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ON A LIMIT BEHAVIOR OF A RANDOM WALK WITH MODIFICATIONS UPON EACH VISIT TO ZERO

We consider the limit behavior of a one-dimensional random walk with unit jumps whose transition probabilities are modified every time the walk hits zero. The invariance principle is proved in the scheme of series where the size of modifications depends on the number of series. For the natural scaling of time and space arguments the limit process is (i) a Brownian motion if modifications are "small", (ii) a linear motion with a random slope if modifications are "large", and (iii) the limit process satisfies an SDE with a local time of unknown process in a drift if modifications are "moderate".

1. Introduction and Main Results

Consider a random walk $\{X_k,\ k \geq 0\}$ on $\mathbb{Z}$ with unit jumps that is constructed in the following way. It behaves as a symmetric random walk until the first visit to 0. After that the probability of the jump to the right becomes equal to $p_1 := 1/2 + \Delta$, and to the left $q_1 := 1/2 - \Delta$, where $\Delta > 0$ is a fixed number. When $\{X_k\}$ secondly visits 0 its transition probabilities to the right and to the left become equal to $p_2 := 1/2 + 2\Delta$ and $q_2 := 1/2 - 2\Delta$, respectively, etc. (if $1/2 + i\Delta > 1$ we set $p_i := 1$).

Let us give the formal definition.

Definition 1.1. A random sequence $\{X_k,\ k \geq 0\}$ with values in $\mathbb{Z}$ is called a random walk with modifications (RWM) upon visits to 0 if

$$\forall k \geq 1\ \forall i_0, i_1, \ldots, i_k,\ |i_{j+1} - i_j| = 1$$

$$\mathrm{P}(X_{k+1} = i_k + 1 \mid X_0 = i_0, X_1 = i_1, \ldots, X_k = i_k) = (\frac{1}{2} + \nu_k \Delta) \wedge 1,$$

$$\mathrm{P}(X_{k+1} = i_k - 1 \mid X_0 = i_0, X_1 = i_1, \ldots, X_k = i_k) = (\frac{1}{2} - \nu_k \Delta) \vee 0,$$

where $\nu_k = |\{j \in \overline{0,k}\ :\ X_j = 0\}| = \sum_{j=0}^k 1\!\!1_{\{X_j = 0\}}$ is the number of visits to 0.

The number $\Delta > 0$ is called the size of modifications.

Set $\mathcal{F}_k := \sigma(X_0, X_1, \ldots, X_k)$. The previous definition is equivalent to

$$P(X_{k+1} = X_k + 1 \mid \mathcal{F}_k) = (\frac{1}{2} + \nu_k \Delta) \wedge 1,$$

$$P(X_{k+1} = X_k - 1 \mid \mathcal{F}_k) = (\frac{1}{2} - \nu_k \Delta) \vee 0.$$

Remark 1.1. The usual random walk with unit jumps and fixed transition probabilities p and $(1-p)$ is non-recurrent if $p \neq 1/2$. So, $1/2 + \nu_\infty \Delta < 1$ with positive probability, where $\nu_\infty := |\{k \geq 0\ :\ X_k = 0\}|$.

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The aim of the paper is to study the limit behavior of the sequence of series $\{X_k^{(n)}\}$ where the size of modifications in the n-th series $\Delta_n \to 0$ as $n \to \infty$.

It is well-known that if $\Delta = 0$, i.e., if $\{X_k\}$ is a symmetric random walk with the unit jumps, then the sequence of processes $\{\frac{X_{[n\cdot]}}{\sqrt{n}}\}$ converges in distribution to a Brownian motion in the space $D([0,\infty))$. So, it is natural to expect that if $\Delta_n \to 0$ fast enough, then the limit of $\{\frac{X^{(n)}_{[n\cdot]}}{\sqrt{n}}\}$ will be a Brownian motion too.

On the other hand, if $\{Y_k\}$ is a random walk with $p_{i,i+1} = p, p_{i,i-1} = 1-p$, then by the law of large numbers we have a.s. convergence

$$\lim_{n\to\infty} \frac{Y_{[nt]}}{n} = (1-2p)t \tag{1}$$

for fixed $t \ge 0$ (and even uniformly on compact sets). Hence, if $\Delta_n \to 0$ "slowly", there is a possibility that some scaling of $X^{(n)}_{[nt]}$ converges to non-zero linear process with a random slope.

The main result of the paper is the following theorem.

Theorem 1.1. *Let $\Delta_n = \frac{c}{n^\alpha}$, where $c > 0, \alpha > 0$. Assume that $X_0^{(n)} = 0$ for all n, and $X_k^{(n)}$ is extended to all $t \ge 0$ by linearity*

$$X_t^{(n)} := X_{[t]}^{(n)} + (X_{[t+1]}^{(n)} - X_{[t]}^{(n)})(t - [t]).$$

- *If $\alpha > 1$, then*

$$\frac{X_{nt}^{(n)}}{\sqrt{n}} \Rightarrow W(t),\ n \to \infty, \tag{2}$$

  where W is a Brownian motion.
- *If $0 < \alpha < 1$, then*

$$\frac{X_{nt}^{(n)}}{n^{1-\frac{\alpha}{2}}} \Rightarrow 2\sqrt{c}\eta t,\ n \to \infty, \tag{3}$$

  where η is a non-negative random variable with the distribution function

$$\mathrm{P}(\eta \le x) = 1 - e^{-x^2}, \quad x \ge 0. \tag{4}$$

- *If $\alpha = 1$, then*

$$\frac{X_{nt}^{(n)}}{\sqrt{n}} \Rightarrow X_\infty(t),\ n \to \infty,$$

  where X_∞ satisfies the SDE

$$X_\infty(t) = 2c\int_0^t l^0_{X_\infty}(s)ds + W(t), t \ge 0, \tag{5}$$

  $l^0_{X_\infty}(t) = \lim_{\varepsilon\to 0+} \frac{1}{2\varepsilon}\int_0^t 1\!\!1_{|X_\infty(s)|\le\varepsilon} ds$ is the local time of X_∞ at 0.

  Here $\Rightarrow$ denotes the weak convergence in the space $C([0,\infty))$.

Remark 1.2. The case $X_0^{(n)} = x_n$ can be treated with the natural modifications.

Remark 1.3. Equation (5) has a unique weak solution due to Girsanov's theorem.

Remark 1.4. The fact that the case $\alpha = 1$ is the critical one can be guessed by the following non-rigorous observations. In some sense the sequence $\{X_k^{(n)}\}$ visits to 0 more rarely than the symmetric random walk with the unit jump (it may not return at all). The number of visits to 0 by the symmetric random walk has a rate $\sqrt{n}$. So, if $\alpha > 1$, then

$$\max_{k=\overline{0,n}} |(\frac{1}{2} + \nu_k\Delta_n) - \frac{1}{2}| = |(\frac{1}{2} + \nu_n\Delta_n) - \frac{1}{2}| = \nu_n\Delta_n = \frac{c\nu_n}{n^\alpha} = O(n^{\frac{1}{2}-\alpha}),\ n \to \infty.$$

If all transition probabilities where constant, i.e.,

$$p_{i,i+1}^{X^{(n)}} = \frac{1}{2} + Kn^{\frac{1}{2}-\alpha},\ p_{i,i-1}^{X^{(n)}} = \frac{1}{2} - Kn^{\frac{1}{2}-\alpha},\ i \in \mathbb{Z}, \tag{6}$$

then it is not difficult to show (2). In some sense transition probabilities of RWM differ from 1/2 even less than above.

On the other hand, if $\alpha < 1$ and if transition probabilities are given in (6), then $\mathrm{E}X_n^{(n)}/\sqrt{n} \to \infty$ as $n \to \infty$. So $n^{\frac{1}{2}}$ is not a natural normalizing factor. We will show that the total number of returns to 0 has a rate $n^{\alpha/2}$ and the instant of the last return to 0 of the process $X_{nt}^{(n)}$ converges to 0 as $n \to \infty$. Therefore the natural choice for scaling is

$$n(\text{ steps }) \times \nu_\infty(\text{ number of modifications }) \times n^{-\alpha}(\text{ size of each modification }) \asymp$$

$$\asymp \frac{n \cdot n^{\alpha/2}}{n^\alpha} = n^{1-\alpha/2},$$

Since transition probabilities do not change after small amount of time (after the last return to 0), the limit process should be linear (compare with (1)).

Remark 1.5. RWM is not a Markov chain because transition probabilities depend on number of visits to 0. The process X_∞ from (5) is not a Markov process too. However the pairs $\{(X_k, \nu_k), k \ge 0\}$, $\{(X_\infty(t), l^0_{X_\infty}(t)), t \ge 0\}$ satisfy the Markov property.

Remark 1.6. For any a we have $\lim_{t\to\infty} \frac{at+W(t)}{t} = a$ a.s. Let X_∞ be a solution of (5). Since the local time is a non-decreasing non-negative function, it can be easily verified that if $\alpha = 1$, then

$$\lim_{t\to\infty} X_\infty(t) = +\infty \ \text{ a.s.}, \quad \mathrm{P}(\exists t_0\ \forall t \ge t_0\ :\ \ l^0_{X_\infty}(t) = l^0_{X_\infty}(t_0)) = 1,$$

and

$$\lim_{t\to\infty} \frac{X_\infty(t)}{t} = \lim_{t\to\infty} l^0_{X_\infty}(t) > 0 \ \text{ a.s.}$$

It can be seen from the proof that the distribution of $\lim_{t\to\infty} l^0_{X_\infty}(t)$ coincides with the distribution of $\sqrt{c}\eta$, where the distribution function of η is given in (4).

Remark 1.7. If the transition probabilities were perturbed only at 0 and would not be changed in time, say $p_{i,i\pm 1} = 1/2$ for $i \ne 0$, $p_{0,1} = p$, $p_{0,-1} = 1-p$, then the weak limit of $\{\frac{X_{nt}}{\sqrt{n}}\}$ may be the skew Brownian motion, i.e., a solution of the SDE

$$dX^{skew}(t) = (2p-1)dl^0_{X^{skew}}(t) + dW(t),$$

see [5] for this particular case, and [6, 9, 10, 11, 12, 15] for further generalizations.

RWM also resembles the multi-excited random walk (but does not equal) that is defined in the following way:

$$\mathrm{P}(X^{ex}_{k+1} = X^{ex}_k + 1 \mid X^{ex}_0, X^{ex}_1, \ldots, X^{ex}_k) =$$

$$1 - \mathrm{P}(X^{ex}_{k+1} = X^{ex}_k - 1 \mid X^{ex}_0, X^{ex}_1, \ldots, X^{ex}_k) = \frac{1}{2} + \varepsilon_{nj},$$

if $j = |\{0 \le i \le k,\ X^{ex}_i = X^{ex}_k\}|$ and $\{\varepsilon_{nj}\}$ are some (may be random) variables.

Under various assumptions on $\{\varepsilon_{nj}\}$ and scaling, limits of multi-excited random walk may be a linear process, or more intricate processes, for example the limits may be a solution of the following stochastic equation

$$dX^{ex}_\infty(t) = \varphi(l^{X^{ex}_\infty(t)}_{X^{ex}_\infty}(t))dt + dW(t)$$

or

$$X^{ex}_\infty(t) = \alpha \max_{s\in[0,t]} X^{ex}_\infty(s) - \beta \min_{s\in[0,t]} X^{ex}_\infty(s) + W(t),$$

see [2, 7, 13, 16] and references therein.

2. Auxiliary lemmas

Let $X_k = X_k^\Delta$ be an RWM, where the modification equals $\Delta > 0$. For simplicity assume that $X_0 = 0$.

Set $\nu_k = \nu_k^\Delta = |\{i = \overline{0,k} \ : \ X_i = 0\}|, \ \nu_\infty = \nu_\infty^\Delta = |\{i \geq 0 \ : \ X_i = 0\}|$.

Lemma 2.1. *For any $k \geq 1$ we have*

$$\mathrm{P}\left(\nu_\infty^\Delta > k\right) = \begin{cases} \prod_{i=1}^{k}(1-2i\Delta), & k \leq (2\Delta)^{-1}, \\ 0, & k > (2\Delta)^{-1}. \end{cases} \tag{7}$$

Proof. It is well known that if $\{S_k\}$ is a random walk with unit jumps, $p_{i,i+1} = 1-p_{i,i-1} = p$, then $\mathrm{P}(\exists k \geq 1 \ S_k = 0 \mid S_0 = 0) = 1 - |p-q| = 1 - |2p-1|$.

So, if $\frac{1}{2} + k\Delta \leq 1$, then

$$\mathrm{P}(\nu_\infty \geq k+1 \mid \nu_\infty \geq k) = 1 - \left(2(\frac{1}{2} + k\Delta) - 1\right) = 1 - 2k\Delta.$$

This implies (7). □

Lemma 2.2. *We have convergence in distribution*

$$\sqrt{\Delta}\nu_\infty^\Delta \Rightarrow \eta, \ \ \Delta \to 0+,$$

where η is a random variable with its distribution function given in (4)*.*

Proof. By the mean value theorem we have

$$\forall y \in (0, \frac{1}{2}) \quad \ln(1-y) = -y - \theta\frac{y^2}{2},$$

where $\theta \in (0,4)$. Let $x \geq 0$ be fixed. Then for some (another) $\theta \in (0,4)$ and $\Delta \leq (16x^2)^{-1}$ we have

$$\ln \mathrm{P}(\nu_\infty^\Delta > \frac{x}{\sqrt{\Delta}}) = \sum_{1 \leq i \leq \frac{x}{\sqrt{\Delta}}} \ln(1-2i\Delta) =$$

$$- \sum_{1 \leq i \leq \frac{x}{\sqrt{\Delta}}} 2i\Delta - \theta \sum_{1 \leq i \leq \frac{x}{\sqrt{\Delta}}} i^2\Delta^2. \tag{8}$$

Consider the first item in (8)

$$\sum_{1 \leq i \leq \frac{x}{\sqrt{\Delta}}} 2i\Delta = \left[\frac{x}{\sqrt{\Delta}}\right]\left(\left[\frac{x}{\sqrt{\Delta}}\right] + 1\right)\Delta \to x^2, \ \ \Delta \to 0+.$$

Consider the second item

$$0 \leq \sum_{1 \leq i \leq \frac{x}{\sqrt{\Delta}}} i^2\Delta^2 \leq \left(\frac{x}{\sqrt{\Delta}}\right)^3 \Delta^2 \to 0, \ \ \Delta \to 0+.$$

Lemma 2.2 is proved. □

Let T_k be the moment of kth return to 0, i.e., $T_1 = 0$ (recall that $X_0 = 0$), $T_k = \inf\{j > T_{k-1} \ : \ X_j = 0\}, \ k \geq 2$. We set by the definition that infimum over the empty set is equal to infinity.

Define $T_\infty := \sup\{k \geq 1 \ : \ X_k = 0\} = \sup\{T_k \ : \ T_k \neq \infty\}$.

Denote by $\tau_k = T_{k+1} - T_k$ the time between successive returns to 0 ($\infty - \infty := \infty$).

Lemma 2.3.

$$\mathrm{E}T_\infty = \sum_{k=1}^{\left[\frac{1}{2\Delta}\right]} \left(\frac{1}{2k\Delta} - 2k\Delta\right)\mathrm{P}(T_k < \infty).$$

Proof. Let $\{S_k\}$ be a random walk with unit jumps, $p_{i,i+1} = p,\ p_{i,i-1} = q = 1-p$, $S_0 = 0$, $\tau_S = \inf\{k \geq 1\ :\ S_k = 0\}$ be the moment of the first return to 0.

It follows from the definition of the RWM that the conditional distribution of τ_k given $\{T_k < \infty\}$ coincides with the distribution of τ_S if $p = p_k = (\frac{1}{2} + k\Delta) \wedge 1$.

Recall that the moment generating function of τ_S equals, see [3],

$$\mathrm{E}s^{\tau_S} 1\!\!1_{\tau_S<\infty} = 1 - \sqrt{1-4pqs^2}.$$

Therefore

$$\mathrm{E}\tau_S 1\!\!1_{\tau_S<\infty} = (1-\sqrt{1-4pqs^2})'|_{s=1} = \frac{8pqs}{2\sqrt{1-4pqs^2}}|_{s=1} = \frac{4pq}{\sqrt{1-4pq}} = \frac{4pq}{|p-q|}. \tag{9}$$

We have

$$T_\infty = \sum_{k=1}^{\left[\frac{1}{2\Delta}\right]} \tau_k 1\!\!1_{\tau_k<\infty} 1\!\!1_{T_k<\infty}.$$

The proof of Lemma 2.3 follows from (9).

□

3. The proof of (3)

Let $\{X_k^{(n)}\}$ be an RWM,

$$X_k^{(n)} = 0,\ \Delta_n = \frac{c}{n^\alpha},\ \alpha \in (0,1),\ T_\infty^{(n)} = \sup\{k \geq 1\ :\ X_k^{(n)} = 0\}.$$

It follows from Lemma 2.3 that

$$\mathrm{E}\frac{T_\infty^{(n)}}{n} \leq (2n)^{-1} \sum_{k=1}^{\left[\frac{1}{2\Delta_n}\right]} \frac{1}{k\Delta_n} = \tag{10}$$

$$(2n)^{-1} \sum_{k=1}^{\left[\frac{n^\alpha}{2c}\right]} \frac{n^\alpha}{ck} = (1+o(1))(2c)^{-1} n^{\alpha-1} \ln\left(\left[\frac{n^\alpha}{2c}\right]\right) \to 0,\ n \to \infty.$$

It follows from the definition of $\{X_k^{(n)}\}$ that

$$\mathrm{E}\left(X_{k+1}^{(n)} \,|\, \mathcal{F}_k\right) = X_k^{(n)} + (p^{(n)}_{\nu_k^{(n)}} - q^{(n)}_{\nu_k^{(n)}}) = X_k^{(n)} + (2\nu_k^{(n)}\Delta_n) \wedge 1,$$

where $\nu_k^{(n)} = |\{0 \leq i \leq k\ :X_i^{(n)} = 0\}|,\ p_i^{(n)} = 1 - q_i^{(n)} = (\frac{1}{2} + i\Delta_n) \wedge 1$.

We have

$$X_k^{(n)} = \sum_{i=0}^{k-1}(X_{i+1}^{(n)} - X_i^{(n)}) = \sum_{i=0}^{k-1}\left(X_{i+1}^{(n)} - \mathrm{E}\left(X_{i+1}^{(n)} \,|\, \mathcal{F}_i\right)\right) + \sum_{i=0}^{k-1}(2c\nu_i^{(n)} n^{-\alpha}) \wedge 1. \tag{11}$$

Let us estimate the second summand on the right hand side of (11) for $k = [nt]$

$$([nt] - T_\infty^{(n)})\left(\frac{2c\nu_\infty^{(n)}}{n^\alpha} \wedge 1\right) = \sum_{i=T_\infty^{(n)}}^{[nt]-1}\left(\frac{2c\nu_\infty^{(n)}}{n^\alpha} \wedge 1\right) \leq \sum_{i=0}^{[nt]-1}\left(\frac{2c\nu_i^{(n)}}{n^\alpha} \wedge 1\right) \leq$$

$$\sum_{i=0}^{[nt]-1}\left(\frac{2c\nu_\infty^{(n)}}{n^\alpha} \wedge 1\right) = [nt]\left(\frac{2c\nu_\infty^{(n)}}{n^\alpha} \wedge 1\right).$$

It follows from the last inequality, Lemma 2.2, and (10) that

$$\frac{\sum_{i=0}^{[nt]-1}(2c\nu_i^{(n)} n^{-\alpha}) \wedge 1}{n^{1-\frac{\alpha}{2}}} \Rightarrow 2\sqrt{c}\eta t,\ n \to \infty$$

in $D([0,\infty))$ with the topology of the uniform convergence on compact sets.

The sequence $M_k^{(n)} := \sum_{i=0}^{k-1}\left(X_{i+1}^{(n)} - \mathrm{E}\left(X_{i+1}^{(n)}\,|\,\mathcal{F}_i\right)\right),\ k\ge 1$ is a martingale difference. It follows from the Doob inequality that

$$\forall\varepsilon>0:\quad \mathrm{P}\left(\max_{k=1,[nt]}\frac{|M_k^{(n)}|}{n^{1-\frac{\alpha}{2}}}\ge\varepsilon\right)\le n^{\alpha-2}\varepsilon^{-2}\mathrm{E}(M_{[nt]}^{(n)})^2 =$$

$$n^{\alpha-2}\varepsilon^{-2}\sum_{i=0}^{[nt]}\mathrm{E}\left(X_{i+1}^{(n)}-\mathrm{E}\left(X_{i+1}^{(n)}\,|\,\mathcal{F}_i\right)\right)^2=$$

$$n^{\alpha-2}\varepsilon^{-2}\sum_{i=0}^{[nt]}\mathrm{E}\left(X_{i+1}^{(n)}-X_i^{(n)}+(2c\nu_i^{(n)}\Delta_n)\wedge 1\right)^2\le 4nt\,n^{\alpha-2}\varepsilon^{-2}=$$

$$4n^{\alpha-1}t\varepsilon^{-2}\to 0,\ n\to\infty.$$

This yields (3).

4. The proof of (2) and (5)

We need the following result on the absolute continuity of the limit.

Lemma 4.1. *Let $\{X_n, n\ge 1\}$ and $\{Y_n, n\ge 1\}$ be sequences of random elements given on the same probability space and taking values in a complete separable metric space E. Assume that*

- $Y_n \xrightarrow{\mathrm{P}} Y_0, n\to\infty$*;*
- *for each $n\ge 1$ we have the absolute continuity of the distributions*
$$\mathrm{P}_{X_n}\ll\mathrm{P}_{Y_n};$$
- *the sequence $\{\rho_n(Y_n), n\ge 1\}$ converges in probability to a random variable p, where $\rho_n=\frac{d\mathrm{P}_{X_n}}{d\mathrm{P}_{Y_n}}$ is the Radon-Nikodym density;*
- $\mathrm{E}p=1$.

Then the sequence of distributions $\{P_{X_n}\}$ converges weakly as $n\to\infty$ to the probability measure $\mathrm{E}(p\,|\,Y_0=y)P_{Y_0}(dy)$.

Similar result was proved by Gikhman and Skorokhod, see [4]. Since their formulation differs slightly from our, for the save of clarity we give a proof.

Proof. Since $\{\rho_n(Y_n), n\ge 1\}$ are non-negative random variables and $\mathrm{E}\rho_n(Y_n)=1, n\ge 1$, the uniform integrability of $\{\rho_n(Y_n), n\ge 1\}$ follows from $\mathrm{E}p=1$, where $p=\lim_{n\to\infty}\rho_n(Y_n)$, see, for example, [14, Chapter II §6].

Hence, the sequence $\{f(Y_n)\rho_n(Y_n), n\ge 1\}$ is uniformly integrable too, and we have

$$\lim_{n\to\infty}\int_E f dP_{X_n}=\lim_{n\to\infty}\mathrm{E}f(X_n)=\lim_{n\to\infty}\mathrm{E}f(Y_n)\rho_n(Y_n)=\mathrm{E}f(Y_0)p=$$

$$\mathrm{E}\left(f(Y_0)\,\mathrm{E}(p\,|\,Y_0)\right)=\int_E f(y)\mathrm{E}(p\,|\,Y_0=y)P_{Y_0}(dy).$$

Lemma 4.1 is proved. □

Let n be fixed, μ be the distribution of $\{X_0^\Delta, X_1^\Delta,\dots,X_n^\Delta\}$ in $\mathbb{R}^{n+1}$, where $\{X_k^\Delta\}$ is an RWM, $X_0^\Delta=0$.

Denote by ν the distribution of a symmetric RW $\{S_0,S_1,\dots,S_n\}$ with unit jumps, $S_0=0,\ S_n=\xi_1+\dots+\xi_n$, where $\{\xi_k\}$ are i.i.d., $\mathrm{P}(\xi_k=\pm1)=1/2$. Then $\mu\ll\nu$ and

$$\frac{d\mu}{d\nu}(i_0,i_1,\dots,i_n)=\frac{\prod_{k=0}^{n-1}\left(p_{\nu_k}1\!\!1_{i_{k+1}=i_k+1}+q_{\nu_k}1\!\!1_{i_{k+1}=i_k-1}\right)}{2^{-n}},$$

where $\nu_k = |\{0 \le j \le k \ : i_j = 0\}|,\ p_i = 1 - q_i = (\frac{1}{2} + i\Delta) \wedge 1.$

So

$$\frac{d\mu}{d\nu}(S_0, S_1, \ldots, S_n) = \frac{\prod_{k=0}^{n-1} \left(p_{\nu_k} 1\!\!1_{\xi_{k+1}=1} + q_{\nu_k} 1\!\!1_{\xi_{k+1}=-1}\right)}{2^{-n}} = \tag{12}$$

$$\prod_{k=0}^{n-1} (1 + ((2\nu_k\Delta) \wedge 1)\xi_k),$$

where $\nu_k = |\{0 \le j \le k \ : S_j = 0\}|$.

Let $M > 0$ be a fixed number and $\Delta \in (0, (2M)^{-2})$. Denote by $\{X_k^{M,\Delta}\}$ a RW with modifications upon visits to 0, where modifications stop changing after $[M/\sqrt{\Delta}]$-th hitting 0:

$$\mathrm{P}(X_{k+1}^{M,\Delta} = X_k^{M,\Delta} + 1 | X_0^{M,\Delta}, X_1^{M,\Delta}, \ldots, X_k^{M,\Delta}) = \frac{1}{2} + (\nu_k^{M,\Delta} \wedge [M/\sqrt{\Delta}])\Delta,$$

$$\mathrm{P}(X_{k+1}^{M,\Delta} = X_k^{M,\Delta} - 1 | X_0^{M,\Delta}, X_1^{M,\Delta}, \ldots, X_k^{M,\Delta}) = \frac{1}{2} - (\nu_k^{M,\Delta} \wedge [M/\sqrt{\Delta}])\Delta,$$

where $\nu_k^{M,\Delta} = |\{0 \le j \le k \ : \ X_j^{M,\Delta} = 0\}|$.

We will assume that $X_0^{M,\Delta} = 0$.

Observe that restriction of the distributions $\mathrm{I}_{\nu_n^{M,\Delta_n} \le [M/\sqrt{\Delta_n}]-1} \mathrm{P}_{\{X_k^{M,\Delta_n}, 0 \le k \le n\}}$ and $1\!\!1_{\nu_n^{\Delta_n} \le [M/\sqrt{\Delta_n}]-1} \mathrm{P}_{\{X_k^{M,\Delta_n}, 0 \le k \le n\}}$ are equal.

Similarly, let X_∞ be a solution of (5), $\tau_M = \inf\{t \ge 0 \ : l^0_{X_\infty}(t) \ge M\}$, and $X_{\infty,M}$ be a solution of

$$X_{\infty,M}(t) = 2\sqrt{c} \int_0^t \left((\sqrt{c} l^0_{X_{\infty,M}}(s)) \wedge M \right) ds + W(t), t \ge 0. \tag{13}$$

Set $\tilde{\tau}_M = \inf\{t \ge 0 \ : l^0_{X_{\infty,M}}(t) \ge M\}$.

Then

$$1\!\!1_{\tilde{\tau}_M \ge T} \mathrm{P}_{\{X_{\infty,M}(t), t \in [0,T]\}} = 1\!\!1_{\tau_M \ge T} \mathrm{P}_{\{X_\infty(t), t \in [0,T]\}}.$$

In view of Lemma 2.2, to prove the Theorem it is sufficient to verify the weak convergence

$$\{\frac{X_{nt}^{M,\Delta_n}}{\sqrt{n}}, t \in [0,T]\} \Rightarrow \{W(t), t \in [0,T]\}$$

if $\alpha > 1$

and

$$\{\frac{X_{nt}^{M,\Delta_n}}{\sqrt{n}}, t \in [0,T]\} \Rightarrow \{X_{\infty,M}(t), t \in [0,T]\}$$

if $\alpha = 1$.

Consider the case $T = 1$ and prove the weak convergence in $C([0,1])$. The case $C([0,T])$, and hence $C([0,\infty))$, can be considered similarly.

Let $S_k = \sum_{i=1}^k \xi_i$, where $\{\xi_k\}$ are i.i.d., $\mathrm{P}(\xi_k = \pm 1) = 1/2$, and $S_t = S_{[t]} + (t - [t])(S_{[t+1]} - S_{[t]})$.

It is possible, see [1], to select copies $\{S_k^{(n)}\}$ of $\{S_k\}$ and a Wiener process W such that

$$\lim_{n\to\infty} \sup_{t\in[0,1]} |\frac{S_{nt}^{(n)}}{\sqrt{n}} - W(t)| = 0, \ \lim_{n\to\infty} \sup_{t\in[0,1]} |\frac{\nu_{[nt]}^n}{\sqrt{n}} - l_W^0(t)| = 0 \ \text{ a.s.}, \tag{14}$$

where $\nu_k^n = |\{0 \le i \le k \ : \ S_i^n = 0\}|$.

Let us apply Lemma 4.1.

Set $E = C([0,1])$, $X_n = \{\frac{X_{nt}^{M,\Delta_n}}{\sqrt{n}}, t \in [0,1]\},\ Y_n = \{\frac{S_{nt}^{(n)}}{\sqrt{n}}, t \in [0,1]\}$.

Recall that $\Delta \in (0,(2M)^{-2})$. Similarly to (12) we get the formula for the Radon-Nikodym density

$$\frac{dP_{X_n}}{dP_{Y_n}}(Y_n) = \frac{dP_{n^{-1/2}X^{M,\Delta_n}_{n\cdot}}}{dP_{n^{-1/2}S^{(n)}_{n\cdot}}}(\frac{S^{(n)}_{n\cdot}}{\sqrt{n}}) = \prod_{k=0}^{n-1}\Big(1+2(\nu^n_k\wedge[M/\sqrt{\Delta_n}])\Delta_n\xi^n_{k+1}\Big),$$

where $\xi^n_k := S^{(n)}_k - S^{(n)}_{k-1}$.

Let us prove the following convergence in probability

$$\lim_{n\to\infty}\ln\left(\prod_{k=0}^{n-1}\Big(1+2(\nu^n_k\wedge[M/\sqrt{cn^{-\alpha}}])cn^{-\alpha}\xi^n_{k+1}\Big)\right) =$$

$$\begin{cases} 0, & \alpha>1,\\ \int_0^1 2\sqrt{c}\left((\sqrt{c}l^0_W(t))\wedge M\right)dW(t) - \int_0^1 2c\left((\sqrt{c}l^0_W(t))\wedge M\right)^2 dt, & \alpha=1.\end{cases}$$

Consider only the case $\alpha = 1$, the case $\alpha > 1$ is similar and simpler.

We have

$$\sum_{k=0}^{n-1}\ln\Big(1+2(\nu^n_k\wedge[M/\sqrt{cn^{-1}}])cn^{-1}\xi^n_{k+1}\Big) =$$

$$\sum_{k=0}^{n-1}2(\nu^n_k\wedge[M/\sqrt{cn^{-1}}])cn^{-1}\xi^n_{k+1} - \frac{1}{2}\sum_{k=0}^{n-1}(2(\nu^n_k\wedge[M/\sqrt{cn^{-1}}])cn^{-1}\xi^n_{k+1})^2 +$$

$$\frac{1}{3}\sum_{k=0}^{n-1}\theta_k\Big(2(\nu^n_k\wedge[M/\sqrt{cn^{-1}}])cn^{-1}\xi^n_{k+1}\Big)^3 =$$

$$\sum_{k=0}^{n-1}2\left(\frac{\sqrt{c}\nu^n_k}{\sqrt{n}}\wedge M\right)\frac{\sqrt{c}}{\sqrt{n}}\xi^n_{k+1} - \sum_{k=0}^{n-1}2c\left(\frac{\sqrt{c}\nu^n_k}{\sqrt{n}}\wedge M\right)^2\frac{1}{n} +$$

$$\frac{1}{3}\sum_{k=0}^{n-1}\theta_k\Big(2(\nu^n_k\wedge[M/\sqrt{cn^{-1}}])cn^{-1}\xi^n_{k+1}\Big)^3 + o(1),$$

where $|\theta_k| \le C = \text{const}$, C is independent of k and n. It will be seen from the proof below that $o(1)\to 0$ as $n\to\infty$ in probability.

The third summand converges to 0 for all ω. Indeed

$$\sum_{k=0}^{n-1}\left|\Big(2(\nu^n_k\wedge[M/\sqrt{cn^{-1}}])cn^{-1}\xi^n_{k+1}\Big)^3\right| \le \sum_{k=0}^{n-1}\Big(2([M/\sqrt{cn^{-1}}])cn^{-1}\Big)^3 =$$

$$n\Big(2([M/\sqrt{cn^{-1}}])cn^{-1}\Big)^3 \to 0,\ n\to\infty.$$

It follows from (14) that the limit of the second term is $\int_0^1 2c\left(\sqrt{c}l^0_W(t)\wedge M\right)^2 dt$.

Consider the first item. Let $\varepsilon > 0$ be fixed. Select $\delta > 0$ and $N \ge 1$ such that

$$\forall n\ge N\quad P\left(\sup_{t\in[0,1]}\Big(|\frac{S^{(n)}_{[nt]}}{\sqrt{n}} - W(t)| + |\frac{\nu^n_{[nt]}}{\sqrt{n}} - l^0_W(t)|\Big) \ge \varepsilon\right) \le \varepsilon;$$

$$P\left(\sup_{s,t\in[0,1],|s-t|\le\delta}|\frac{\nu^n_{[nt]}}{\sqrt{n}} - \frac{\nu^n_{[ns]}}{\sqrt{n}}| \ge \varepsilon\right) \le \varepsilon;\ P\left(\sup_{s,t\in[0,1],|s-t|\le\delta}|l^0_W(t) - l^0_W(s)| \ge \varepsilon\right) \le \varepsilon.$$

Set $m = [\frac{1}{\delta}] + 1$. For simplicity assume that n/m is integer. Then

$$I_n := \left|\sum_{k=0}^{n-1}\left(\frac{\sqrt{c}\nu^n_k}{\sqrt{n}}\wedge M\right)\frac{\xi^n_{k+1}}{\sqrt{n}} - \int_0^1\left((\sqrt{c}l^0_W(t))\wedge M\right)dW(t)\right| \le$$

$$\left|\sum_{j=0}^{m-1}\sum_{k=jn/m}^{(j+1)n/m-1}\left(\frac{\sqrt{c}\nu_k^n}{\sqrt{n}}\wedge M-\frac{\sqrt{c}\nu_{jn/m}^n}{\sqrt{n}}\wedge M\right)\frac{\xi_{k+1}^n}{\sqrt{n}}\right|+$$

$$\left|\sum_{j=0}^{m-1}\left(\frac{\sqrt{c}\nu_{jn/m}^n}{\sqrt{n}}\wedge M\right)\left(\Big(\sum_{k=jn/m}^{(j+1)n/m-1}\frac{\xi_{k+1}^n}{\sqrt{n}}\Big)-\Big(W(\frac{j+1}{m})-W(\frac{j}{m})\Big)\right)\right|+$$

$$\left|\sum_{j=0}^{m-1}\Big(\frac{\sqrt{c}\nu_{jn/m}^n}{\sqrt{n}}\wedge M-(\sqrt{c}l_W^0(\frac{j}{m}))\wedge M\Big)\Big(W(\frac{j+1}{m})-W(\frac{j}{m})\Big)\right|+$$

$$\left|\sum_{j=0}^{m-1}\int_{\frac{j}{m}}^{\frac{j+1}{m}}\left((\sqrt{c}l_W^0(\frac{j}{m}))\wedge M-(\sqrt{c}l_W^0(t))\wedge M\right)dW(t)\right|=I_1^{n,m}+I_2^{n,m}+I_3^{n,m}+I_4^m.$$

It follows from (14) and Lebesgue dominated convergence theorem that

$$\lim_{n\to\infty}\mathrm{E}(I_1^{n,m})^2=\lim_{n\to\infty}\frac{1}{n}\,\mathrm{E}\sum_{j=0}^{m-1}\sum_{k=jn/m}^{(j+1)n/m-1}\left(\frac{\sqrt{c}\nu_k^n}{\sqrt{n}}\wedge M-\frac{\sqrt{c}\nu_{jn/m}^n}{\sqrt{n}}\wedge M\right)^2=$$

$$\sum_{j=0}^{m-1}\mathrm{E}\int_{\frac{j}{m}}^{\frac{j+1}{m}}\left((\sqrt{c}l_W^0(\frac{j}{m}))\wedge M-(\sqrt{c}l_W^0(t))\wedge M\right)^2dt.$$

The last expression also equals I_4^m.

It follows from (14) that $\lim\limits_{n\to\infty}I_2^{n,m}=\lim\limits_{n\to\infty}I_3^{n,m}=0$ a.s. for each fixed m. Since the second moments of $I_2^{n,m},I_3^{n,m}$ are uniformly bounded we have convergence

$$\forall m\geq 1\quad \lim_{n\to\infty}\mathrm{E}|I_2^{n,m}|=\lim_{n\to\infty}\mathrm{E}|I_3^{n,m}|=0.$$

So for any $m\geq 1$

$$\limsup_{n\to\infty}\mathrm{E}\left|\sum_{k=0}^{n-1}\left(\frac{\sqrt{c}\nu_k^n}{\sqrt{n}}\wedge M\right)\frac{\xi_{k+1}^n}{\sqrt{n}}-\int_0^1\big(\sqrt{c}l_W^0(t)\wedge M\big)\,dW(t)\right|\leq$$

$$2\left(\sum_{j=0}^{m-1}\mathrm{E}\int_{\frac{j}{m}}^{\frac{j+1}{m}}\left((\sqrt{c}l_W^0(\frac{j}{m}))\wedge M-(\sqrt{c}l_W^0(t))\wedge M\right)^2dt\right)^{1/2}.$$

Letting $m\to\infty$ we get

$$\lim_{n\to\infty}\mathrm{E}|I_n|=0.$$

To apply Lemma 4.1 it remains to prove that

$$\mathrm{E}\exp\left\{\int_0^1 2\sqrt{c}\,\big(\sqrt{c}l_W^0(t)\wedge M\big)\,dW(t)-\int_0^1 2c\,\big(\sqrt{c}l_W^0(t)\wedge M\big)^2dt\right\}=1.$$

The last equality follows from the Novikov theorem because the integrands are bounded, see [8, Theorem 6.1, Chapter VI]. Hence, the sequence of processes $\{\frac{X_{nt}^{M,\Delta_n}}{\sqrt{n}},t\in[0,1]\}_{n\geq 1}$ converges in distribution to a process, whose distribution has a density

$$\exp\left\{\int_0^1 2\sqrt{c}\,\big(\sqrt{c}l_W^0(t)\wedge M\big)\,dW(t)-\int_0^1 2c\,\big(\sqrt{c}l_W^0(t)\wedge M\big)^2dt\right\}$$

with respect to the Wiener measure.

By the Girsanov theorem, this process is a weak solution to the SDE (13).

The Theorem is proved.

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