbb{S}^2$ is a stochastic vector. We define the following stochastic vectors and matrices

$$\mathbf{r}_1 = \frac{p_3}{p_1}\mathbf{r}_3 + (1 - \frac{p_3}{p_1})\mathbf{r}, \qquad \mathbf{r}_2 = \frac{p_3}{p_2}\mathbf{r}_3 + (1 - \frac{p_3}{p_2})\mathbf{r},$$

$$\mathbb{Q}_1 = \begin{pmatrix} \mathbf{r}_1 \\ \mathbf{r} \\ \mathbf{r} \end{pmatrix}, \qquad \mathbb{Q}_2 = \begin{pmatrix} \mathbf{r} \\ \mathbf{r}_2 \\ \mathbf{r} \end{pmatrix}, \qquad \mathbb{Q}_3 = \begin{pmatrix} \mathbf{r} \\ \mathbf{r} \\ \mathbf{r}_3 \end{pmatrix},$$

where $\mathbf{r}_1, \mathbf{r}$, and $\mathbf{r}$ are respectively the first, the second, the third row vectors of $\mathbb{Q}_1$ and so on. Due to the construction of stochastic vectors $\mathbf{r_1}, \mathbf{r_2}, \mathbf{r_3}, \mathbf{r}$, the square stochastic matrices $\mathbb{Q}_1, \mathbb{Q}_2, \mathbb{Q}_3$ have a common fixed point $\mathbf{p}$, i.e, $\mathbb{Q}_1, \mathbb{Q}_2, \mathbb{Q}_3 \in \mathsf{SM}[\mathbf{p}]$. Consequently, by means of the square stochastic matrices $\mathbb{Q}_1, \mathbb{Q}_2, \mathbb{Q}_3$, we define an operator $\mathcal{Q}_{\mathbf{r_1},\mathbf{r_2},\mathbf{r_3},\mathbf{r}} : \mathbb{S}^2 \to \mathbb{S}^2$ as follows

$$\mathcal{Q}_{\mathbf{r_1},\mathbf{r_2},\mathbf{r_3},\mathbf{r}}(\mathbf{x}) = x_1^2\mathbf{r}_1 + x_2^2\mathbf{r}_2 + x_3^2\mathbf{r}_3 + 2(x_1x_2 + x_1x_3 + x_2x_3)\mathbf{r}$$

This operator is the $\mathbf{p}-$majorizing QSO.

These examples show that there are a plenty of $\mathbf{p}-$majorizing QSO. The reader may refer to the paper [19] for some other examples.

4. Regularity of $\mathbf{p}$−Majorizing Quadratic Stochastic Operators

Let $\mathcal{Q} = (q_{ijk})_{i,j,k=1}^{3}$ be a cubic stochastic matrix and

$$\mathbb{Q}_i = (q_{ijk})_{j,k=1}^{3}, \quad 1 \leq i \leq 3, \qquad \mathbb{Q}_{\mathbf{x}} = \mathbb{Q}_1 x_1 + \mathbb{Q}_2 x_2 + \mathbb{Q}_3 x_3, \quad \forall\, \mathbf{x} \in \mathbb{S}^2.$$

Proposition 4.1. *The following statements hold true*

(i) *One has that* $\mathbb{Q}_{\lambda\mathbf{x}+(1-\lambda)\mathbf{y}} = \lambda\mathbb{Q}_{\mathbf{x}} + (1-\lambda)\mathbb{Q}_{\mathbf{y}}$ *for any* $\lambda \in [0,1]$ *and* $\mathbf{x}, \mathbf{y} \in \mathbb{S}^2$;

(ii) *One has that* $\mathbf{x}\mathbb{Q}_{\mathbf{y}} = \mathbf{y}\mathbb{Q}_{\mathbf{x}}$ *for any* $\mathbf{x}, \mathbf{y} \in \mathbb{S}^2$.

Proof. It is clear that

$$\begin{aligned}
\mathbb{Q}_{\lambda\mathbf{x}+(1-\lambda)\mathbf{y}} &= \mathbb{Q}_1(\lambda x_1 + (1-\lambda)y_1) + \mathbb{Q}_2(\lambda x_2 + (1-\lambda)y_2) + \mathbb{Q}_3(\lambda x_3 + (1-\lambda)y_3) \\
&= \lambda\left(\mathbb{Q}_1 x_1 + \mathbb{Q}_2 x_2 + \mathbb{Q}_3 x_3\right) + (1-\lambda)\left(\mathbb{Q}_1 y_1 + \mathbb{Q}_2 y_2 + \mathbb{Q}_3 y_3\right) \\
&= \lambda\mathbb{Q}_{\mathbf{x}} + (1-\lambda)\mathbb{Q}_{\mathbf{y}}.
\end{aligned}$$

Moreover, due to formula (7), we have that

$$\begin{aligned}
\mathbf{x}\mathbb{Q}_{\mathbf{y}} = \mathbf{x}\begin{pmatrix} \mathbf{y}\mathbb{Q}_1 \\ \mathbf{y}\mathbb{Q}_2 \\ \mathbf{y}\mathbb{Q}_3 \end{pmatrix} &= x_1 \cdot \mathbf{y}\mathbb{Q}_1 + x_2 \cdot \mathbf{y}\mathbb{Q}_2 + x_3 \cdot \mathbf{y}\mathbb{Q}_3 \\
&= \mathbf{y}\left(\mathbb{Q}_1 x_1 + \mathbb{Q}_2 x_2 + \mathbb{Q}_3 x_3\right) = \mathbf{y}\mathbb{Q}_{\mathbf{x}}
\end{aligned}$$

This completes the proof. □

Let $\mathcal{Q} : \mathbb{S}^2 \to \mathbb{S}^2$ be the QSO associated with a cubic stochastic matrix $\mathcal{Q} = (q_{ijk})_{i,j,k=1}^{3}$. For any $\mathbf{x}^{(0)} \in \mathbb{S}^2$, we define the trajectory $\{\mathbf{x}^{(n)}\}$ of the QSO starting from $\mathbf{x}^{(0)}$ as follows

$$\mathbf{x}^{(n)} = \mathcal{Q}(\mathbf{x}^{(n-1)}) = \mathbf{x}^{(n-1)}\mathbb{Q}_{\mathbf{x}^{(n-1)}}, \quad \forall \;\; n \in \mathbb{N}.$$

Recall that a mapping $\varphi : \mathbb{S}^2 \to \mathbb{R}$ is called a Lyapunov functional if $\left\{\varphi\left(\mathbf{x}^{(n)}\right)\right\}$ is a decreasing sequence for any $\mathbf{x}^{(0)} \in \mathbb{S}^2$. Let $int\,\mathbb{S}^2 = \{\mathbf{x} \in \mathbb{S}^2 : \ \mathbf{x} > 0\}$ and $\partial\mathbb{S}^2 = \{\mathbf{x} \in \mathbb{S}^2 : \ x_1x_2x_3 = 0\}$.

Proposition 4.2. *Let* $\mathcal{Q} : \mathbb{S}^2 \to \mathbb{S}^2$ *be the* $\mathbf{p}$*−majorizing QSO with respect to a stochastic vector* $\mathbf{p} > 0$*. Then the following statements hold true.*

(i) *One has that* $\mathcal{Q}(int\,\mathbb{S}^2) \subset int\,\mathbb{S}^2$;

(ii) $\varphi_{\mathbf{p}}(\mathbf{x}) = |x_1 - p_1| + |x_2 - p_2| + |x_3 - p_3|$ *is a Lyapunov functional;*

Proof. Let $\mathcal{Q} : \mathbb{S}^2 \to \mathbb{S}^2$ be the $\mathbf{p}$−majorizing QSO with respect to a stochastic vector $\mathbf{p} > 0$. This means that $\mathbb{Q}_1, \mathbb{Q}_2, \mathbb{Q}_3 \in \mathsf{SM}[\mathbf{p}]$.

(i) We then have

$$\mathbb{Q}_{\mathbf{p}} = \mathbb{Q}_1 p_1 + \mathbb{Q}_2 p_2 + \mathbb{Q}_3 p_3 = \begin{pmatrix} \mathbf{p}\mathbb{Q}_1 \\ \mathbf{p}\mathbb{Q}_2 \\ \mathbf{p}\mathbb{Q}_3 \end{pmatrix} = \begin{pmatrix} \mathbf{p} \\ \mathbf{p} \\ \mathbf{p} \end{pmatrix} > 0.$$

Since $\mathbf{p} > 0$, we get that $\mathbb{Q}_1 + \mathbb{Q}_2 + \mathbb{Q}_3 > 0$. Moreover, we obtain that $\mathbb{Q}_{\mathbf{x}} > 0$ for any $\mathbf{x} > 0$. This means that, for any $\mathbf{x} \in \mathbb{S}^2$ with $\mathbf{x} > 0$, the stochastic matrix $\mathbb{Q}_{\mathbf{x}}$ is positive. Consequently, we have that $\mathcal{Q}(\mathbf{x}) = \mathbf{x}\mathbb{Q}_{\mathbf{x}} > 0$ for any $\mathbf{x} > 0$. This means that $\mathcal{Q}(int\,\mathbb{S}^2) \subset int\,\mathbb{S}^2$.

(ii) It is clear that

$$\mathbf{p}\mathbb{Q}_{\mathbf{x}} = \mathbf{x}\mathbb{Q}_{\mathbf{p}} = \mathbf{x}\begin{pmatrix} \mathbf{p} \\ \mathbf{p} \\ \mathbf{p} \end{pmatrix} = \mathbf{p}.$$

Therefore, the stochastic matrix $\mathbb{Q}_{\mathbf{x}}$ has a fixed point $\mathbf{p}$, i.e., $\mathbb{Q}_{\mathbf{x}} \in \mathsf{SM}[\mathbf{p}]$ for any $\mathbf{x} \in \mathbb{S}^2$.

Since $\mathbf{x}^{(n+1)} = \mathbf{x}^{(n)}\mathbb{Q}_{\mathbf{x}^{(n)}}$, due to Theorem 1.1, ($i$) and ($ii$), we get that

$$|x_1^{(n+1)} - tp_1| + |x_2^{(n+1)} - tp_2| + |x_3^{(n+1)} - tp_3| \leq |x_1^{(n)} - tp_1| + |x_2^{(n)} - tp_2| + |x_3^{(n)} - tp_3|$$

for any $t \in \mathbb{R}$. If we let $t = 1$ then we obtain that $\varphi_{\mathbf{p}}(\mathbf{x}^{(n+1)}) \leq \varphi_{\mathbf{p}}(\mathbf{x}^{(n)})$ for any $n \in \mathbb{N}$. This means that $\varphi_{\mathbf{p}}(\mathbf{x})$ is the Lyapunov functional. This completes the proof. $\square$

Let $\mathbf{e}_i = (\delta_{i1}, \delta_{i2}, \delta_{i3})$ and $\mathbf{e}_i^{(n+1)} = \mathbf{e}_i^{(n)} \mathbb{Q}_{\mathbf{e}_i^{(n)}}$, where δ_{ij} is Kronecker's delta symbol.

Theorem 4.1. *Let $\mathcal{Q} : \mathbb{S}^2 \to \mathbb{S}^2$ be the $\mathbf{p}-$majorizing QSO with respect to a stochastic vector $\mathbf{p} > 0$. Then the following conditions are mutually equivalent:*

- (i) *One has $\mathbf{e}_1^{(k)}, \mathbf{e}_2^{(k)}, \mathbf{e}_3^{(k)} \in int\, \mathbb{S}^2$ for some $k \in \mathbb{N}$;*
- (ii) *One has $\overline{\mathcal{Q}^{(k)}\left(\mathbb{S}^2\right)} \subset int\, \mathbb{S}^2$ for some $k \in \mathbb{N}$;*
- (iii) *One has $\overline{\mathcal{Q}^{(k)}\left(int\, \mathbb{S}^2\right)} \subset int\, \mathbb{S}^2$ for some $k \in \mathbb{N}$;*
- (iv) *The trajectory $\{\mathbf{x}^{(n)}\}$ starting from any initial point $\mathbf{x}^{(0)} \in \mathbb{S}^2$ converges to the unique fixed point $\mathbf{p}$.*

Proof. Let $\mathcal{Q} : \mathbb{S}^2 \to \mathbb{S}^2$ be the $\mathbf{p}-$majorizing QSO with respect to a stochastic vector $\mathbf{p} > 0$. This means that $\mathbb{Q}_1, \mathbb{Q}_2, \mathbb{Q}_3 \in \mathsf{SM}[\mathbf{p}]$. We will prove the following implications $(i) \Rightarrow (ii) \Leftrightarrow (iii) \Rightarrow (iv) \Rightarrow (i)$.

$(i) \Rightarrow (ii)$. Let $\mathbf{e}_1^{(k)}, \mathbf{e}_2^{(k)}, \mathbf{e}_3^{(k)} \in int\, \mathbb{S}^2$ for some $k \in \mathbb{N}$, where $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ are vertices of the simplex $\mathbb{S}^2$. Since $\mathbb{Q}_{\lambda\mathbf{x}+(1-\lambda)\mathbf{y}} = \lambda\mathbb{Q}_{\mathbf{x}} + (1-\lambda)\mathbb{Q}_{\mathbf{y}}$ for any $\lambda \in [0,1]$ and $\mathbf{x}, \mathbf{y} \in \mathbb{S}^2$, we have

$$
\begin{aligned}
\mathbf{x}^{(1)} &= \mathbf{x}^{(0)}\mathbb{Q}_{\mathbf{x}^{(0)}} = x_1^{(0)}\mathbf{e}_1\mathbb{Q}_{\mathbf{x}^{(0)}} + x_2^{(0)}\mathbf{e}_2\mathbb{Q}_{\mathbf{x}^{(0)}} + x_3^{(0)}\mathbf{e}_3\mathbb{Q}_{\mathbf{x}^{(0)}} \\
&= \left(x_1^{(0)}\right)^2 \mathbf{e}_1\mathbb{Q}_{\mathbf{e}_1} + x_1^{(0)}x_2^{(0)}\mathbf{e}_1\mathbb{Q}_{\mathbf{e}_2} + x_1^{(0)}x_3^{(0)}\mathbf{e}_1\mathbb{Q}_{\mathbf{e}_3} + x_1^{(0)}x_2^{(0)}\mathbf{e}_2\mathbb{Q}_{\mathbf{e}_1} + \left(x_2^{(0)}\right)^2 \mathbf{e}_2\mathbb{Q}_{\mathbf{e}_2} \\
&\qquad + x_2^{(0)}x_3^{(0)}\mathbf{e}_2\mathbb{Q}_{\mathbf{e}_3} + x_1^{(0)}x_3^{(0)}\mathbf{e}_3\mathbb{Q}_{\mathbf{e}_1} + x_2^{(0)}x_3^{(0)}\mathbf{e}_3\mathbb{Q}_{\mathbf{e}_2} + \left(x_3^{(0)}\right)^2 \mathbf{e}_3\mathbb{Q}_{\mathbf{e}_3}.
\end{aligned}
$$

Let $\mathbf{e}_{ij}^{(00)} = \mathbf{e}_i\mathbb{Q}_{\mathbf{e}_j}$ for any $i \neq j$ for which $\mathbf{e}_{ij}^{(00)} = \mathbf{e}_{ji}^{(00)}$ for all i, j. We then obtain that

$$
\begin{aligned}
\mathbf{x}^{(1)} &= \left(x_1^{(0)}\right)^2 \mathbf{e}_1^{(1)} + \left(x_2^{(0)}\right)^2 \mathbf{e}_2^{(1)} + \left(x_3^{(0)}\right)^2 \mathbf{e}_3^{(1)} + x_1^{(0)}x_2^{(0)}\mathbf{e}_{12}^{(00)} + x_1^{(0)}x_2^{(0)}\mathbf{e}_{21}^{(00)} \\
&\qquad + x_1^{(0)}x_3^{(0)}\mathbf{e}_{13}^{(00)} + x_1^{(0)}x_3^{(0)}\mathbf{e}_{31}^{(00)} + x_2^{(0)}x_3^{(0)}\mathbf{e}_{23}^{(00)} + x_2^{(0)}x_3^{(0)}\mathbf{e}_{32}^{(00)}.
\end{aligned}
$$

Similarly, we may get that

$$
\begin{aligned}
\mathbf{x}^{(2)} &= \mathbf{x}^{(1)}\mathbb{Q}_{\mathbf{x}^{(1)}} = \left(x_1^{(0)}\right)^2 \mathbf{e}_1^{(1)}\mathbb{Q}_{\mathbf{x}^{(1)}} + \left(x_2^{(0)}\right)^2 \mathbf{e}_2^{(1)}\mathbb{Q}_{\mathbf{x}^{(1)}} + \left(x_3^{(0)}\right)^2 \mathbf{e}_3^{(1)}\mathbb{Q}_{\mathbf{x}^{(1)}} + \cdots \\
&= \left(x_1^{(0)}\right)^4 \mathbf{e}_1^{(1)}\mathbb{Q}_{\mathbf{e}_1^{(1)}} + \left(x_1^{(0)}\right)^2 \left(x_2^{(0)}\right)^2 \mathbf{e}_1^{(1)}\mathbb{Q}_{\mathbf{e}_2^{(1)}} + \left(x_1^{(0)}\right)^2 \left(x_3^{(0)}\right)^2 \mathbf{e}_1^{(1)}\mathbb{Q}_{\mathbf{e}_3^{(1)}} + \cdots \\
&\quad + \left(x_1^{(0)}\right)^2 \left(x_2^{(0)}\right)^2 \mathbf{e}_2^{(1)}\mathbb{Q}_{\mathbf{e}_1^{(1)}} + \left(x_2^{(0)}\right)^4 \mathbf{e}_2^{(1)}\mathbb{Q}_{\mathbf{e}_2^{(1)}} + \left(x_2^{(0)}\right)^2 \left(x_3^{(0)}\right)^2 \mathbf{e}_2^{(1)}\mathbb{Q}_{\mathbf{e}_3^{(1)}} + \cdots \\
&\quad + \left(x_1^{(0)}\right)^2 \left(x_3^{(0)}\right)^2 \mathbf{e}_3^{(1)}\mathbb{Q}_{\mathbf{e}_1^{(1)}} + \left(x_2^{(0)}\right)^2 \left(x_3^{(0)}\right)^2 \mathbf{e}_3^{(1)}\mathbb{Q}_{\mathbf{e}_2^{(1)}} + \left(x_3^{(0)}\right)^4 \mathbf{e}_3^{(1)}\mathbb{Q}_{\mathbf{e}_3^{(1)}} + \cdots .
\end{aligned}
$$

Therefore, we have that

$$
\mathbf{x}^{(2)} = \left(x_1^{(0)}\right)^4 \mathbf{e}_1^{(2)} + \left(x_2^{(0)}\right)^4 \mathbf{e}_2^{(2)} + \left(x_3^{(0)}\right)^4 \mathbf{e}_3^{(2)} + \cdots . \tag{8}
$$

Analogously, we can show for any $n \in \mathbb{N}$ that

$$
\mathbf{x}^{(n)} = \left(x_1^{(0)}\right)^{2^n} \mathbf{e}_1^{(n)} + \left(x_2^{(0)}\right)^{2^n} \mathbf{e}_2^{(n)} + \left(x_3^{(0)}\right)^{2^n} \mathbf{e}_3^{(n)} + \cdots . \tag{9}
$$

Hence, due to the formula (9), if one has $\mathbf{e}_1^{(k)}, \mathbf{e}_2^{(k)}, \mathbf{e}_3^{(k)} \in int\, \mathbb{S}^2$ for some $k \in \mathbb{N}$ then so does $\mathbf{x}^{(k)} \in int\, \mathbb{S}^2$ for any $\mathbf{x}^{(0)} \in \mathbb{S}^2$. This means that $\mathcal{Q}^{(k)}\left(\mathbb{S}^2\right) \subset int\, \mathbb{S}^2$.

$(ii) \Leftrightarrow (iii)$. Since $int\,\mathbb{S}^2 \subset \mathbb{S}^2$, $\mathcal{Q}$ is continuous and $\mathbb{S}^2$ is compact (so does $\mathcal{Q}^{(k)}\left(\mathbb{S}^2\right)$), we get that

$$\overline{\mathcal{Q}^{(k)}\left(int\,\mathbb{S}^2\right)} \subset \overline{\mathcal{Q}^{(k)}\left(\mathbb{S}^2\right)} = \mathcal{Q}^{(k)}\left(\mathbb{S}^2\right) = \mathcal{Q}^{(k)}\left(\overline{int\,\mathbb{S}^2}\right) \subset \overline{\mathcal{Q}^{(k)}\left(int\,\mathbb{S}^2\right)}.$$

Therefore, one has $\mathcal{Q}^{(k)}\left(\mathbb{S}^2\right) \subset int\,\mathbb{S}^2$ if and only if $\overline{\mathcal{Q}^{(k)}\left(int\,\mathbb{S}^2\right)} \subset int\,\mathbb{S}^2$.

$(iii) \Rightarrow (iv)$. Let $\overline{\mathcal{Q}^{(k)}\left(int\,\mathbb{S}^2\right)} \subset int\,\mathbb{S}^2$ (or equivalently $\mathcal{Q}^{(k)}\left(\mathbb{S}^2\right) \subset int\,\mathbb{S}^2$) for some $k \in \mathbb{N}$. Then one has $\mathcal{Q}^{(n)}\left(\mathbb{S}^2\right) \subset int\,\mathbb{S}^2$ for any $n \geq k$. On the other hand, we have the following inclusion

$$\cdots \subset \mathcal{Q}^{(n)}\left(\mathbb{S}^2\right) \subset \cdots \subset \mathcal{Q}^{(3)}\left(\mathbb{S}^2\right) \subset \mathcal{Q}^{(2)}\left(\mathbb{S}^2\right) \subset \mathcal{Q}\left(\mathbb{S}^2\right) \subset \mathbb{S}^2.$$

Hence, the nested closed (compact) sets

$$\cdots \subset \mathcal{Q}^{(k+2)}\left(\mathbb{S}^2\right) \subset \mathcal{Q}^{(k+1)}\left(\mathbb{S}^2\right) \subset \mathcal{Q}^{(k)}\left(\mathbb{S}^2\right) \subset int\,\mathbb{S}^2$$

are separated from the boundary $\partial\mathbb{S}^2$ of the simplex. Therefore, there exists $\alpha > 0$ such that $x_1^{(n)}, x_2^{(n)}, x_3^{(n)} > \alpha$ for any $n \geq k$. Consequently, we have $\omega\left(\mathbf{x}^{(n)}\right) \subset int\,\mathbb{S}^2$.

As we showed in the proof of Proposition 4.2, (i) that $\mathbb{Q}_{\mathbf{x}} > 0$ for any $\mathbf{x} > 0$. Particularly, $\mathbb{Q}_{\mathbf{x}^*} > 0$ for any $\mathbf{x}^* \in \omega\left(\mathbf{x}^{(n)}\right)$. Therefore, $\mathbf{p}$ is the unique fixed point of $\mathbb{Q}_{\mathbf{x}}$ for any $\mathbf{x} \in \mathbb{S}^2$. In its own turn, this means that $\mathbf{p}$ is also the unique fixed point of QSO in the interior of the simplex. Thus, we have that $\mathbb{Q}_{\mathbf{x}^*} \in \mathsf{SSM} \bigcap \mathsf{SM}[\mathbf{p}]$ for any $\mathbf{x}^* \in \omega\left(\mathbf{x}^{(n)}\right)$. On the other hand, since $\mathbb{Q}_{\mathbf{x}} = \mathbb{Q}_1 x_1 + \mathbb{Q}_2 x_2 + \mathbb{Q}_3 x_3$, we get that $\omega\left(\{\mathbb{Q}_{\mathbf{x}^{(n)}}\}\right) = \left\{\mathbb{Q}_{\omega(\mathbf{x}^{(n)})}\right\}$. Consequently, we obtain that $\omega\left(\{\mathbb{Q}_{\mathbf{x}^{(n)}}\}\right) \subset \mathsf{SSM} \bigcap \mathsf{SM}[\mathbf{p}]$. Due to Theorem 1.2, the sequence $\{\mathbb{Q}_{\mathbf{x}^{(n)}}\}$ is strongly ergodic. This means that the sequence $\left\{\mathbb{Q}^{[0,n]}\right\}$, where $\mathbb{Q}^{[0,n]} := \mathbb{Q}_{\mathbf{x}^{(0)}}\mathbb{Q}_{\mathbf{x}^{(1)}} \cdots \mathbb{Q}_{\mathbf{x}^{(n)}}$ converges to a stable matrix $\mathbb{Q}^*$ with identical rows $\mathbf{q}$.

On the other hand, we know that

$$\mathbf{x}^{(n+1)} = \mathbf{x}^{(n)}\mathbb{Q}_{\mathbf{x}^{(n)}} = \mathbf{x}^{(0)}\mathbb{Q}_{\mathbf{x}^{(0)}}\mathbb{Q}_{\mathbf{x}^{(1)}} \cdots \mathbb{Q}_{\mathbf{x}^{(n)}} = \mathbf{x}^{(0)}\mathbb{Q}^{[0,n]}.$$

Therefore, $\mathbf{x}^{(n)}$ converges to $\mathbf{q}$ which must be a fixed point. Since $\mathbf{p}$ is the unique fixed point, we get that $\mathbf{q} = \mathbf{p}$. Thus, the trajectory $\{\mathbf{x}^{(n)}\}$ starting from any point $\mathbf{x}^{(0)} \in \mathbb{S}^2$ converges to the unique fixed point $\mathbf{p}$.

$(iv) \Rightarrow (i)$. Let the trajectory $\{\mathbf{x}^{(n)}\}$ starting from any initial point $\mathbf{x}^{(0)} \in \mathbb{S}^2$ converge to the unique fixed point $\mathbf{p}$. If we choose $\mathbf{x}^{(\mathbf{0})} = \mathbf{e}_i$, where $i = 1, 2, 3$ then $\mathbf{e}_i^{(n)}$ converges to $\mathbf{p}$. Since $\mathbf{p} > 0$, there is k_i (depending on i) such that $\mathbf{e}_i^{(k_i)} > 0$. We know that $\mathcal{Q}(int\,\mathbb{S}^2) \subset int\,\mathbb{S}^2$. Therefore, $\mathbf{e}_i^{(k)} > 0$ for all $i = 1, 2, 3$ where $k = \max\{k_1, k_2, k_3\}$, i.e., $\mathbf{e}_i^{(k)} \in int\,\mathbb{S}^2$ for all $i = 1, 2, 3$. This completes the proof. □

Remark 4.1. Let $\mathcal{L}(\mathbf{x}) = \mathbf{x}\mathbb{P}$ be a linear stochastic operator associated with a square stochastic matrix $\mathbb{P}$. We say that $\mathcal{L}$ is $\mathbf{p}-$majorizing with respect to a stochastic vector $\mathbf{p} > 0$ if $\mathbf{p}\mathbb{P} = \mathbf{p}$. The classical result in the Markov chain theory states that the trajectory of the $\mathbf{p}-$majorizing linear stochastic operator starting from any initial point converges to the fixed point $\mathbf{p} > 0$ if and only if there exists $k \in \mathbb{N}$ such that $\mathbf{e}_1^{(k)}, \mathbf{e}_2^{(k)}, \mathbf{e}_3^{(k)} \in int\,\mathbb{S}^2$. Theorem 4.1 is a generalization of this result in the nonlinear setting.

Corollary 4.1 ([12])**.** *Let $\mathcal{Q} : \mathbb{S}^2 \to \mathbb{S}^2$ be a scrambling (positive) $\mathbf{p}-$majorizing QSO with respect to a stochastic vector $\mathbf{p} > 0$. Then its trajectory $\{\mathbf{x}^{(n)}\}$ starting from any initial point $\mathbf{x}^{(0)} \in \mathbb{S}^2$ converges to the unique fixed point $\mathbf{p}$.*

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