Theory of Stochastic Processes
Vol. 22 (38), no. 1, 2017, pp. 62–70

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SCALE PARAMETER ESTIMATION OF DISCRETE SCALE INVARIANT PROCESSES

Estimating the scale parameter of a continuous-time discrete scale invariant (DSI) process is one of the fundamental problems in the literature. We present an efficient estimation method which is based on transforming the DSI process into a vector-valued self-similar process and allows us to obtain the structure of the covariance matrix, which is the product of a scale and a block-Toeplitz matrices, and its block size depends on the unknown scale parameter. Therefore, we sweep the block size and obtain the maximum likelihood (ML) estimate of the scale parameter. We show that, the ML estimator does not directly solve the scale estimation problem. Hence, we penalize the likelihood following an information theoretic approach. The performance of the estimation method is studied via simulation. Finally this method is applied to the real data of S&P500 and Dow Jones indices for some special periods.

1. Introduction

Scale invariance (or self-similarity), as an important feature, often are used as a fundamental property to interpret many natural and man-made phenomena like turbulence of fluids, textures in geophysics, telecommunications of network traffic, image processing, fluctuations of stock market, $\cdots$ [3]. Scale invariance is often described as a symmetry of the system relatively to a transformation of a scale, that is mainly a dilation or a contraction (up to some re-normalization) of the system parameters [3]. Discrete scale invariance (DSI) is a property which requires invariance by dilation for certain preferred scaling factors [18]. This characteristic feature of such process is the invariance of its finite dimensional distributions by certain dilation for specific scaling factor. Burnecki et.al. [6] and Borgnat et.al. [4] have studied the property of DSI and its relation to periodically correlated processes by means of the Lamperti transformation. It is known that DSI leads to log-periodic corrections to scaling. Log-periodic oscillations have been used to predict price trends, turbulent time series, multi-fractal measures and crashes on financial markets [8], [9]. For a wide reference in this regard see [10].

Estimating the scale parameter of DSI processes is one of the fundamental problems in the literature. Modarresi and Rezakhah [13] presented an iterative method for scale estimation of DSI processes with stationary increments. There are various open issues and vivid discussions about the estimation of scale parameter but still there is not a universal method which could be considered as the most promising method to find the best approximation of scale parameter in all cases [13].

This paper provides a new method to estimate the scale parameter λ of a continuous-time DSI process with scale grater than one. The estimation method is based on the work of Ramirez et.al [14] in estimating the cycle period of a periodically correlated process. To estimate the scale parameter λ, first we consider some geometrical sampling of a continuous-time DSI process at points $\alpha^k, k \in \mathbb{W} = \{0, 1, \cdots\}$, such that $\lambda = \alpha^q$ [11], and q is the number of sample points in scale intervals $[\lambda^{n-1}, \lambda^n), n \in \mathbb{N}$. The

2000 *Mathematics Subject Classification.* 60G18 ; 62F30.
Key words and phrases. Discrete scale invariance, Multi-dimensional self-similar process, Scale parameter estimation.

estimation method is based on obtaining the ML estimate of the scale parameter λ for a fix α. Then, it is shown that the optimal α, in the sense of minimizing the mean squares of errors (MSE), is between 1.1 and 1.2.

By imposing the sampling scheme, we provide some discretization of a continuous-time DSI process. This sampling has the advantage to have a corresponding multi-dimensional self-similar process [11]. Then, by stacking N realizations of the multi-dimensional self-similar process into a vector, we obtain the structure of the covariance matrix, which is a Schur product of a scale matrix to a block-Toeplitz matrix. By sweeping the block size of the block-Toeplitz matrix, which depends on the number of sample points, q, in successive scale intervals, we obtain the ML estimate of the scale parameter. However, it is shown that the ML estimator does not directly solve the estimation problem, and it is therefore necessary to add a penalty term. Following [14], we use an information theoretic criterion to penalize large values of the scale parameter. Specifically, we used the minimum description length criterion [17], which provides us a better estimate than the ML approach.

Let $\{X(t), t \in \mathbb{R}^+\}$ be a continuous-time DSI process with scale $\lambda > 1$. By considering the geometrical sampling scheme, which enables one to have q sample points at $1, \alpha, \cdots, \alpha^{q-1}$ in the first scale interval $[1, \lambda)$, where $\lambda = \alpha^q$, and following the sampling at corresponding points $\{\alpha^{nq+j}, n \in \mathbb{N}, j = 0, \cdots, q-1\}$ in the other scale intervals $[\lambda^{n-1}, \lambda^n)$, we would have the sampled DSI process $\{X(t), t \in \check{T}\}$ where $\check{T} = \{\alpha^{nq+j}, n \in \mathbb{W}, j = 0, \cdots, q-1\}$. By embedding the sampled DSI process $X(.)$ in q columns, an embedded multi-dimensional self-similar process is obtained which is denoted by $U(\lambda^n) = (U^0(\lambda^n), \cdots, U^{q-1}(\lambda^n))$ where $U^j(\lambda^n) \equiv X(\alpha^{nq+j})$. To facilitate such study, the subsidiary multi-dimensional self-similar process [12] is considered which is obtained by re-indexing consecutive observation of the embedded multi-dimensional self-similar process with successive positive integers as $V(n) = (V^0(n), \cdots, V^{q-1}(n))$, where $V^j(n) \equiv U^j(\lambda^n)$. These arrangements provide a suitable platform to obtain the ML estimate of λ. The main tool we use in the estimation method, is the relationship between the sampled DSI process $X(.)$ and the subsidiary multi-dimensional self-similar process $V(.)$. However, the ML estimator does not directly solve the scale estimation problem. By an example we show that, there are some peaks in the log-likelihood function for some values of q. Hence, we penalize the likelihood following an information theoretic approach, similar to that in [14].

This paper is organized as follows. Section 2, provides a background on embedded and subsidiary multi-dimensional self-similar processes and their covariance structures. In Section 3, the estimation method for scale parameter λ is presented. In Section 4, the performance of the estimation method is studied via simulation. Finally this method is applied to the real data of S&P500 and Dow Jones indices for some special periods.

2. Preliminaries

In this section we review the definition of embedded and subsidiary multi-dimensional self-similar processes and their covariance matrix structures.

Definition 2.1. A process $\{X(t), t \in \mathbb{R}^+\}$ is said to be self-similar (or scale invariant), if for any $\lambda > 0$

$$\{\lambda^{-H} X(\lambda t), t \in \mathbb{R}^+\} \equiv \{X(t), t \in \mathbb{R}^+\} \tag{1}$$

where $\equiv$ means equality in all finite dimensional distributions. The process is said to be DSI of index H and scaling factor $\lambda_0 > 0$, if (1) holds for $\lambda = \lambda_0$.

For a continuous-time DSI process $\{X(t), t \in \mathbb{R}^+\}$ with scale $\lambda > 1$, if we consider sampling of the process at points of set $\check{T} = \{\alpha^{nq+j} : n \in \mathbb{W}, j = 0, \cdots, q-1\}$, then $X(.)$ with parameter space $\check{T}$ is called sampled DSI process [11].

Definition 2.2. The process $U(t) = (U^0(t), U^1(t), \cdots, U^{q-1}(t))$, with parameter space $\breve{T} = \{\lambda^n, n \in \mathbb{W}\}$ is a multi-dimensional self-similar process, where
(i) $\{U^j(.)\}$ for every $j = 0, \cdots, q-1$ is self-similar with parameter space $\breve{T} = \{\lambda^n, n \in \mathbb{W}\}$.
(ii) For every $n, \tau \in \mathbb{Z}, j, k = 0, \cdots, q-1$

$$Cov(U^j(\lambda^{n+\tau}), U^k(\lambda^n)) = \lambda^{2nH} Cov(U^j(\lambda^\tau), U^k(1)).$$

For a sampled DSI process $\{X(t), t \in \breve{T}\}$ with scale $\lambda > 1$ and parameter space $\breve{T}$, the process $U(\lambda^n) = (U^0(\lambda^n), U^1(\lambda^n), \cdots, U^{q-1}(\lambda^n))$ is called an embedded multi-dimensional self-similar process, where $U^j(\lambda^n) \equiv X(\alpha^{nq+j})$ [12]. Corresponding to such an embedded multi-dimensional self-similar process, Modarresi and Rezakhah [12] defined a subsidiary q-dimensional self-similar process $V(n)$ as

$$V(n) = (V^0(n), V^1(n), \cdots, V^{q-1}(n)), \qquad n \in \mathbb{W},$$

where $V(n) \equiv U(\lambda^n)$.

Remark 2.1. The covariance matrix of $V(n)$ is denoted by

$$R^H(n, \tau) = [R^H_{j,k}(n, \tau)]_{j,k=0,\cdots,q-1}$$

where

$$R^H_{j,k}(n, \tau) = E[V^j(n+\tau)V^{k*}(n)] = E[X(\alpha^{(n+\tau)q+j})X^*(\alpha^{nq+k})].$$

By the DSI property of the process $X(.)$ we have that

$$R^H_{j,k}(n, \tau) = \lambda^{2nH} E[X(\alpha^{\tau q+j})X^*(\alpha^k)] = \lambda^{2nH} R^H_{j,k}(\tau), \tag{2}$$

where $R^H_{j,k}(\tau) = R^H_{j,k}(0, \tau) = E(V^j(\tau)V^{k*}(0))$.

See [12].

3. Scale Parameter Estimation

In this section, we present a new method for scale parameter estimation of a continuous-time DSI process with scale $\lambda > 1$. The estimation method is compatible for real world data, and its motivation comes from the cycle period estimation of a periodically correlated process, proposed by Ramirez et.al. [14]. In this method, we obtain the ML estimator of λ. Then, by an example of a Weierstrass-Mandelbrot process, we show that the estimator cannot be directly applied. Therefore, we penalize the likelihood following an information theoretic approach, similar to that in [14].

Let the continuous-time DSI process $\{X(t), t \in \mathbb{R}^+\}$, is zero-mean complex-valued Gaussian. By imposing the sampling scheme, which provides samples at q points $1, \alpha, \cdots, \alpha^{q-1}$ in the first scale interval $[1, \lambda)$, where $\lambda = \alpha^q$, and at multiple λ^n of such points in the scale intervals $[\lambda^n, \lambda^{n+1}), n \in \mathbb{N}$, we have the sampled DSI process $\{X(t), t \in \breve{T}\}$ where $\breve{T} = \{\alpha^{nq+j}, n \in \mathbb{W}, j = 0, \cdots, q-1\}$.

First, for a fixed α, we would like to derive the ML estimator of λ. Thus, the estimation problem is reduced to finding the ML estimator of q. To this end, we first assume that the number of sample points in each scale interval, q, is known. Then, we arrange the sampled DSI process $X(t)$ in blocks of size q, which provides an embedded multi-dimensional self-similar process $U(\lambda^n) = (U^0(\lambda^n), \cdots, U^{q-1}(\lambda^n)), n \in \mathbb{W}$, where $U^j(\lambda^n) \equiv X(\alpha^{nq+j})$. By re-indexing consecutive observation of the embedded multi-dimensional self-similar process with successive positive integers, the subsidiary q-dimensional self-similar process $V(n) = (V^0(n), \cdots, V^{q-1}(n))$ is obtained where $V^j(n) \equiv U^j(\lambda^n)$ [12]. Now, we stack N realizations of $V(n)$ into the vector

$$z = [V'(0) \quad V'(1) \quad \cdots \quad V'(N-1)]',$$

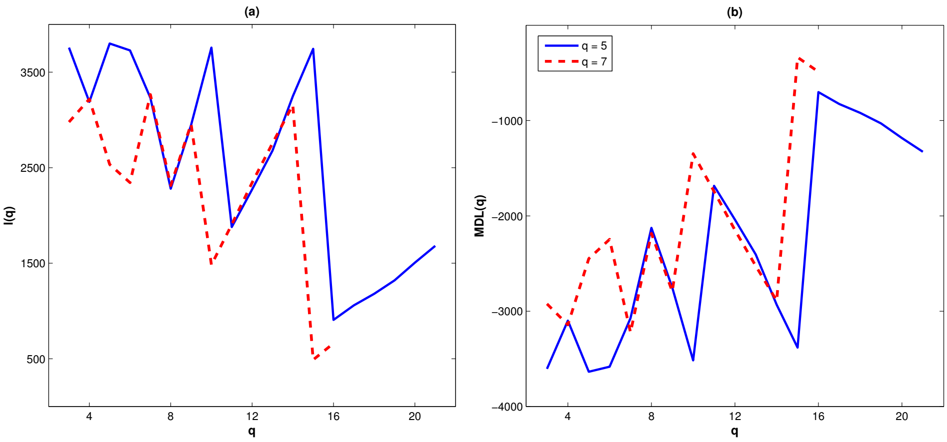

FIGURE 1. Likelihood function (a), and MDL criterion (b) for $q = 5$ (blue line) and $q = 7$ (red dashed line).

which is the stack of Nq samples of $X(t)$. Thus, the covariance matrix of z is of the form

$$\mathbf{C} = \begin{pmatrix} R^H(0) & R^H(-1) & . \quad . \quad . & R^H(-N+1) \\ R^H(1) & \lambda^{2H}R^H(0) & . \quad . \quad . & \lambda^{2H}R^H(-N+2) \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ R^H(N-1) & \lambda^{2H}R^H(N-2) & . \quad . \quad . & \lambda^{2(N-1)H}R^H(0) \end{pmatrix},$$

where $R^H(\tau) = [R^H_{j,k}(\tau)]_{j,k=0,\cdots q-1}$, and $R^H_{j,k}(\tau) = E(V^j(\tau)V^{k*}(0))$.

The covariance matrix $\mathbf{C}$ can be represented as a Schur product of a scale matrix Λ to a block-Toeplitz matrix $\mathbf{C}_0$ with block-size q, as $\mathbf{C} = \Lambda \circ \mathbf{C}_0$, where $\Lambda = (\Lambda_0 \otimes I_q)$, and I_q is identity matrix of size q:

$$\Lambda_0 = \begin{pmatrix} 1 & 1 & . \quad . \quad . & 1 \\ 1 & \lambda^{2H} & . \quad . \quad . & \lambda^{2H} \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ 1 & \lambda^{2H} & . \quad . \quad . & \lambda^{2(N-1)H} \end{pmatrix},$$

$$\mathbf{C}_0 = \begin{pmatrix} R^H(0) & R^H(-1) & . \quad . \quad . & R^H(-N+1) \\ R^H(1) & R^H(0) & . \quad . \quad . & R^H(-N+2) \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ R^H(N-1) & R^H(N-2) & . \quad . \quad . & R^H(0) \end{pmatrix}.$$

Furthermore, the observations z are distributed as

$$f(z; q, \mathbf{C}) = \frac{1}{(2\pi)^{Nq/2}\,|\mathbf{C}|^{1/2}} \exp\{-z'\mathbf{C}^{-1}z/2\}.$$

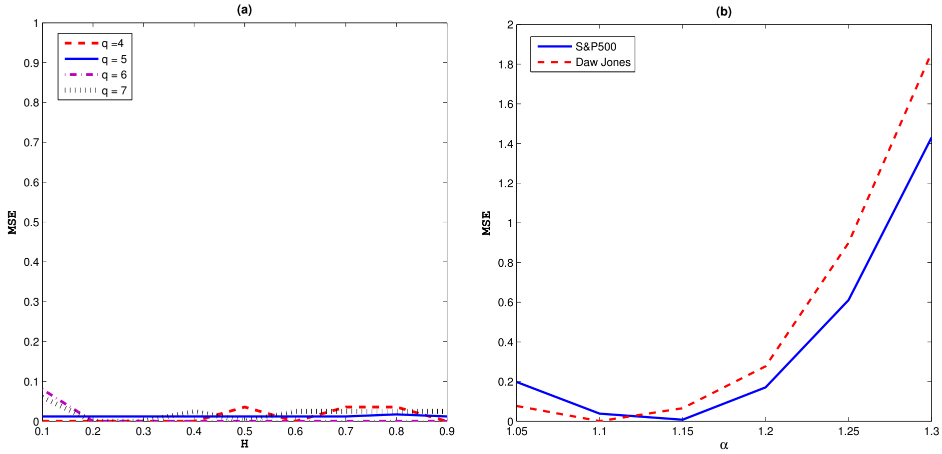

FIGURE 2. (a) Mean square error of scale parameter estimator in 100 repetitions of a Weierstrass-Mandelbrot process in Example 1 for $\alpha = 1.15$ and different values of q. (b) MSE of scale parameter estimation for S&P500 indices (blue line) from 16/10/2000 until 23/7/2002, where λ was evaluated approximately with 1.66 in [16]; and Dow Jones indices from 6/3/2009 until 14/11/2012 where λ was evaluated approximately with 1.493 in [16] (red dashed line), versus different values of α. The plot shows that the optimal α in the sense of minimizing the mean squares of errors, is between 1.1 and 1.2.

3.1. **Maximum Likelihood and Minimum Description Length estimators.** Now, to derive the ML estimator, we proceed as follows: By assuming M independent and identically distributed realizations of z, $\{z_m\}_{m=0}^{M-1}$, we compute the likelihood function for a fixed q. Then by sweeping q, we obtain the value that maximizes the likelihood

$$L(q, \mathbf{C}) := \prod_{m=0}^{M} f(z_m; q, \mathbf{C}) = \frac{1}{(2\pi)^{NqM/2}\,|\mathbf{C}|^{M/2}} \exp\{-1/2 \sum_{m=0}^{M} z_m' \mathbf{C}^{-1} z_m\}.$$

Thus, we have that

$$l(q) := \ln L(q, \hat{\mathbf{C}}) = -\frac{NMq}{2}(\ln(2\pi) + 1) - \frac{M}{2} \ln |\hat{\mathbf{C}}|, \tag{3}$$

where $\hat{\mathbf{C}}$ is the sample covariance matrix $\hat{\mathbf{C}} = \frac{1}{M}\sum_{m=0}^{M-1} z_m z_m^*$. Hence, the ML estimator of q is given by the value of q that maximizes the log-likelihood (3).

However, the ML estimator does not directly solve the scale estimation problem. The following example shows that, there are some peaks in the log-likelihood function for some values of q. Hence, we penalize the likelihood following an information theoretic approach, similar to that in [14].

Example 3.1. Consider a Weierstrass-Mandelbrot process

$$X(t) = \sum_{n=-\infty}^{\infty} \lambda^{-nH}(1 - e^{i\lambda^n t})e^{i\phi_n}, \tag{4}$$

with scale $\lambda > 1$ and $0 < H < 1$, where the ϕ_n's are i.i.d. random variables uniformly distributed on $[0, 2\pi)$ [3]. With the indicated restrictions on λ and H, the series (4) converges almost surely [2], [5]. Imposing the sampling scheme with $\alpha = 1.2$, we apply the method to estimate q in two cases: $q = 5, 7$, with $H = 0.5$, $N = 100$ and $M = 25$. First, we consider a case where $q = 5$: As we may see in Figure 1 (a) with blue line, there are clear peaks at $q = 5$ and its multiples $10, 15$. Also, for the case $q = 7$ there

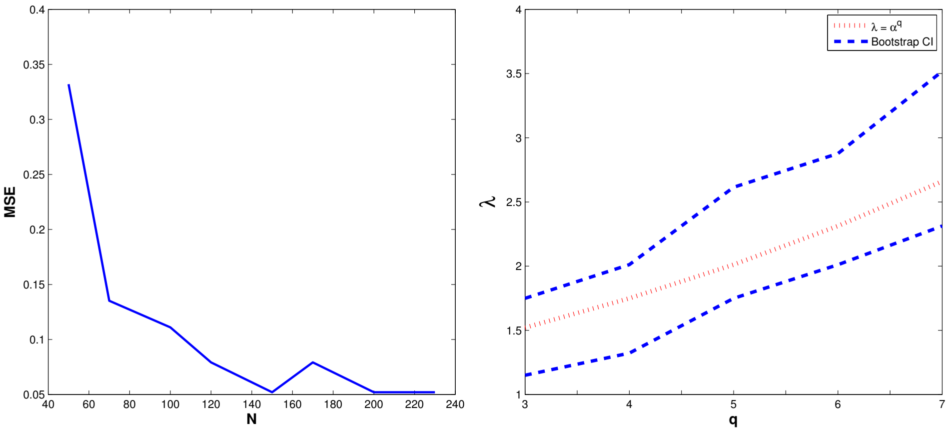

FIGURE 3. (a) Mean square error of the MDL scale parameter estimator of a Weierstrass-Mandelbrot process in Example 1, for $q = 4, H = 0.5, \alpha = 1.15$ and different values of N. As the sample size N grows, the MDL estimator converges to true value of λ. (b) The non-overlapping block bootstrap confidence interval of the MDL estimator of a Weierstrass-Mandelbrot process in Example 1, for $H = 0.5, \alpha = 1.15$ and different values of q (blue dashed lines), and true value of λ (red dotted line).

are some peaks at $q = 4, 7, 9, 14$, indicated with red dashed line. Hence, to obtain the true estimation of q, a penalty term is necessary to consider. Following [14], we use an information theoretic criterion to penalize large values of q. Specifically, we use the minimum description length (MDL) criterion [17], which provides us a better estimate than the ML approach. The MDL criterion for our problem is given by

$$(5) \qquad MDL(q) = -l(q) + \frac{1}{2}\beta \ln M$$

$$= NMq/2(\ln(2\pi) + 1) + M/2 \ln |\hat{\mathbf{C}}| + \frac{1}{2}\beta \ln M.$$

where β is the number of degrees of freedom of the model, which is $\beta = Nq^2$ [14]. Thus, the MDL-based estimator is

$$(6) \qquad \hat{q} = \arg\min_{q=1,\cdots,q_{max}} MDL(q).$$

The blue line in Figure 1 (b), shows the MDL criterion for the case $q = 5$. In this Figure, we can observe that the q that minimizes the criterion is $q = 5$. Also, with red dashed line, it can be seen that $q = 7$ minimizes the MDL which is coincide with the true value of q.

4. Simulations and Empirical Data

The accuracy of the method is investigated for simulated and empirical data. In the simulation case, to visualize efficiency of the estimation method, we have simulated the Weierstrass-Mandelbrot process with different scale parameters and presented the graph of the MSE for different Hurst indices. As it is shown by Figure 2 (a), the method gives an accurate estimation. Moreover, the consistency of the MDL estimator is illustrated in Figure 3 (a). In this Figure, the MSE of estimator is computed for different values of N. Evidently, as the sample size N grows, the MDL estimator converges to true value of $\lambda = \alpha^q$. Furthermore, the non-overlapping block bootstrap confidence interval for MDL estimator is depicted in Figure 3 (b) for $\alpha = 1.15, H = 0.5$ and different values of q. The

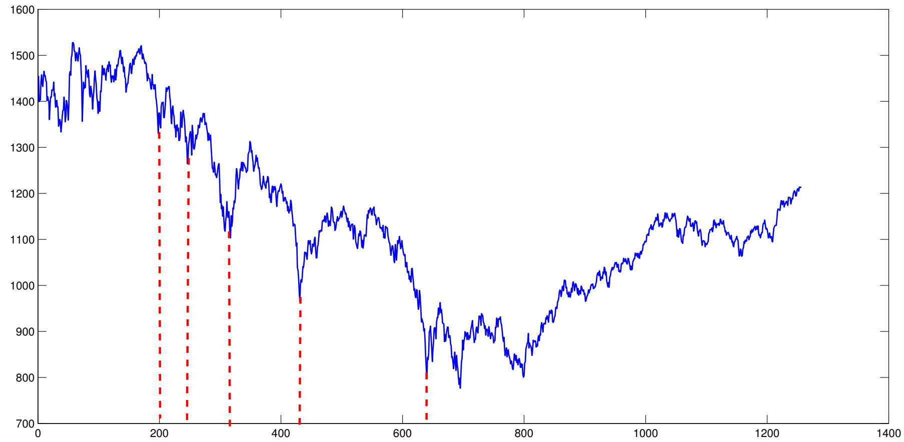

FIGURE 4. S&P500 indices from 1/1/2000 until 31/12/2004. The existence of a DSI behavior is justified from 16/10/2000 until 23/7/2002 in four scale intervals which are indicated with red lines. The scale of the process for the periods is evaluated approximately with 1.66 [16].

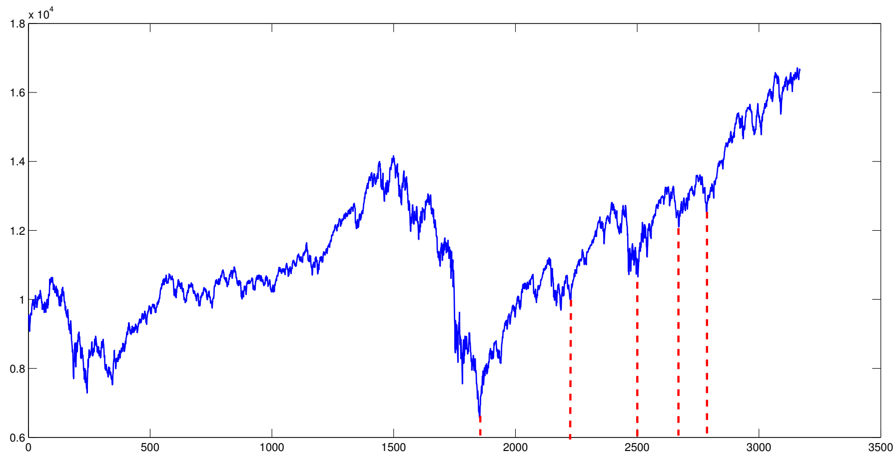

FIGURE 5. Dow Jones indices from 25/10/2001 until 28/5/2014. The existence of a DSI behavior is justified from 6/3/2009 until 14/11/2012 in four scale intervals which are indicated with red lines. The scale of the process for the periods is evaluated approximately with 1.493 [16].

superiority of the method is also investigated for empirical data. To this end, we study the daily indices of two stock markets: S&P500 and Dow Jones, for some special periods. First, we consider daily indices of S&P500 from the first January 2000 till the end of 2004. As there is not any index on Saturdays, Sundays and holidays, the available data for the selected period are 1256 days. The time series of these indices is shown in Figure 4. These data are also studied by Bartolozzi et.al. [1], Rezakhah and Maleki [16] where the existence of a DSI behavior, in some periods of data has been justified. The indices from 16th October 2000 until 23th July 2002, which the DSI behavior can be seen in four scale intervals, was considered by the author in [16], and the preferred scaling factor of the process for the periods was evaluated approximately with 1.66. Now, we apply the proposed method to estimate λ. The scale parameter for $\alpha = 1.1$ and 1.15 is estimated about 1.464 and 1.749 respectively.

As an another example, we consider daily indices of Dow Jones from 25th October 2001 till 28th May 2014. Same as the S&P500 indices, there is not any index on Saturdays, Sundays and holidays, the available data for the selected period are 3168 days. These indices are plotted in Figure 5, where the existence of a DSI behavior in a period from 6th March 2009 until 14th November 2012 has been justified in four scale intervals, and the scale parameter λ was evaluated approximately with 1.493 [16]. By applying the estimation method, the scale parameter λ for $\alpha = 1.1$ and 1.15 is estimated about 1.335 and 1.520, respectively, which give a good estimations for λ.

5. Conclusion

This paper provides a new method for the scale parameter estimation of a continuous-time DSI process. In this method, by imposing a geometrical sampling scheme, we obtain some discretization of the continuous-time process, and then by some precise method we provide the ML estimate of the scale parameter. The main tool in this method, is the relationship between the sampled DSI process and a subsidiary multi-dimensional self-similar process, obtained by arranging the sampled DSI process in blocks of size given by the number of sample points in scale intervals. However, we showed that the ML estimator is not a valid estimator. This issue is solved by penalizing large values of the scale parameter using the MDL criterion. There are various discussions about the scale parameter estimation but still there is not a universal method which could be considered as the most promising method to find the best approximation of scale parameter in all cases. Simulations and numerical evaluations clarified that the proposed method provided a good estimation for scale parameter. Moreover the estimation method is easily implemented and computationally fast.

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