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A LIMIT THEOREM FOR BOUNDARY LOCAL TIME OF A SYMMETRIC REFLECTED DIFFUSION

Let X be a symmetric diffusion reflecting in a $\mathcal{C}^3$-bounded domain $D \subset \mathbb{R}^d$, $d \geq 1$, with a $\mathcal{C}^2$-bounded and non-degenerate matrix a. For $t > 0$ and $n, k \in \mathbb{N}$ let $N(n,t)$ be the number of dyadic intervals $I_{n,k}$ of length 2^{-n}, $k \geq 0$, that contain a time $s \leq t$ s.t. $X(s) \in \partial D$. For a suitable normalizing factor $H(t)$ we prove, extending the one dimensional case, that a.s. for all $t > 0$ the entropy functional $N(n,t)/H(2^{-n})$ converges to the boundary local time $L(t)$ as $n \to \infty$. Applications include boundary value problems in PDE theory, efficient Monte Carlo simulations and Finance.

1. Introduction and statement of the results

Let D be a bounded $\mathcal{C}^3$-domain in $\mathbb{R}^d$, $d \geq 1$, and $X(t)$ a symmetric SDE reflecting in $\bar{D}$ in the conormal direction. The matrix $[a^{ij}]$ of X is $\mathcal{C}^2$ and satisfies $c_1 \|\xi\|^2 \leq (a(y)\xi, \xi) \leq c_2 \|\xi\|^2$ for some positive constants c_1, c_2 and all y, ξ in $\mathbb{R}^d$.

It is both of great theoretical and practical interests to devise limit theorems for the boundary local time L to be exploited e.g. in boundary value problems in PDEs, mathematical Finance and Monte Carlo simulations. We extend the entropy strong law in [18] to the case of the random closed set $Z^\partial(t)$ of times $s \leq t$ s.t. $X(s) \in \partial D$, i.e. for non-negative integers n and k let $I_{n,k} = [k2^{-n}, (k+1)2^{-n})$ and consider for $t > 0$ the number $N(n,t)$ of intervals $I_{n,k}$ that intersect $Z^\partial(t)$; we look for a scale H s.t. the entropy functional $N(n,t)/H(2^{-n})$ has a non-trivial limit as $n \to \infty$. In the multidimensional situation (for convenience X will start a.s. from $\alpha_0 \in \partial D$) for $\alpha \in \partial D$, let $e(t)$ be an excursion starting at α, $l(e)$ be its duration and let $Q^\alpha(e(t) \in dx, l(e) > t)$ be the α-excursion law, see [15]. Let $Q^\alpha(t) = Q^\alpha(e(t) \in \bar{D}, l(e) > t)$ be the Lévy-Hsu quantity, then we have now

$$H(t) = \frac{1}{t|\partial D|} \int_{[0,t]\times\partial D} Q^\alpha(s) ds d\alpha$$

with $H(t) \sim \sqrt{2/\pi t}$ near 0 (there is a misprint in formula (8.9) on p. 261 of [15] where the constant should read $\sqrt{2/\pi}$ instead of $2\sqrt{2/\pi}$, see also Section 1.1 below). Our result is still a corollary of the fact that $N(n,\tau)$ the time-boxes counting functional while running the inverse local time $S(\tau) = \inf\{t > 0 | L(t) > \tau\}$ (Greek letters are reserved for local time scale matters) converges to τ as $n \to \infty$ for all $\tau > 0$ a.s.-P^{α_0}. We proceed as in [18] mainly using a classical Borel-Cantelli argument based on sharp estimates of the first and second moments of $N(n,\tau)$.

The new difficulties to overcome are, beside controlling the tricky remainder R_3 in the Euler-McLaurin formula, to provide a formula for the hitting probabilities of the *non-Markovian* process S, see Theorem 15 below. In the Markovian homogeneous context, hitting probabilities follow from Hunt-type formulas with potential kernels and Choquet capacities, see e.g. [9]. The fundamental observation is that conditionally on the trace of X on the boundary, noted X^∂, see [22] and [17], S becomes an additive process

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(with independent but not homogeneous increments) with a (random) Lévy measure $\gamma(\omega)(\tau, dt)$. In the terminology of [24] S is locally homogeneous. It is instructive to use here a concrete discretization method to deal with our hitting probabilities. Since in dimension one the mean of $\gamma((s,\infty))$ over $(0, 2^{-n})$ is the right factor, it was to be expected that the right factor here is not only the mean over $(0, 2^{-n})$ but also the mean over the boundary of the above Lévy-Hsu quantity. Indeed, a limit holds true, roughly speaking, because the boundary ∂D is "quickly sewn together" so as to become a "single point" thanks to ergodic phenomena of the trace of X on the boundary established in [3] and therefore the situation almost boils down to that in [18].

Various technical difficulties (e.g. great effort is made to watch for monotonicities in quantities related to improper infinite integrals) demand estimates, for both small and large times, on the transition densities $p^D(t, x, y)$ of the killed diffusion X_t^D. We establish estimates, using sometimes non-trivial results of PDE theory, on the time derivatives of these densities and note that more powerful versions are known to hold in the case without boundary, see e.g. [14]. We accordingly need a workable basis for this endeavour and improve upon [23] concerning the definition of the primary "raw" candidate for p^D (given on p. 596 therein) which is processed in the E. E. Levi fundamental iterative method and leads in the end to p^D.

1.1. **Comparison with previous works.** There are, to the best of our knowledge, a result concerning reflecting Brownian motion and related works for a free Brownian motion Y. It is proved in [15], in a $\mathcal{C}^3$-domain, that $N_3(h,t)$ the number of excursions s.t. $l(e) > h$ accomplished by X before t satisfies $\sqrt{h} N_3(h,t) \to (1/\sqrt{2\pi}) L(t)$ as $h \to 0$ a.s.. Consider now the local time of Y as a continuous additive functional with the area measure on ∂D as the Revuz measure. Theorem 4.1 p. 67 in [5] considers, though in a Lipschitz domain, seven functionals generated by the following integers $N_k(h,t)$, $k = 1...7$, where $N_k(h,t)$ is the number of excursions s.t. respectively: $\left\| e(h^2) - e(0) \right\| > h$; $l(e) > h^2$ and $\| e(\infty) - e(0) \| > h$; $N_3(h,t)$ etc. We have in probability $\lim h N_k(h,t) = d_k L(t)$ where d_k are constants. In [1] the number of excursions with $diam(e) > h$ is further studied. All the above papers use Lévy-system-like master formulas. Variants of the representation of the local time as an occupation density are proposed in [10] for essentially symmetric diffusions. Eq. (24) of [26], in connection with the method of the random walks on the boundary in Monte Carlo simulations in PDEs, gives the following heuristic equation $L(t) = c \int_0^t I_{\partial D}(X_s) \sqrt{ds}$ for reflected Brownian motion.

In comparison with these constructions, our entropy limit law is obviously not a by-product of excursion statistics and is intrinsic in that it depends only on the set $Z^{\partial}(t)$. Our result provides a rigorous intrinsic justification for the above equation of [26]. The relevance to numerical computations by Monte Carlo methods is another major contribution of this paper because of the well known obstructions due to the boundary and this is perhaps the reward for working with a discontinuous non-adapted functional.

1.2. **Structure of the paper and Notations.** Basic topics are, for the convenience of the reader, discussed in Section 2. Section 3 is technical and contains results about the function $p^D(t, x, y)$ and related quantities. The fundamental Section 4 is actually of independent interest and is about the law of the hitting time of an interval by an increasing additive process. The key technical Lemma 16 in Section 5 (a perturbed non-homogeneous potential kernel) allows us to finish off.

Our notations are essentially standard. Hats are for Fourier transforms and the subscripts s and c are for sine and cosine transforms The class of N-times continuously differentiable functions on E with derivatives tending to 0 at infinity is $\mathcal{C}_0^N(E)$. The m-th order ordinary derivative of $f(x)$ is noted $f^{(m)}(x)$. Before actually differentiating a composed function $f(g(x))$ we write $\partial_i[f(g)]$ but once the derivation has been performed we write $\partial_i f(g)$. The Lebesgue measure of E is $|E|$. The mean value of a boundary function φ is noted $\bar{\varphi}$, i.e. $\bar{\varphi} = (1/|\partial D|)\int_{\partial D}\varphi(\alpha)d\alpha$. To stress the dependence of a constant c on some variable x we write $c = c(x)$. Unimportant constants will be denoted by c, c', ... and they may vary from line to line while proofs are in process.

2. Some facts

For the convenience of the reader, we gather here some topics that will be needed below. The papers [15], [22], [23] and [3] are essential; the book [16] is also useful. The celebrated Euler-Maclaurin summation formula, see e.g. [8], is

Lemma 1. *Let $f(s)$ be a function in $\mathcal{C}^1(\mathbb{R}_+)$ and $h, t > 0$; we have for any integer m*

$$h\sum_{k=1}^{m} f(t+kh) = \int_0^{t+mh} f(s)ds - \int_0^{t+h} f(s)ds + \frac{h}{2}[f(t+h) + f(t+mh)] - h\int_h^{mh} P_1(\frac{s}{h})f^{(1)}(t+s)ds,$$

where $P_1(s)$ is the Bernoulli function with Fourier expansion $\sum_{k=1}^{\infty}\sin(2\pi ks)/\pi k$. Provided the sum and integrals converge, m can be taken infinite. The last three error terms on the r.h.s. above will be noted respectively R_i, $i = 1, 2, 3$.

2.1. The boundary processes and inverse local time S.

The boundary process X^∂ is a symmetric Hunt process with infinite life time whose generator A^∂ (A stands for that of X) is the so called Dirichlet-to-Neumann map, see e.g. [15] and [3]. Its semigroup P_τ^∂ converges in the total variation norm exponentially fast to its unique invariant probability measure $d\alpha/|\partial D|$ and we have, see Lemma 3 p. 337 in [3],

Theorem 2. *Let φ be a bounded boundary function with $\bar{\varphi} = 0$ (see the notations), then $\left\|P_\tau^\partial\varphi\right\|_\infty \le c\left\|\varphi\right\|_\infty \exp -c'\tau$ for some absolute constants c, c' and all $\tau \ge 0$.*

It follows from our strong non-degeneracy assumptions that the extended Markov process $X^{\partial+} = (S, X(S))$, i.e. the trace of the time-space *degenerate* diffusion $(t, X(t))$, is (in fact strong) Feller (see Section 5 of [4] who uses the Malliavin calculus and this is also implicit in the Appendix of [25] who use PDEs).

Now let $\mathcal{F}^{\partial,0} = \sigma(X_\tau^\partial, \tau \ge 0)$ and $\mathcal{F}^\partial$ be the completion of $\mathcal{F}^{\partial,0}$ with respect to the usual family of measures P^μ in the sixtuple Markov set up of X. Then there exists, see [22] and also [17], a regular conditional probability $P(\omega)$ of $P(\cdot|\mathcal{F}^\partial)$ on $\sigma(S(\tau), X_\tau^\partial, e_\tau, \tau \ge 0)$ under which S is an additive process. Let $N((\theta,\tau], dt)$ be the number of jumps of magnitude in $dt \subset (0,\infty)$ in $(\theta,\tau]$. If S is stochastically continuous, then a Lévy-Itô decomposition holds

$$E(\omega)\exp i\lambda(S(\tau) - S(\theta)) = \exp - \int_0^\infty (1 - \exp i\lambda t)\gamma^{\theta,\tau}(\omega, dt), \tag{1}$$

where $\gamma^{\theta,\tau}(\omega, dt) = E(\omega)N((\theta,\tau], dt)$ is the Lévy measure. Moreover, for a.e. ω the (non-homogeneous) convolution semigroup of probability measures on $\mathbb{R}_+$ with Fourier transform (1) and indexed by $0 \le \theta \le \tau$ generates the family of two-parameter transition kernels $P^{\theta,\tau}(s, dt) = E(\omega)(s + S(\tau) - S(\theta) \in dt)$. It is convenient to build S on the standard Skorokhod space in the formalism of [19] which parallels the homogeneous context. Although such an interpretation suffers from some drawbacks, it is convenient for our purposes here because the strong Markov property reads the usual way.

From now on, when dealing with S we shall suppress ω for convenience and write $N((\theta,\tau], t) = N((\theta,\tau], (t,\infty))$ and respectively $\gamma^{\theta,\tau}(t)$ and $\Psi^{\theta,\tau}(\lambda)$ for the tail of the Lévy measure $\gamma^{\theta,\tau}((t,\infty))$ and the Lévy exponent in eq. (1).

2.2. **The point process of excursions.** Let $e(\alpha,\beta)$ be the excursion which starts at α and ends at β and set $\mathcal{E}^\alpha$ for the set of e's that start from $\alpha \in \partial D$. Let $\mathcal{N}$ be the compensating measure of the corresponding point process. Concrete calculations are carried out in [15] p. 251 (in the particular case $A = \Delta$ but which generalize immediately) and $\mathcal{N}((\theta,\tau]\times E)$ is explicitly given by $\int_{(\theta,\tau)} Q^{X^\partial(\eta)}(E\cap\{e(0)=X^\partial(\eta)\})d\eta$ where Q^α is the excursion law, for $E \subset \cup_\alpha \mathcal{E}^\alpha$. We shall write $Q^\alpha(t)$ for $Q^\alpha(e(t)\in\bar{D}, l(e)>t)$ so that $Q^\alpha(t) = (1/2)\int_D \partial_{n(a)(\alpha)} p^D(t,x,\alpha)\,dx$.

3. Preparatory results

3.1. The additive structure under conditioning.

Lemma 3. *For P^{α_0}-a.e. ω the process $S(\omega)(\tau,\omega')$, $\tau\geq 0$, is stochastically continuous and for all $\tau,t>0$ we have*

$$\gamma^{0,\tau}(\omega)(t) = \int_0^\tau d\eta Q^{X^\partial(\eta,\omega)}(t).$$

Proof. We first check that a.s. ω our additive process S is continuous in probability with respect to $P(\omega)$. By Theorem 3.3 p. 596 of [22] it is equivalent to show that for all $\mathcal{F}^\partial_\tau$ stopping times θ we have $S_{\theta-}=S_\theta$ a.s. where of course here S is the free unconditioned process. It indeed suffices to observe that the extended boundary process $X^{\partial+}$ (i.e. we are adding the component X^∂) is quasi-left continuous because it is Feller, see Section 2.

The quantity $N(\tau,t)$ (defined in Section 2.1) is integrable since we have by [15]

$$E^{\alpha_0}N(\tau,t) = E^{\alpha_0}[\sum_{\eta\in\mathcal{J}^{0,\tau}} I_{\{l>t\}}(e_\eta)] = E^{\alpha_0}[\int_0^\tau d\eta Q^{X^\partial(\eta)}(t)] < \infty.$$

On the other hand, $N(\tau,t)=\int_{(0,\tau)} d\eta Q^{X^\partial(\eta)}(t) + M_\tau$ where M_τ is an $\mathcal{F}_{S(\tau)}$-martingale. Let $F\in\mathcal{F}^\partial$; since $X^\partial_\tau \in \mathcal{F}_{S(\tau)}$ (suitably completed), see propositions 31 and 30 in [7], then $I_F M_\tau$ is an $\mathcal{F}_{S(\tau)}$-martingale and we have

$$\int_F N(\tau,t)P^{\alpha_0}(d\omega) = \int_F P^{\alpha_0}(d\omega)\int_0^\tau d\eta Q^{X^\partial(\eta)}(t)$$

from which the Lemma follows. □

3.2. **Asymptotics related to $Q^\alpha(t)$.** It seems convenient to pass to the Riemann structure induced by the symmetric positive definite matrix $a^{-1}=[a_{ij}]$ to build the fundamental solution $p^D(t,x,y)$ using the parametrix method, see [16] and [23]. For $x\in\bar{D}$ and $\xi\in\mathbb{R}^d$, let $K(x,\xi)=a_{ij}(x)\xi^i\xi^j$ and set $\Gamma(t,x,y)=(2\pi t)^{-d/2}\exp -K(y,x-y)/2t$. In the canonical patch W_z of Lemma 6.1 of [16], inspired by [15], set $x_*=(x^1,...x^{d-1},-x^d)$, $\Gamma^*(t,x,y)=\Gamma(t,x_*,y)$ and $q_z(t,x,y)=\Gamma(t,x,y)-\Gamma^*(t,x,y)$. In the associated $\mathcal{C}^3$-partition of unity $w^2_{z,\nu}$, $z\in\bar{D}$ and $\nu=1\ ...\ n(z)$, let the "raw" $q(t,x,y)$ be $\sum_{z,\nu} w_{z,\nu}(x)q_z(t,x,y)w_{z,\nu}(y)$ and put

$$p^D(t,x,y) = q(t,x,y) + \int_0^t ds\int_D q(t-s,x,z)f(s,z,y)m(dz) \tag{2}$$

where f satisfies an integral equation. Since a slight modification of the initial solution will not affect the final outcome p^D, the idea is to use the matrix $[a^{ij}_*(y)]$ whose entries are the same as those of $[a^{ij}(y)]$ except for the elements $a^{dj}_*(y)=a^{jd}_*(y)=-a^{dj}(y)$ when $j<d$. Then the substitute for the last but one eq. at the bottom of p. 26 in [16] includes the relation $\partial_t\Gamma^*(t,x,y)=(1/2)a^{ij}_*(y)\partial_{x^ix^j}\Gamma^*(t,x,y)$.

Lemma 4. *Set $K'_i(x,\xi) = (1/2)\partial_{\xi^i} K(x,\xi)$, $i = 1...d$. For all integers $N \geq 0$, there are two polynomials $P_{N,1}(\cdot)$ and $P_{N,2}(\cdot)$ of the ratio K/t with degrees N and coefficients only depending on d s.t. for $i,j = 1...d$ (for convenience, arguments are suppressed below)*

$$\partial_{x^i t^N}\Gamma = t^{-(N+1/2)}\frac{K'_i}{\sqrt{t}}P_{N,1}(\frac{K}{t})\Gamma, \tag{3}$$

$$\partial_{x^i x^j t^N}\Gamma = t^{-(N+1)}\left[-a_{ij}(y)P_{N,1}(\frac{K}{t}) + \frac{K'_i K'_j}{t}P_{N,2}(\frac{K}{t})\right]\Gamma \tag{4}$$

and similar formulas hold for Γ^* when $i \neq d$ by just replacing x by x_*, but in $\partial_d \Gamma^*$ there is moreover a change of sign. This will happen each time we take a derivative in $K(y, y - x_*)$ with respect to x^d.

Proof. Clearly we have $\partial_{x^i}\Gamma = (K'_i/t)\Gamma$, $\partial_{x^i x^j}\Gamma = \left((-a_{ij}(y) + (K'_i K'_j)/t)/t\right)\Gamma$ and $\partial_t \Gamma = ((-d + (K/t))/2t)\,\Gamma$ for all $x, y \in D$, $t > 0$. Concerning Γ^*, when $i \neq d$ we have the same formulas by just replacing x by x_*, but in $\partial_d \Gamma^*$ there is moreover a change of sign because $y^d - x_*^d = y^d + x^d$. We then use induction. □

We now begin with moderate time asymptotics.

Lemma 5. *For all $N \geq 0$ and $T > 0$ there is a $c = c(N,T)$ s.t. for all $x \in D$, $t \leq T$*

$$\int_D m(dy)\left|\partial_t^N f(t,x,y)\right| \leq ct^{-(N+1/2)}.$$

Proof. Let $t \leq T$. We are inspired by the proof of the inequalities (a.8,9) in p. 598 of [23] where the inequality on top of the same page is fundamental. It follows from our construction that $f_0(t,x,y)$ is given by

$$\begin{aligned}
&\sum_{z,\nu} w_{z,\nu}(x)(A_x - \partial_t)(\Gamma - \Gamma^*)w_{z,\nu}(y)\\
&+\sum_{z,\nu} a^{ij}(x)\left[\partial_{x^j} w_{z,\nu}(x)\partial_{x^i}(\Gamma - \Gamma^*) + \partial_{x^i} w_{z,\nu}(x)\partial_{x^j}(\Gamma - \Gamma^*)\right]w_{z,\nu}(y)\\
&+\sum_{z,\nu}(Aw_{z,\nu}(x))(\Gamma - \Gamma^*)w_{z,\nu}(y)
\end{aligned}$$

and that $(A_x - \partial_t)(\Gamma - \Gamma^*)$ is equal to

$$\frac{1}{2}\left[(a^{ij}(x) - a^{ij}(y))\partial_{x^i x^j}\Gamma - (a^{ij}(x) - a_*^{ij}(y))\partial_{x^i x^j}\Gamma^*\right] + b^i(x)\partial_{x^i}(\Gamma - \Gamma^*).$$

Hence by Lemma 4 we have for $n = 0$ and all $N \geq 0$

$$\int_D m(dy)\left|\partial_t^N f_n(t,x,y)\right| \leq t^{-N}(2^N + N2^N)^n C^{n+1}\Gamma(\frac{n+1}{2})^{-1}t^{(n-1)/2}, \tag{5}$$

Suppose, using induction on n, that the inequality (5) is valid for f_n (and all $N \geq 0$). Before using the Leibnitz formula for the N-th derivative of a product, the upper argument of the ds-integral in the eq. for f_{n+1} is freed from t thanks to an obvious change of variables. By the dominated convergence theorem we have for $N \geq 1$ (the case $N = 0$ is immediate)

$$\begin{aligned}
&\int_D m(dy)\partial_t^N f_{n+1}\\
= \;& t^{-N}\Big(\sum_{k=0}^{N}\binom{N}{k}\int_0^t ds\int_D m(dz)\,(t-s)^k\, f_0^{(k)}\int_D m(dy)s^{N-k}f_n^{(N-k)}\\
&+N\sum_{k=0}^{N-1}\binom{N-1}{k}\int_0^t ds\int_D m(dz)\,(t-s)^k\, f_0^{(k)}\int_D m(dy)s^{N-1-k}f_n^{(N-1-k)}\Big).
\end{aligned}$$

It is not difficult to see that we have by monotonicity

$$\int_D m(dy)\left|\partial_t^N f_{n+1}(t,x,y)\right| \le \left(\frac{2^N}{t^N}+\frac{N2^{N-1}}{t^N}\right)(2^N+N2^N)^n C^{n+2}\Gamma(\frac{n+2}{2})^{-1}t^{n/2},$$

whence the lemma by the standard parametrix expansion for f. □

As far as large time asymptotics are concerned we have the

Lemma 6. *When $t > 4T$ (where T is arbitrary but positive), there are for all integers $N \ge 0$ two constants $c = c(a, D, N, T)$ and an absolute c' s.t. for all x, $y \in D$*

$$|\partial_t^N p^D(t,x,y)| \le c\exp -c't.$$

Proof. It follows immediately from Corollary 1 on p. 160 of [11] that there exist absolute constants c' and T and a constant $c = c(a, D, f)$ s.t. for all $x \in D$ and $t > T$

$$|P_t^D f(x)| \le c\exp -c't, \tag{6}$$

where $f(\cdot)$ is uniformly Hölder continuous of some index in $(0, 1)$. By changing the constant c, T can be made arbitrary but positive. Hence (for f constant $= 1$) our result is valid for $N = 0$ by the fact that for some $c = c(T)$ and all $x, y \in D$ we have $p^D(T, x, y) \le c$ and by the semigroup property. Assume that it holds up to N. We again have by the semigroup property, the Leibnitz formula and the dominated convergence theorem

$$\partial_t^{N+1}p^D(t,x,y) = \partial_t\sum_{k=0}^{N}\binom{N}{k}\int_D p^{(k)D}(\frac{t}{2},x,z)p^{(N-k)D}(\frac{t}{2},z,y)m(dz).$$

The induction hypothesis settles all the terms on the r.h.s. where $k \ne 0, N$. Consider e.g. the case $k = 0$ where the novelty concerns only the quantity

$$\int_D p^D(\frac{t}{2},x,z)p^{(N+1)D}(\frac{t}{2},z,y)m(dz);$$

we have

$$\begin{aligned}
p^{(N+1)D}(\frac{t}{2},z,y) &= \partial_t^N\int_D p^D(\frac{T}{2},z,z')\partial_t\left[p^D(\frac{t-T}{2},z',y)\right]m(dz')\\
&= \frac{1}{2}\int_D p^D(\frac{T}{2},z,z')\partial_t^N\left[A_{z'}p^D(\frac{t-T}{2},z',y)\right]m(dz').
\end{aligned}$$

By the dominated convergence theorem and the smoothness of our data we have

$$\partial_t^N[A_{z'}p^D(\frac{t-T}{2},z',y)] = 2^{-N}\int_D A_{z'}p^D(\frac{T}{2},z',z'')\partial_t^N p^D(\frac{t-2T}{2},z'',y)m(dz''),$$

so that the induction hypothesis works again. □

We are now in the position to prove the following fundamental Lemma.

Lemma 7. *For all $\alpha \in \partial D$ and $t > 0$, the monotone function $Q^\alpha(t)$ is given by*

$$Q^\alpha(t) = Q_1^\alpha(t) + Q_2^\alpha(t), \tag{7}$$

where

$$\begin{aligned}
Q_1^\alpha(t) &= \frac{1}{2}\int_D \partial_{n(\alpha)}q(t,\alpha,y)m(dy),\\
Q_2^\alpha(t) &= \frac{1}{2}\int_0^t ds\int_D m(dz)\partial_{n(\alpha)}q\,(t-s,\alpha,z)\int_D f\,(s,z,y)\,m(dy).
\end{aligned}$$

For all $\alpha \in \partial D$, the function $Q^\alpha(t)$ is $\mathcal{C}^\infty$ in t and for all non-negative integers N, there are constants $T > 0$, $c = c(N,T)$ and $c(\alpha, N)$ s.t.

$$\left|Q_2^{(N)\alpha}(t)\right| \leq ct^{-N} \text{ if } t \leq T \tag{8}$$

$$Q^{(N)\alpha}(t) \sim c(\alpha, N)t^{-(N+1/2)} \text{ as } t \to 0 \tag{9}$$

where $c(\alpha, 0) = 1/\sqrt{2\pi}$ is independent of α and $c(\alpha, N)$ is bounded on ∂D. Moreover, the boundary function $Q^{\cdot}(t)$ is Lipschitz continuous, i.e. there is a constant $c = c(T)$ s.t. for all $\alpha, \beta \in \partial D$ and all $t \leq T$ we have

$$|Q^\alpha(t) - Q^\beta(t)| \leq ct^{-1}\|\alpha - \beta\|. \tag{10}$$

On the other hand, there are constants $c = c(D, N, T)$ and an absolute c' s.t. for all $t > T$ and $\alpha \in \partial D$

$$|Q^{(N)\alpha}(t)| \leq c \exp(-c't). \tag{11}$$

Proof. Let $\alpha \in \partial D$. It is trivial that $Q^\alpha(t)$ in Section (2.2) is monotone decreasing (see [15] or the Corollary on p. 63 in [16]). The case $N = 0$, needed below in deriving a lower bound for $\widehat{Q}_s^\alpha(\lambda)$, is special and treated apart so that the developments just below are not strictly necessary for now. Take moderate t's, i.e. consider $t \leq T$. The inequality (8) for $N = 0$ follows from Lemma 5 and the leading term near zero is Q_1^α. Now set

$$G(z, \xi) = (a_{di}\,(z)\,\xi^i)\sqrt{|a^{-1}(z)|}\exp -K(z, \xi)/2$$

and note that by our assumptions $G(z, \xi)$ is smooth in both variables with bounded ξ-derivatives (recall that $a_{di}(\alpha) = \delta_{di}$ and $K(\alpha, \xi) = a_{ij}(\alpha)\xi^i\xi^j + (\xi^d)^2$ where $i, j < d$).

By construction of the partition of unity $w_{z,\nu}^2$ there is an integer $n_0 = n_0(D)$ s.t. for all $\alpha \in \partial D$ there are (with an unambiguous abuse of notation) at most n_0 open balls $B(\beta(\nu), \delta(\nu))$, $\nu = 1 \dots n_0$, that pass through α. Let δ be so small that $B(\alpha, \delta)$ is in all the balls $B(\beta(\nu), \delta(\nu))$, $\nu = 1 \dots n(\alpha)$, that contain α and set $B_\nu^c(\alpha) = B(\beta(\nu), \delta(\nu))\backslash B(\alpha, \delta)$. Since $\partial_n w_{z,\nu}(\beta) = 0$ for all $\beta \in \partial D$, we have

$$\begin{aligned} Q_1^\alpha(t) &= \frac{1}{2}\sum_{\nu=1}^{n(\alpha)} w_{\beta,\nu}(\alpha)\int_{B^+(\beta,\delta)(\nu)} \partial_{d(\alpha)}q_\beta\,(t, \alpha, y)\,w_{\beta,\nu}(y)m(dy) \\ &= \sum_{\nu=1}^{n(\alpha)} w_{\beta,\nu}(\alpha)\int_{B^+(\alpha,\delta)} \partial_{d(\alpha)}\Gamma\,(t, \alpha, y)\,w_{\beta,\nu}(y)m(dy) + Q_{12}^\alpha(t) \end{aligned}$$

where $Q_{12}^\alpha(t)$ is the middle sum above but the integrals are now over $B_\nu^c(\alpha) \cap \mathbb{R}_+^d$. Since for all $y \in D$ and $\nu = 1 \dots n(\alpha)$ we have $K(y, y - \alpha) \geq c(\delta)$ on $B_\nu^c(\alpha)$, then by Lemma 4 it holds that for some $c = c(\delta, n_0, D)$

$$|Q_{12}^\alpha(t)| \leq ct^{-(d+1)/2}\exp(-c'/t) \tag{12}$$

so that $\sqrt{t}Q_{12}^\alpha(t) \to 0$ as $t \to 0$. Let us now deal with the main term above, say $Q_{11}^\alpha(t)$. We have by a change of variables (centers at 0 are suppressed)

$$\begin{aligned} &\sqrt{t}Q_{11}^\alpha(t) = \frac{1}{(2\pi)^{d/2}}\int_{B^+(\delta)/\sqrt{t}} G(\alpha, y)dy \\ &+ c\sum_{\nu=1}^{n(\alpha)} w_{\beta,\nu}(\alpha)\int_{B^+(\delta)/\sqrt{t}} \left[w_{\beta,\nu}(\cdot)G(\cdot, y)|_\alpha^{\alpha+y\sqrt{t}}\right]dy = Q_0^\alpha(t) + Q_{13}^\alpha(t). \end{aligned} \tag{13}$$

As $t \to 0$, the half-balls $B^+(\delta)/\sqrt{t}$ increase to $\mathbb{R}_+^d$ and by the monotone convergence theorem the main term $Q_0^\alpha(t)$ increases to $1/\sqrt{2\pi}$. By the elementary Lagrange formula we have for $\nu = 1 \ldots n(\alpha)$

$$w_{\beta,\nu}(\cdot)G(\cdot,y)|_{\alpha}^{\alpha+y\sqrt{t}} = \sqrt{t}y^k \int_0^1 \partial_{z^k}\left(G(\cdot,y)w_{\beta,\nu}(\cdot)\right)(\alpha+hy\sqrt{t})dh,$$

so that $Q_{13}^{\alpha}(t) \to 0$ as $t \to 0$ and the equivalence of the lemma is proved for $N = 0$.

For $N > 0$, the inequality (8) is established as in the proof of Lemma 5 by just replacing the couple (f_0, f_n) with $(\partial_n q, f)$. Now, the following quantity

$$t^{N+1/2} \sum_{\nu=1}^{n(\alpha)} w_{\beta,\nu}(\alpha) \int_{B^+(\beta,\delta)(\nu)} \partial_{d(\alpha)t^N} \Gamma(t,\alpha,y)\, w_{\beta,\nu}(y) m(dy)$$

is the sum of a main and error terms, as in the case $N = 0$, where the integrations are respectively over the half-ball $B^+(\alpha,\delta)$ and the set $B_\nu^c(\alpha) \cap \mathbb{R}_+^d$ and we see that by Lemma 4 and the dominated convergence theorem (we stress that we don't need to use again the finer decomposition of eq. (13)) the main term converges to

$$(2\pi)^{-d/2} \int_{\mathbb{R}_+^d} P_{N,1}(K(\alpha,y))G(\alpha,y)dy$$

whereas the remainder is treated as $Q_{12}^{\alpha}(t)$ above. The equivalence (9) follows from the relation (8).

Next, note that whereas the first order space derivatives $\partial_{x^i} p^D(t,x,y)$, $i = 1 \ldots d$, can be estimated directly, see (a.13,14) p. 600 in [23] where $\int_D \left|\partial_{x^i} p^D(t,x,y)\right| m(dy) \le c/\sqrt{t}$, the second order ones are not straightforward. An ingeneous use of the single layer potentials, see relation (2.3) of Theorem 2.1 p. 247 of [12], gives $\mathcal{C}^2$ estimates up to the boundary and the inequality (10) follows by our assumptions on the data.

Now we pass on to $t > T$ and inequality (11). By the semigroup property, Lemma 6 and the dominated convergence theorem we have

$$\left|\partial_t^N \partial_{n(\alpha)} p^D(t,y,\alpha)\right| \le \int_D \left|\partial_t^N p^D(t-T,y,z)\right| \partial_{n(\alpha)} p^D(T,z,\alpha) m(dz) \le c \exp -c't$$

where $c = c(N, D, t)$. □

3.3. **Asymptotics related to the exponent of S.** To deal with improper infinite integrals we need the

Lemma 8. *Let f be an integrable function on $[u,v]$, $u, v \in \mathbb{R}$, and g a non-negative and non-decreasing (respectively non-increasing) function on $[u,v]$, then we have for some $c \in [u,v]$ by Bonnet's Theorem*

$$\int_u^v f(t)g(t)dt = g(v)\int_c^v f(t)dt \;\; (\textit{respectively } = g(u)\int_u^c f(t)dt).$$

If g is a function in $\mathcal{C}^1([u,v])$ and $\lambda > 0$, then we have the inequality

$$\left|\int_u^v \sin(\lambda t)g(t)dt\right| \le \frac{1}{\lambda}(|g(u)| + \int_u^v |g'(t)|\,dt). \tag{14}$$

Moreover when $g \in \mathcal{C}^N(\mathbb{R}_+)$, $N \ge 1$, and the functions $g_k(t) = t^k g^{(k)}(t)$, $k = 0 \ldots N$, are in $L^1(\mathbb{R}_+)$, then there are absolute integers c_k s.t. for all $h > 0$

$$\partial_h^N \int_{\mathbb{R}_+} dt \sin(t) g(\frac{t}{h}) = \frac{(-1)^N}{h^N} \int_{\mathbb{R}_+} dt \sin(t) \sum_{k=1}^N c_k g_k(\frac{t}{h}). \tag{15}$$

This formula is also considered to hold for $N = 0$ if we take $g_0 = g^{(0)} = g$ and $c_0 = 1$. Similar considerations hold for cosine.

Proof. The inequality follows, by Bonnet, from splitting $g^{(1)}$ as the difference of its positive and negative parts. Next, it suffices to note that

$$tg_k^{(1)}(t) = kt^k g^{(k)}(t) + t^{k+1} g^{(k+1)}(t), \tag{16}$$

and use induction and the dominated convergence theorem. □

The Lemma 8 will be used when g has different behaviours for moderate and arbitrarily large values of t. The improper integrals will be typically split into one small interval near zero for technical reasons and other more natural intervals according to different behaviours of g. We are now in the position to prove the

Lemma 9. *Let $\tau > \theta \geq 0$. The exponent of S is the complex boundary functional*

$$\Psi^{\theta,\tau}(\lambda) = \Psi_s^{\theta,\tau}(\lambda) - i\Psi_c^{\theta,\tau}(\lambda) = \int_\theta^\tau d\eta V(\lambda, X_\eta^\partial)$$

where $\Psi_s^{\theta,\tau}(\lambda) = \lambda\widehat{Q}_s^\alpha(\lambda)$, $\Psi_c^{\theta,\tau}(\lambda) = \lambda\widehat{Q}_c^\alpha(\lambda)$ and $V(\lambda,\alpha) = -i\lambda\widehat{Q}^\alpha(\lambda)$.

For all $N \geq 0$, there exist a λ_0 and constants c_0 and c s.t. for any $\lambda \geq \lambda_0$ and $\tau > \theta \geq 0$ we have the following lower and upper bounds

$$\Psi_s^{\theta,\tau}(\lambda) \geq c_0(\tau-\theta)\sqrt{\lambda}, \tag{17}$$

$$|\Psi_s^{\theta,\tau(N)}(\lambda)| + |\Psi_c^{\theta,\tau(N)}(\lambda)| \leq c(\tau-\theta)\lambda^{1/2-N}. \tag{18}$$

Proof. By Lemma 7 $Q^\alpha(t) \in L^1((0,\infty))$ for all $\alpha \in \partial D$. By extending $Q^\alpha(t)$ as 0 to the left of the origin, the new function (still noted Q^α) is in $L^1(\mathbb{R})$. We then recover the exponent $\Psi^{\theta,\tau}(\lambda)$ in (1) as a bona fide Fourier transform thanks to an elementary integration by parts $\Psi^{\theta,\tau}(\lambda) = -i\lambda\widehat{\gamma}^{\theta,\tau}(\lambda)$. We now need Fubini in the context of improper integrals. Since $Q^\alpha(\cdot)$ is decreasing, its sine integrals over the periods $[2\pi k/\lambda, 2\pi(k+1)/\lambda]$, $k \geq 0$, are positive. We can apply Fubini over all the periods $[2\pi k/\lambda, 2\pi(k+1)/\lambda]$, $k \geq 0$, and the global Fubini follows by monotone convergence. The cosine integral is treated similarly but after isolating the first interval $[0, 3\pi/2\lambda]$ after which we are reduced to the situation of a decreasing positive function to be integrated against $\cos\lambda(t + 3\pi/2\lambda) = \sin\lambda t$, $t \geq 0$.

To treat the lower bound on the sine transform of $Q^\alpha(t)$ (as well as estimates on the sine and cosine transform derivatives) the improper integrals will be split into three parts. Lemmas 8 and 7 will be used several times below and they will not be explicitly mentioned. We have to be very careful and watch for monotonicities.

For all $N \geq 0$ there are three constants $c \in \mathbb{R}$, $c' \in \mathbb{R}$ and $t_1 > 0$ s.t. for all $\alpha \in \partial D$ and $t \leq t_1$ we have

$$c \leq t^{N+1/2} Q^{(N)\alpha}(t) \leq c'$$

and $c > 0$ when $N = 0$. On the other hand, there exists a t_2 s.t. for all $t \leq t_2$ we have the inequality $K(y, y-\alpha)/t > (d+2)$ (see Lemma 4 where $P_{1,1}(u) = (u-(d+2))/2$) on the set $B_\nu^c(\delta)$ of the proof of Lemma 7. Take the base $t_0 = \min\{t_1, t_2\}$ which will be yet again subject to further constraints. Recall that the terms $Q_0^\alpha(t)$, $Q_{12}^\alpha(t)$ and $Q_{13}^\alpha(t)$ appear in the proof of Lemma 7. Recall also that c which appears in Lemma 8 is any argument in $[u,v]$ which realizes the intermediate value theorem (used in the proof of the second mean value theorem on p. 193 of [2]) when the primitive $\int_u^t f(s)ds$ runs between its minimum and maximum values in $t \in [u,v]$. Now set $F(t) = \int_{(0,t)} ds \sin(s)/\sqrt{s}$ and note that $F(t)$ is positive, strictly increases on $[0,\pi]$ to its global maximum which is reached at $t = \pi$ and converges to $\sqrt{\pi/2}$ as $t \to \infty$; moreover its (strictly positive) global minimum after $t = \pi$ is reached at $t = 2\pi$ (facts known as Titchmarsh's Lemma which follows e.g. from the Leibnitz alternating series criterion). The term $Q_0^\alpha(t)$, see eq. (13), can be extended to a continuous function, still noted $Q_0^\alpha(t)$, to the closed interval $[0, t_0]$ by setting $Q_0^\alpha(0) = 1/\sqrt{2\pi}$. Let $\delta > 0$ be some small number to be fixed below and set

$\lambda_0' = \delta/t_0$, there exists a $\lambda_0 \geq \lambda_0'$ s.t. for all $\lambda \geq \lambda_0$ ($Q_0^\alpha(\cdot/\lambda)$ is obviously monotone decreasing for all $\lambda > 0$ and $\alpha \in D$)

$$\lambda \int_0^{\delta/\lambda} dt \frac{\sin(\lambda t)}{\sqrt{t}} Q_0^\alpha(t) = \sqrt{\frac{\lambda}{2\pi}} F(c(\delta, \lambda))$$

where $c(\delta, \lambda) \in [0, \delta]$ with of course $F(c(\delta, \lambda)) \in [0, F(\pi)]$. On $[\delta/\lambda, t_0]$, let us write

$$\begin{aligned} \lambda \int_{\delta/\lambda}^{t_0} dt \frac{\sin(\lambda t)}{\sqrt{t}} Q_0^\alpha(t) &= \sqrt{\frac{\lambda}{2\pi}} \int_0^{\lambda t_0} \frac{\sin t}{\sqrt{t}} dt \\ &\quad -\sqrt{\lambda} \left[\frac{1}{\sqrt{2\pi}} \int_0^\delta \frac{\sin t}{\sqrt{t}} dt + \left(\frac{1}{\sqrt{2\pi}} - Q_0^\alpha(t_0) \right) \int_{c'(\delta,\lambda)}^{\lambda t_0} \frac{\sin t}{\sqrt{t}} dt \right] \end{aligned}$$

where $c'(\delta, \lambda)$ can in fact be chosen in $[\delta, \pi]$. For λ_0 large enough and $\lambda \geq \lambda_0$ the first term on the r.h.s. above is $\geq F(2\pi)\sqrt{\lambda/2\pi}$ and we can choose t_0 and δ so small that $F(\delta) < F(2\pi)/4$ and $2F(\pi)(1/\sqrt{2\pi} - Q_0^\alpha(t_0)) < F(2\pi)/(4\sqrt{2\pi})$. Hence for $c_0 = F(2\pi)/(2\sqrt{2\pi})$ the r.h.s. above is $> c_0\sqrt{\lambda}$ but we have not yet finished with $Q_1^\alpha(t)$. As far as Q_{12}^α is concerned we have, without constraints on λ, by construction of t_0 for some $c(\lambda) \in [0, \pi]$

$$\lambda \int_0^{t_0} \sin(\lambda t) Q_{12}^\alpha(t) dt = \sqrt{\lambda} Q_{12}^\alpha(t_0) \sqrt{t_0} \int_{c(\lambda)}^{\lambda t_0} \frac{\sin t}{\sqrt{t}} dt,$$

so that by eq. (12) the contribution of Q_{12}^α in absolute value falls below $\sqrt{\lambda}c_0/2$ for small enough t_0. Next, write for some large $\delta' = \delta'(F)$, to be fixed below and all $\lambda \geq \lambda_0 \geq \lambda_0' = \max\{\delta/t_0, \delta'/t_0\}$

$$\lambda \int_0^{t_0} \frac{\sin(\lambda t)}{\sqrt{t}} Q_{13}^\alpha(t) dt = \lambda \left(\int_0^{\delta'/\lambda} + \int_{\delta'/\lambda}^{t_0} \right) \sin(\lambda t) \frac{Q_{13}^\alpha(t)}{\sqrt{t}} dt;$$

regarding the first integral, by the proof of Lemma 7 and the properties of the partition of unity w the quantity $Q_{13}^\alpha(t)/\sqrt{t}$ is bounded over $(0, t_0)$ so that the first integral contributes less than $c\delta'$ for some $c > 0$; in order to track derivatives with respect to t in $Q_{13}^\alpha(t)/\sqrt{t}$ let us use spherical coordinates in $B^+(\delta)/\sqrt{t}$. We have for $y \neq 0$, $y = \|y\| (y/\|y\|)$ and so $y = r(\theta^1, \theta^2, ..., \theta^d)$ where the θ^i's are the director cosines, i.e. the cosines of the angles between y and the vectors of the canonical basis, and $r \in (0, \delta/\sqrt{t})$ so that our half-ball is transformed into the rectangle $[-1, 1]^{d-1} \times [0, \delta/\sqrt{t}]$. The Jacobian of the transformation $y = y(\theta^1, \theta^2, ..., \theta^{d-1}, r)$ is given by $r^{d-1}/\left|\theta^d\right|$ and we have by another Leibnitz formula and a change of variables

$$\begin{aligned} \frac{d}{dt}\left(Q_{13}^\alpha(t)/\sqrt{t}\right) &= ct^{-1/2} \int_0^1 h dh \int_{B_\nu^+(\delta)} dy \sum_{k,l=1}^d \partial_{z^k z^l} [G(\cdot, y) w_{\beta,\nu}(\cdot)](\alpha + hy\sqrt{t}) \\ &\quad -c't^{-3/2} \int_0^1 dh \int_{\left[-\delta/\sqrt{t}, \delta/\sqrt{t}\right]^{d-1}} \frac{d\theta^1 ... d\theta^{d-1}}{\sqrt{1 - t\delta^{-2}(\theta_1^2 + ...\theta_{d-1}^2)}} \\ &\quad \sum_{k=1}^d \theta^k \partial_{z^k} [G(\cdot, \theta) w_{\beta,\nu}(\cdot)](\alpha + h\theta\sqrt{t}) \end{aligned}$$

for some constants c and c'. Hence from our assumptions and the exponential inequality of [23] used in the proof of Lemma 5 (concerning θ^d) it follows that $\left|d\left(Q_{13}^\alpha(t)/\sqrt{t}\right)/dt\right| \leq ct^{-3/2}$, where $c = c(\delta, t_0)$, so that if δ' is sufficiently large and then for a λ_0 large enough we have

$$\left| \lambda \int_0^{t_0} \frac{\sin(\lambda t)}{\sqrt{t}} Q_{13}^\alpha(t) dt \right| \leq c\sqrt{\lambda},$$

for all $\lambda \geq \lambda_0$, where $c \leq c_0/4$. To sum up, there is (by abuse of notation) a $c_0 > 0$ and a λ_0 so large that for all $\lambda \geq \lambda_0$

$$\lambda \int_0^{t_0} \sin(\lambda t) Q_1^\alpha(t) dt > c_0 \sqrt{\lambda}.$$

Since the derivative of the term $Q_2^\alpha(t)$ is bounded by c/t, the above inequality is still valid for $Q^\alpha(t)$. For a sequence $m \to \infty$ write

$$\left| \lambda \int_{t_0}^{m} \sin(\lambda t) Q^\alpha(t) dt \right| \leq Q^\alpha(t_0) + c \int_{t_0}^{m} \exp(-c's) ds < c(t_0) < \infty,$$

so that the infinite part of the sine improper integral will not contribute against $c_0\sqrt{\lambda}$ for all sufficiently large λ and the lower bound (17) is established.

Let us now deal with the inequality (18). We consider only the cosine term since the sine one is similar. By a change of variables we have

$$\lambda \widehat{Q}_c^\alpha(\lambda) = \int_0^\infty \cos(t) Q^\alpha(\frac{t}{\lambda}) dt$$

and we are in the situation of Lemma 8. Take $N \geq 1$ (the result is trivial when $N = 0$) and set (the c_k's are those of Lemma 18)

$$G_N(\cdot) = \sum_{k=1}^{N} c_k Q_k^\alpha(\cdot), \tag{19}$$

for the above λ_0 and all $\lambda \geq \lambda_0$, on $[t_0, \lambda t_0]$ we have $|G_N(t/\lambda)| \leq c(N)\sqrt{\lambda/t}$. By relation (16) (in the proof of Lemma 8) we have

$$\partial_t \left[G_N(\frac{t}{\lambda}) \right] = \frac{1}{\lambda} \sum_{k=1}^{N} c_k Q_k^{(1)\alpha}(\frac{t}{\lambda}) = \frac{1}{t} \sum_{k=1}^{N+1} c_k Q_k^\alpha(\frac{t}{\lambda}),$$

so that $|\partial_t G_N(t/\lambda)| \leq c\sqrt{\lambda} t^{-3/2}$ and

$$\left| \int_{t_0}^{\lambda t_0} \cos(t) G_N(\frac{t}{\lambda}) dt \right| \leq c(t_0)\sqrt{\lambda} + c\sqrt{\lambda} \int_{t_0}^{\lambda t_0} t^{-3/2} dt \leq c(N, t_0)\sqrt{\lambda}.$$

On $[0, t_0]$ we comfortably write

$$\left| \int_0^{t_0} \cos(t) G_N(\frac{t}{\lambda}) dt \right| \leq c(N) \int_0^{t_0} \sqrt{\lambda/t}\, dt \leq c(N, t_0)\sqrt{\lambda}$$

and as usual on $(\lambda t_0, m)$ we have for all $\lambda \geq \lambda_0$

$$\begin{aligned} \left| \int_{\lambda t_0}^{m} \cos(t) G_N(\frac{t}{\lambda}) dt \right| &\leq c(N, t_0) + \int_{t_0}^{m/\lambda} \sum_{k=1}^{N+1} c_k t^{k-1} \left| Q^{(k)\alpha}(t) \right| dt \\ &\leq c(N, t_0) \leq c(N, t_0)\sqrt{\lambda} \end{aligned}$$

whence the result. □

4. Hitting an interval

We establish here the fundamental Theorem 15 below.

4.1. **A formula and estimates for the transition density of S.**

Lemma 10. *Recall we are working in the formalism of* [19]. *Under E_s^θ the r.v. $S(\tau)$, $\tau \geq 0$, is absolutely continuous with density $p^{\theta,\theta+\tau}(\omega)(s,t)$, $t \geq 0$, in $\mathcal{C}_0^\infty$ given by*

$$p^{\theta,\theta+\tau}(s,t) = \frac{1}{2\pi}\int_{\mathbb{R}} d\lambda \exp(-i(t-s)\lambda)\exp -\Psi^{\theta,\theta+\tau}(\lambda) \tag{20}$$

$$= \frac{1}{\pi}\int_0^\infty d\lambda[\cos((t-s)\lambda)f_c^{\theta,\theta+\tau}(\lambda) + \sin((t-s)\lambda)f_s^{\theta,\theta+\tau}(\lambda)] \tag{21}$$

where (see Lemma 9)

$$\begin{aligned} f_c^{\theta,\theta+\tau}(\lambda) &= \exp(-\Psi_s^{\theta,\theta+\tau}(\lambda))\cos(\Psi_c^{\theta,\theta+\tau}(\lambda)), \\ f_s^{\theta,\theta+\tau}(\lambda) &= \exp(-\Psi_s^{\theta,\theta+\tau}(\lambda))\sin(\Psi_c^{\theta,\theta+\tau}(\lambda)). \end{aligned}$$

Proof. Fix a time parameter $\theta \geq 0$ and a space position $s \geq 0$. The process S moves by adding non-negative (space) jumps to s. The increment $S(\tau_2) - S(\tau_1)$, $\theta \leq \tau_1 < \tau_2$, is a r.v. with the Fourier transform $\exp -\Psi^{\tau_1,\tau_2}(\lambda)$ and, by Lemma 9, $\lambda^N \exp -\Psi^{\tau_1,\tau_2}(\lambda)$ is absolutely integrable for all integers $N \geq 0$ whence the well known smoothness and representation result (20). Passing from the Fourier transform to the sine and cosine transforms, thanks to dominated convergence, the other identity then follows. □

Lemma 11. *Under the notations of the previous Lemma, let $N \geq 1$ (the result for $N = 0$ is trivial). There is a λ_0 and a constant c s.t. for all $\lambda \geq \lambda_0$ we have*

$$\left|\partial_\lambda^N f_{s,c}^{\theta,\theta+\tau}(\lambda)\right| \leq c\frac{\sum_{k=1}^N (\tau\sqrt{\lambda})^k}{\lambda^N}\exp -c_0\tau\sqrt{\lambda}.$$

Moreover, if $\lambda \leq \lambda_0$ there is a $c = c(\lambda_0)$ s.t. $\left|\partial_\lambda^N f_{s,c}^{\theta,\theta+\tau}(\lambda)\right| \leq c(\lambda_0)$.

Proof. We first pass, thanks to the elementary Euler relations, to differentiating the complex exponential functions $\exp(-\Psi_s^{\theta,\theta+\tau}(\lambda) \pm i\Psi_c^{\theta,\theta+\tau}(\lambda))$. We use the explicit Bruno formula for the N-th derivative of the latter composite exponential functions (see the topics related to Partitions and the Bell polynomials in virtually any textbook on combinatorics) combined with Lemma 9. The second claim follows by dominated convergence, Lemma 7 and Leibnitz. □

Lemma 12. *There are constants c, c' and $c'' = c''(\tau)$ which is bounded on compacts of $\mathbb{R}_+$ s.t.*

$$p^{\theta,\theta+\tau}(s,t) \leq c + c'(t-s)^{-1} + c''(t-s)^{-2}.$$

Proof. For convenience, take $\theta = 0$ and $s = 0$. Let us consider only the cosine term in eq. (21) since the other term is similar. For the λ_0 of the above Lemma and for $m \to \infty$ we have thanks to a change of variables and an integration by parts

$$\begin{aligned} &\frac{1}{\tau^2}\int_{\tau^2\lambda_0}^m d\lambda \cos(\frac{t}{\tau^2}\lambda)f_c^{0,\tau}(\frac{\lambda}{\tau^2}) \\ = \ &\frac{1}{t}\left(\sin(\frac{t}{\tau^2}\cdot)f_c^{0,\tau}(\frac{\cdot}{\tau^2})|_{\tau^2\lambda_0}^m\right) - \frac{1}{t\tau^2}\int_{\tau^2\lambda_0}^m d\lambda \sin(\frac{t}{\tau^2}\lambda)\partial_\lambda f_c^{0,\tau}(\frac{\lambda}{\tau^2}) \end{aligned}$$

and the first term on the r.h.s. above is (in modulus) less than $1/t$; concerning the second one we apply Lemma 8 and the corresponding first term in the inequality (14) is, by Lemma 11, less than $c(\lambda_0)\tau_0 t^{-2}$. Again by Lemma 11, a change of variable and an integration by parts we have for the corresponding second term in the same inequality

$$\begin{aligned}\frac{1}{t^2\tau^2}\int_{\tau^2\lambda_0}^{m}\left|\partial_\lambda^2 f_c^\tau(\frac{\lambda}{\tau^2})\right|d\lambda &\le c\frac{\tau^2}{t^2}\int_{\tau^2\lambda_0}^{m} d\lambda(\lambda^{-3/2}+\lambda^{-1})\exp -c_0\sqrt{\lambda}\\ &= c\frac{\tau^2}{t^2}\int_{\tau\sqrt{\lambda_0}}^{\sqrt{m}} d\lambda(\lambda^{-2}+\lambda^{-1})\exp -c_0\lambda\\ &\le \frac{c(\lambda_0)}{t^2}(\tau^2+\tau^2\left|\log\tau\right|+\tau+1).\end{aligned}$$

The integral over $(0,\tau^2\lambda_0)$ gives only $c(\lambda_0)$ which is independent of both t and τ. □

4.2. **A non-homogeneous Lévy kernel.** We are inspired by a device in [24] p. 160.

Lemma 13. *Let φ be a smooth function with compact (connected) support in $(0,\infty)$ and $0<t<\inf Supp(\varphi)$. We have for all $\theta>0$*

$$\lim_{\tau\to 0}\frac{P^{\theta,\theta+\tau}\varphi(t)}{\tau}=\int_0^\infty ds\varphi^{(1)}(t+s)Q^{X_\theta^\partial}(s).$$

Proof. Since φ is smooth with compact support, we can make $N\ge 0$ successive integrations by parts regarding the exponential in $\widehat{\varphi}$ and we easily deduce that both φ and $\widehat{\varphi}$ are in $L^1(\mathbb{R})$ with

$$|\widehat{\varphi}(\lambda)|\le\lambda^{-N}\left\|\varphi^{(N)}\right\|_{L^1}. \tag{22}$$

Now when $t<\inf Supp(\varphi)$ we have $\varphi(t)=0$ and by Lemma 10, the inequality (22) with $N=0$ near the origin and $N=2$ at infinity, and Fubini we can write

$$\frac{P^{\theta,\theta+\tau}\varphi(t)}{\tau}=\frac{1}{2\pi}\int_{\mathbb{R}}d\lambda\exp(i\lambda t)\widehat{\varphi}(-\lambda)\frac{\exp(-\Psi^{\theta,\theta+\tau}(\lambda))-1}{\tau}. \tag{23}$$

For fixed $z\in\mathbb{C}$ we have by the Taylor formula $\exp(-z)=1-z+R(z)$ where $R(z)=\exp(-c(z)z)z^2/2$ for some real $c(z)\in(0,1)$. Therefore, expressing the exponent as a full Fourier transform as in the proof of Lemma 9, we have by essentially Lemma 9 and Cauchy-Schwarz for all $\lambda\in\mathbb{R}$

$$\begin{aligned}\left|R(\Psi^{\theta,\theta+\tau}(\lambda))\right| &\le c\lambda\left|\Psi^{\theta,\theta+\tau}(\lambda)\right|^2\int_\theta^{\theta+\tau}d\eta\int_{\mathbb{R}}ds\sin(\lambda s)Q^{X_\eta^\partial}(s)\exp -c\Psi^{\theta,\theta+\tau}(\lambda)\\ &\le c\lambda^2\tau\int_\theta^{\theta+\tau}d\eta\left|\widehat{Q}^{X_\eta^\partial}(\lambda)\right|^2\le c\lambda^2\tau^2,\end{aligned}$$

so that by inequality (22) with $N=4$ for large λ's, the contribution of $R(\Psi)$ in eq. (23) is $\le c\tau$ for some absolute constant c. For small λ's it suffices to take $N=0$. Again by inequality (22), Fubini and by Proposition 5.1.13 p. 193 of [6] (beware of their different conventions) the main term in eq. (23) is equal to

$$\begin{aligned}&\frac{1}{2\pi}\int_{\mathbb{R}}d\lambda\exp(i\lambda t)\widehat{\varphi}(-\lambda)\frac{i\lambda\int_\theta^{\theta+\tau}d\eta\widehat{Q}^{X_\eta^\partial}(\lambda)}{\tau}\\ =\;&\frac{1}{2\pi\tau}\int_\theta^{\theta+\tau}d\eta\int_{\mathbb{R}}d\lambda\exp(i\lambda t)\widehat{\varphi^{(1)}}(-\lambda)\widehat{Q}^{X_\eta^\partial}(\lambda)\\ =\;&\frac{1}{\tau}\int_\theta^{\theta+\tau}d\eta\int_0^\infty ds\varphi^{(1)}(t+s)Q^{X_\eta^\partial}(s)\\ =\;&\int_0^\infty ds\varphi^{(1)}(t+s)Q^{X_\theta^\partial}(s)+R(\theta,\tau)\end{aligned}$$

where in the above last two lines the $Q^{\cdot}(\cdot)$ is the original unextended one on $(0,\infty)$ and

$$R(\theta,\tau)=\frac{1}{\tau}\int_{\theta}^{\theta+\tau}d\eta\int_{0}^{\infty}ds\varphi^{(1)}(t+s)(Q^{X_{\eta}^{\partial}}-Q^{X_{\theta}^{\partial}})(s) \tag{24}$$

which converges to 0 by the right continuity of $X_{\cdot}^{\partial}(\omega)$ and the dominated convergence theorem. □

4.3. **Main result.** Before we state our key theorem, we shall need the following

Lemma 14. *For all $\gamma\in(0,1/2)$, there exists a constant c s.t. for all $\tau>\theta\geq 0$ we have the variation*

$$E^{\alpha_0}\left\|X^{\partial}(\tau)-X^{\partial}(\theta)\right\|\leq c(\tau-\theta)^{1/(1+2\gamma)}.$$

Proof. For $i=1\ ...\ d$ define the boundary functions $\varphi_i(\alpha)=\alpha_i$ (which are of course in the domain of A^{∂}) and let u_i be the solution of the corresponding Dirichlet problem. By Theorem 6.14 p. 107 of [13] and our assumptions it follows that for some c we have $|\partial_k u_i(x)|\leq c$ on $\bar{D}$ for all $i,k=1\ ...\ d$. By the Itô formula, the Burkholder-Davis-Gundy inequality and well known inequalities we have for some $c=c(a,D,\gamma)$ and all $\tau>\theta\geq 0$

$$\begin{aligned}E^{\alpha_0}\left|X_i^{\partial}(\tau)-X_i^{\partial}(\theta)\right| &\leq \left(E^{\alpha_0}\left|X_i^{\partial}(\tau)-X_i^{\partial}(\theta)\right|^{1+2\gamma}\right)^{1/(1+2\gamma)}\\ &\leq c\left(E^{\alpha_0}\left(S(\tau)-S(\theta)\right)^{1/2+\gamma}+(\tau-\theta)\right)^{1/(1+2\gamma)}.\end{aligned}$$

On the other hand, consider Theorem 2.1 p. 55 of [20] which gives moment estimates for a (stationary) Lévy process in terms of its Lévy measure (thanks to a clever Laplace transform device). If we follow, on p. 57, the indications of the step d) of its proof and also the remarks below and replace the product of the time parameter by the Lévy measure therein by our time dependent Lévy measure, we have for separate ω's by Lemma 7 that for some absolute c

$$E_{\delta_0}^{0}(\omega)\left(S(\tau)-S(\theta)\right)^{1/2+\gamma}\leq\int_0^{\infty}s^{1/2+\gamma}\gamma^{\theta,\tau}(\omega,ds)<c(\tau-\theta)$$

and the Lemma follows by conditioning. □

Theorem 15. *Let $\alpha_0\in\partial D$, $\tau>0$ and $I=(b,b')$, $b'>b>0$. Under the above notations, the hitting probability of I by the (progressive) inverse local time S before τ is given by*

$$P^{\alpha_0}\left\{\exists\theta<\tau|S_\theta\in I\right\}=\int_0^b dtE^{\alpha_0}\int_0^{\tau}d\theta Q^{X_{\theta}^{\partial}}(I-t)p^{0,\theta}(0,t).$$

Proof. The basic observation is that as the càdlàg process S is strictly increasing, the passage from below b into I occurs only once in $(0,\tau)$. It follows from our data and [15] that a.s.-$P^{\alpha_0}(d\omega)$ the precise time b will be the beginning or the end of no excursion e and $X(b)\notin\partial D$. Moreover, since the extended boundary process $X^{\partial+}$ is Feller, the law of the hitting time θ_I of the slice $I\times\partial D$ is continuous, see e.g. Theorem 4.7 p. 11 in [21], so that I can't be hit by S at exactly τ, i.e. the passage into I occurs only via a unique (space) jump recorded at some time $\theta<\tau$. To deal with singularities and avoid above all the short time asymptotics of the transition densities of S which arise e.g. when trying to use another Euler-McLaurin on the sum over k below, let us consider two small (unrelated) time and space numbers τ_0 and ϵ decreasing to zero and write by Lemma 3 and the monotone convergence theorem (as $\alpha_0\in\partial D$, for a fixed "environment" ω we are working under $E^0_{\delta_0}(\omega)$)

$$\begin{aligned} P^{\alpha_0}\{\theta_I < \tau\} &= E^{\alpha_0}[P^{\alpha_0}(\theta_I < \tau|\mathcal{F}^\partial)(\omega)] \\ &= E^{\alpha_0}P^0_{\delta_0}(\omega)\{\exists\theta < \tau|S_\theta \in I\} \\ &= \lim_{\tau_0\downarrow 0} E^{\alpha_0}P^0_{\delta_0}(\omega)\{\exists\theta \in (\tau_0,\tau)|S_{\theta-}(\omega') < b, S_\theta(\omega') \in I\}. \end{aligned}$$

Set $\Omega'(\tau_0)$ for the event under $P^0_{\delta_0}$ (we are dropping ω). For any integers $m \geq 1$ and $k = 0 \ldots 2^m$ let $\Delta\theta = (\tau - \tau_0)2^{-m}$, $\theta_k = \tau_0 + k\Delta\theta$ and $\Delta\theta_k = \theta_k - \theta_{k-1} = \Delta\theta$ (for $k \geq 1$). Set now $\Omega'(\tau_0, m) = \{\exists k = 1 \ldots 2^m | S(\theta_{k-1}) < b, S(\theta_k) \in I\}$. Let us write for $\epsilon \downarrow 0$ to be fixed below as some function of $\Delta\theta$

$$\Omega'(\tau_0, m) = \{\exists k = 1 \ldots 2^m | S(\theta_{k-1}) \leq b - \epsilon, S(\theta_k) \in I\} \cup \Omega'_0(\tau_0, m, \epsilon)$$

where the small set $\Omega'_0(\tau_0, m, \epsilon) = \{\exists k = 1 \ldots 2^m | S(\theta_{k-1}) \in (b-\epsilon, b), S(\theta_k) \in I\}$ which is inside $\{\exists\theta > 0 | S(\theta) \in (b - \epsilon, b)\}$. We have therefore by the right-continuity of S

$$E^{\alpha_0}P^0_{\delta_0}(\omega)(\Omega'_0(\tau_0, m, \epsilon)) \leq P^{\alpha_0}\{L(b) > L(b-\epsilon)\} \to 0 \text{ as } \epsilon \to 0.$$

Let $\Omega'(\tau_0, m, \epsilon)$ stand for the main set in $\Omega'(\tau_0, m)$ and write

$$\begin{aligned} P^0_{\delta_0}(\Omega'(\tau_0, m, \epsilon)) &= \sum_{k=1}^{2^m} P^0_{\delta_0}\{S(\theta_{k-1}) \leq b - \epsilon, S(\theta_k) \in I\} \\ &= \int_0^{b-\epsilon} dt \sum_{k=1}^{2^m} \frac{P^{\theta_{k-1},\theta_k}(t, I)}{\Delta\theta} p^{0,\theta_{k-1}}(0,t)\Delta\theta_k \qquad (25) \\ &= \int_0^b dt I_{(0,b-\epsilon)} \sum_{k=1}^{2^m} Q^{X^\partial_{\theta_{k-1}}}(I - t) p^{0,\theta_{k-1}}(0,t)\Delta\theta_k + R(\tau_0, m, \epsilon) \end{aligned}$$

where the remainder

$$R(\tau_0, m, \epsilon) = \int_0^{b-\epsilon} dt \sum_{k=1}^{2^m} \left[\frac{P^{\theta_{k-1},\theta_k}(t, I)}{\Delta\theta} - Q^{X^\partial_{\theta_{k-1}}}(I - t)\right] p^{0,\theta_{k-1}}(0,t)\Delta\theta_k$$

which we now study. Lemma 13 only gives convergence at separate θ_k's and we need a uniform bound; its proof shows that the rest $R(\theta_{k-1}, \Delta\theta)$ at eq. (24) is the main difficulty. For $\delta \in (0, \epsilon/2)$, let φ^δ be a smooth function s.t. $\varphi^\delta = 1$ on I, $\varphi^\delta = 0$ on $C(b - \delta, b' + \delta)$ and $\varphi^\delta \leq 1$ elsewhere and let us write

$$\frac{P^{\theta_{k-1},\theta_k}(t, I)}{\Delta\theta} = \frac{P^{\theta_{k-1},\theta_k}\varphi^\delta(t)}{\Delta\theta} - \frac{1}{\Delta\theta}\left(\int_{b-\delta}^b + \int_{b'}^{b'+\delta}\right)\varphi^\delta(s)p^{\theta_{k-1},\theta_k}(t,s)ds;$$

by Lemma 12, $p^{\theta_{k-1},\theta_k}(t,s) \leq c(s-t)^{-2}$ where $c = c(b, b')$, so that (minus) the second term on the r.h.s. above is $\leq c\Delta\theta^{-1}\delta\epsilon^{-2}$ where $c = c(b, b')$. Concerning the main term on the r.h.s. just above, Lemma 7 and an integration by parts give

(26)

$$\left|Q^{X^\partial_{\theta_{k-1}}}(I - t) + \int_0^\infty ds\varphi^\delta(t+s)\partial_s Q^{X^\partial_{\theta_{k-1}}}(s)\right| \leq -\left(\int_{b-t-\delta}^{b-t} + \int_{b'-t}^{b'-t+\delta}\right)\partial_s Q^{X^\partial_{\theta_{k-1}}}(s)ds;$$

the r.h.s. above is, by Lemma 7, less than $c\delta\epsilon^{-3/2}$ where $c = c(b, b')$. Let us now deal with the quantity $R(\theta_{k-1}, \Delta\theta)$. We have

$$\begin{aligned} R(\theta_{k-1}, \Delta\theta) &= -\frac{1}{\Delta\theta}\int_{\theta_{k-1}}^{\theta_k} d\eta \int_{I-t} \partial_s(Q^{X^\partial_\eta} - Q^{X^\partial_{\theta_{k-1}}})(s)ds \\ &\quad -\frac{1}{\Delta\theta}\int_{\theta_{k-1}}^{\theta_k} d\eta \left(\int_{b-t-\delta}^{b-t} + \int_{b'-t}^{b'-t+\delta}\right) \varphi^\delta(t+s)\partial_s(Q^{X^\partial_\eta} - Q^{X^\partial_{\theta_{k-1}}})(s)ds \\ &= R_1(\theta_{k-1}, \Delta\theta) + R_2(\theta_{k-1}, \Delta\theta). \end{aligned}$$

Concerning R_1, by eq. (10) for $s = b - t$, $b' - t$ and by Lemma 14 with e.g. $\gamma = 1/4$ we have $|R_1| \le c\Delta\theta^{2/3}\epsilon^{-1}$ for some $c = c(b, b')$; the other rest R_2 is treated as in relation (26) and therefore for small enough ϵ

$$|R(\theta_{k-1}, \Delta\theta)| \le c(\Delta\theta^{2/3} + \delta)\epsilon^{-2}.$$

To sum up, if we choose $\epsilon = \Delta\theta^{1/6}$ and $\delta = \Delta\theta^2$ then for all $t \in (0, b - \epsilon)$

$$\left|\frac{P^{\theta_{k-1},\theta_k}(t, I)}{\Delta\theta} - Q^{X^\partial_{\theta_{k-1}}}(I - t)\right| \le c\Delta\theta^{1/3},$$

for some $c = c(b, b')$. In order to settle the rest $R(\tau_0, m, \epsilon)$, it suffices to set a bound on the transition density involved (but *not* that of Lemma 12 !). Indeed, for a sequence $m' \to \infty$ and the λ_0 of Lemma 9 we have for $\theta \in (\tau_0, \tau)$ by Lemma 10

$$\left|\int_0^{m'} d\lambda \left(\cos(t\lambda) f_c^{0,\theta}(\lambda) + \sin(t\lambda) f_s^{0,\theta}(\lambda)\right)\right| \le 2\lambda_0 + 2\int_{\lambda_0}^\infty d\lambda \exp -c_0\tau_0\sqrt{\lambda} = c(\tau_0) \tag{27}$$

so that $R(\tau_0, m, \epsilon) \to 0$ as $m \to \infty$ (for fixed τ_0). Let us then return to the Riemann sum inside the integral in the main term in eq. (25). It is equal to

$$\begin{aligned} &\int_{\tau_0}^{\tau} d\theta Q^{X^\partial_\theta}(I - t)p^{0,\theta}(0, t) \\ &-\sum_{k=1}^{2^m}\int_{\theta_{k-1}}^{\theta} d\theta Q^{X^\partial_{\theta_{k-1}}}(I - t)\left(p^{0,\theta}(0, t) - p^{0,\theta_{k-1}}(0, t)\right) \\ &-\sum_{k=1}^{2^m}\int_{\theta_{k-1}}^{\theta} d\theta \left(Q^{X^\partial_\theta}(I - t) - Q^{X^\partial_{\theta_{k-1}}}(I - t)\right)p^{0,\theta}(0, t). \end{aligned}$$

Since

$$\begin{aligned} p^{0,\theta}(0, t) - p^{0,\theta_{k-1}}(0, t) &= \frac{1}{2\pi}\int_{\mathbb{R}} d\lambda \exp(-it\lambda)\left(\exp -\Psi^{0,\theta}(\lambda) - \exp -\Psi^{0,\theta_{k-1}}(\lambda)\right) \\ = \frac{1}{2\pi}\int_{\mathbb{R}} d\lambda \exp(-it\lambda)&\left(\exp -\Psi^{\theta_{k-1},\theta}(\lambda) - 1\right)\exp -\Psi^{0,\theta_{k-1}}(\lambda), \end{aligned}$$

then by the proof of Lemma 13 and by Lemma 9 it follows that $|(p^{0,\theta} - p^{0,\theta_{k-1}})(0, t)| \le c(\tau_0)\Delta\theta$. Therefore, by our choice of ϵ and by the above estimates the (integrals over $(0, b - \epsilon)$ of the) second and third rests above converge to 0 as $m \to \infty$ by dominated convergence. Our Theorem follows upon taking the limit as $\tau_0 \downarrow 0$ by monotone convergence. □

5. The entropy limit law

Before attacking the moments we need some preliminary calculations concerning a variant of the time functional in Theorem 15. The following result is the crux of nearly all the preparatory work.

Lemma 16. *Let $t > s \geq 0$ and θ, $\tau \geq 0$, suppose that $\tau - \theta \leq \tau_0$, $\tau_0 > 0$ and set*

$$U_r^{\theta,\tau}(s,t) = \int_\theta^\tau Q^{X_\eta^\partial}(r) p^{\theta,\eta}(s,t) d\eta,$$

then for all $N \geq 0$ there is a $c = c(\tau_0)$ s.t.

$$\left|\partial_t^N U_r^{\theta,\tau}(s,t)\right| \leq c \sup_\alpha Q^\alpha(r)(t-s)^{-(N+1/2)}.$$

Proof. For convenience take $\theta = 0$ and $s = 0$. We need Lemma 10 and Fubini to rewrite the functional (with the obvious notation) $U_r^\tau(t)$ and this translates into justifying a dominated convergence. Observe in this respect that $Q^{X^\partial(\eta)}(r)$ can be majorized uniformly in $\eta \geq 0$ and so by Lemma 12 Fubini is indeed valid. Set

$$u_{s,c}^\tau(r,\lambda) = \int_0^\tau d\eta Q^{X^\partial(\eta)}(r) f_{s,c}^\eta(\lambda).$$

After Fubini, the Leibnitz formula and Lemma 8 we can write (with the understanding that once $N - k \neq 0$ then $c_0 = 0$)

$$\partial_t^N U_r^\tau(t) = \frac{1}{\pi t^N} \sum_{k=0}^N \binom{N}{k} \frac{(-1)^k}{t} \int_0^\infty d\lambda \sum_{l=0}^{N-k} c_l \left(\cos(\lambda) u_{c,k}^\tau(r, \frac{\lambda}{t}) + \sin(\lambda) u_{s,k}^\tau(r, \frac{\lambda}{t}) \right).$$

The integrals will be split at the λ_0 (and *not* $t\lambda_0$!) of the previous Lemmas. Proceeding as in the proof of the inequality (18) of Lemma 9 we have by the dominated convergence theorem for all integers $k = 0 \ldots N$

$$\left|u_{s,c,k}^\tau(r,\lambda)\right| \leq \int_0^\infty d\eta Q^{X^\partial(\eta)}(r) \left|f_{s,c}^\eta(\lambda)\right| \leq c \sup_\alpha Q^\alpha(r)/\sqrt{\lambda}$$

whence the result. Concerning the part on $(0, \lambda_0)$, thanks to a change of variables we have to study the following quantity

$$\int_0^{\lambda_0/t} d\lambda \sum_{l=0}^{N-k} c_l \left(\cos(t\lambda) u_{c,k}^\tau(r,\lambda) + \sin(t\lambda) u_{s,k}^\tau(r,\lambda) \right).$$

Consider first $t < 1$ so that the integral on the r.h.s. above can be split into the two parts $(0, \lambda_0)$ and $(\lambda_0, \lambda_0/t)$. The first integral is treated by suppressing altogether the sine, cosine and the rest (by the second assertion of Lemma 11) and it is majorized (in modulus) by $c(\lambda_0, \tau_0) \sup \{Q^\alpha(r) | \alpha \in D\} \leq c(\lambda_0, \tau_0) \sup \{Q^\alpha(r) | \alpha \in D\} / \sqrt{t}$. On $(\lambda_0, \lambda_0/t)$ we are in the first situation of Lemma 11 and suppressing again the sine and cosine (only) the integral is in modulus less than $c\sqrt{\lambda_0/t} \sup \{Q^\alpha(r) | \alpha \in D\}$. If $t \geq 1$ we have by Lemma 8 and (the second part of) Lemma 11 that our integral over $(0, \lambda_0/t)$ is in modulus less than $c(\lambda_0, \tau_0) \sup \{Q^\alpha(r) | \alpha \in D\} / \sqrt{t}$. □

Lemma 17. *Let $\alpha_0 \in \partial D$, $\tau > 0$ and for small $t > 0$ set $H(t) = (1/t) \int_0^t \bar{Q}(s) ds$. As $t \to 0$ we have $(1/t) \int_0^t ds \int_0^\tau d\theta E^{\alpha_0} Q^{X^\partial(\theta)}(s) = \tau H(t) + o(H(t))$*

Proof. We have by Theorem 2

$$E^{\alpha_0} Q^{X_\theta^\partial}(s) = \int_{\partial D} Q^\alpha(s) p_\theta^\partial(\alpha_0, \alpha) d\alpha = \bar{Q}(s) + R(\theta, s),$$

where $|R(\theta, s)|$ is equal to

$$\left| \int_{\partial D} (Q^\alpha(s) - \bar{Q}(s)) p_\theta^\partial(\alpha_0, \alpha) d\alpha \right|$$
$$\leq c \sup_{\partial D} |Q^\alpha(s) - \bar{Q}(s)| \exp(-c'\theta) \leq c \sup_{\partial D} \int_{\partial D} |Q^\alpha(s) - Q^\beta(s)| d\beta.$$

It is possible to select an absolute distribution of N_D boundary points α_i, $i = 1 \ldots N_D$, and a $\delta > 0$ (see the proof of Lemma 7) s.t. given any α, β in ∂D we can move from α to β through a chain of points in the distribution in such a way that all pairs of consecutive points α_i, α_{i+1} fall within the same ball $B_\nu(z)$ of our partition of the unity and B_ν contains both of $B(\alpha_i, \delta)$ and $B(\alpha_{i+1}, \delta)$. As a result of this, the singularity $1/\sqrt{s}$ will successively cancel out and the result follows by the properties of the form $K(z, y)$ on the boundary points of the patch W, the boundedness (in s) of $\dot{Q_2}(s)$ and the proof of Lemma 9. ☐

5.1. **The first moment of** $N(n, \tau)$**.**

Lemma 18. *Let* $\alpha_0 \in \partial D$*, as* $n \to \infty$ *we have* $E^{\alpha_0} N(n, \tau) = \tau H(2^{-n}) + o(H(2^{-n}))$.

Proof. We have by Theorem 15, Fubini and monotone convergence

$$E^{\alpha_0} N(n, \tau) = 1 + 2^n \int_0^{2^{-n}} dt E^{\alpha_0} \int_0^\tau d\theta Q^{X_\theta^\partial}(2^{-n} - t)[2^{-n} \sum_{k \geq 0} p^{0,\theta}(0, t + k2^{-n})].$$

Let us first apply the Euler Maclaurin formula on the finite intervals $[0, m]$. We obviously have

$$E^{\alpha_0} N(n, \tau) = 2^n \int_0^{2^{-n}} dt \int_0^\tau d\theta E^{\alpha_0} Q^{X_\theta^\partial}(2^{-n} - t) + \sum_{i=1}^3 \lim_m R_i(m) + 1,$$

where the rests $R_i(m)$, $i = 1, 2, 3$, correspond by abuse of notation to the rests in lemma 1 but after integrations. The main term is dealt with by Lemma 17 and we now turn to the error terms.

For sufficiently large n, we have by Lemma 16 and monotonicity

$$\begin{aligned} |R_1(m)| &= 2^n \int_0^{2^{-n}} dt E^{\alpha_0} \int_0^\tau d\theta [Q^{X_\theta^\partial}(2^{-n} - t) \int_0^{t+2^{-n}} ds p^{0,\theta}(0, s)] \\ &= 2^n \int_0^{2^{-n}} dt \int_0^{t+2^{-n}} ds E^{\alpha_0} U^{0,\tau}_{2^{-n}-t}(0, s) \leq c 2^{n/2} \int_0^{2^{-n}} \frac{dt}{\sqrt{2^{-n} - t}} = c. \end{aligned}$$

Next, by the change of variables $s = t + 2^{-n}k$ for $k = 1$ and $k = m$ we have by Theorem 15 $R_2(m) \leq 1/2 + 1/2$ for all m. The delicate $R_3(m)$ is treated by a Riemann-Lebesgue argument since it is an oscillatory-like integral. By the well known fact that (see e.g. exercise 11 p. 52 of [6])

$$\sup_{m,s} \left| \sum_{k=1}^m \frac{\sin(2\pi k s)}{\pi k} \right| < \infty,$$

by Lemma 10 and by the dominated convergence theorem we can write

$$\begin{aligned} R_3(m) &= -\int_0^{2^{-n}} dt E^{\alpha_0} \int_0^\tau d\theta Q^{X_\theta^\partial}(2^{-n} - t) \int_{2^{-n}}^{m2^{-n}} ds P_1(s2^n) \partial_s p^{0,\theta}(0, t + s) \\ &= -\int_0^{2^{-n}} dt \int_{2^{-n}}^{m2^{-n}} ds P_1(s2^n) E^{\alpha_0} \partial_s U^{0,\tau}_{2^{-n}-t}(0, t + s) \\ &= -\sum_{k=1}^\infty \frac{1}{\pi k} \int_0^{2^{-n}} dt \int_{2^{-n}}^{m2^{-n}} ds \sin(\pi k s 2^{n+1}) E^{\alpha_0} \partial_s U^{0,\tau}_{2^{-n}-t}(0, t + s); \end{aligned}$$

Lemma 16 and monotonicity give for some c independent of m

$$|R_3(m)| \leq c \sum_{k=1}^\infty \frac{2^{n/2}}{k^2} \int_0^{2^{-n}} \frac{dt}{\sqrt{2^{-n} - t}} = c.$$

☐

5.2. **The second moment of $N(n,\tau)$.**

Lemma 19. *Let $\alpha_0 \in \partial D$, as $n \to \infty$ we have $E^{\alpha_0} N^2(n,\tau) = [\tau H(2^{-n})]^2 + O(H(2^{-n}))$.*

Proof. Clearly we have

$$\begin{aligned} E^{\alpha_0} N^2(n,\tau) &= 2 \sum_{1\le k<l} P^{\alpha_0}(\exists \tau_1,\, \tau_2 < \tau | S(\tau_1) \in I_{n,k} \, and \, S(\tau_2) \in I_{n,l}) \\ +3 \sum_{k\ge 1} P^{\alpha_0}(\exists \tau_1 &< \tau | S(\tau_1) \in I_{n,k}\,) + 1. \end{aligned}$$

Note that the second sum on the r.h.s. above is only $O(H(2^{-n})$. Let P_{kl} be the quantity under the double sum over k and l and set θ_k for the hitting time of $I_{n,k}$. Since we are working under $P^{\alpha_0}(d\omega)$, the inverse local time S starts a.s. from the space 0 at the time parameter 0. By the strong Markov property (in its version given by [9] p. 179 which is easily adapted to our non-homogeneous context here following [19]) P_{kl} is given by

$$E^{\alpha_0}\left[E^0_{\delta_0}\left((\theta_k < \tau) \int_{S(\theta_k)}^{l2^{-n}} dt \int_{\theta_k}^{\tau} d\eta p^{\theta_k,\eta}(S(\theta_k), t) Q^{X^\partial_\eta}(I_{n,l} - t) \right)\right].$$

Let $dF_k = F_k(d\theta, ds)$ be the joint law of the couple $(\theta_k, S(\theta_k))$ (under $E^0_{\delta_0}$). For $k \ge 1$ and $l \ge 0$ let $s^* = (k+1)2^{-n} - s$ and $r(l) = s^* + (l+1)2^{-n} - t$, then by the same manipulations as in the first moment, the sum $\sum_{l \ge k+1} P_{kl}$ is equal to

$$\begin{aligned} & E^{\alpha_0} \int_{[0,\tau]\times I_{n,k}} dF_k \left(\sum_{l\ge k+1} \int_0^{l2^{-n}-s} dt \int_\theta^\tau d\eta p^{\theta,\eta}(s, s+t) Q^{X^\partial_\eta}(I_{n,l} - s - t)\right) \\ = \;& E^{\alpha_0} \int_{[0,\tau]\times I_{n,k}} dF_k \sum_{l \ge 0} \int_{I_{n,l}+s^*} dt\, U^{\theta,\tau}_{r(l)}(s, s+t) + R^k_0 \end{aligned}$$

where

$$R^k_0 = E^{\alpha_0} \int_{[0,\tau]\times I_{n,k}} dF_k \int_0^{s^*} dt U^{\theta,\tau}_{s^*-t}(s, s+t)$$

and by Theorem 15 and Lemma 18 $\left|\sum_k R^k_0\right| = O(H(2^{-n}))$. Let us then pursue with the main term. By another change of variables the main term in the sum $\sum_{l\ge k+1} P_{kl}$ becomes

$$\begin{aligned} & E^{\alpha_0} \int_{[0,\tau]\times I_{n,k}} dF_k \int_0^{2^{-n}} dt \int_\theta^\tau d\eta \left(\sum_{l\ge 0} p^{\theta,\eta}(s, s+s^*+t+l2^{-n}) \right) Q^{X^\partial_\eta}(2^{-n} - t) \\ = \;& E^{\alpha_0} \int_{[0,\tau]\times I_{n,k}} dF_k \left(2^n \int_0^{2^{-n}} dt \int_\theta^\tau d\eta Q^{X^\partial_\eta}(2^{-n} - t) \right) + \sum_{i=1}^3 R^k_i(\infty) \end{aligned}$$

where the rests $R^k_i(\infty)$, $i = 1, 2, 3$, correspond for $m = \infty$ by abuse of notation to the rests in Lemma 1 but after four integrations. The latter are treated as in the proof of Lemma 18 and we have $\left|\sum_k R^k_i(\infty)\right| = O(H(2^{-n}))$ for $i = 1, 2, 3$. After the disappearence of the argument s, an integration by parts (recall that the boundary process is right continuous) gives for the main term in the second moment

$$\begin{aligned} & 2\sum_{k\geq 1} E^{\alpha_0}\int_0^\tau F_k(d\theta, I_{n,k})\left(2^n\int_0^{2^{-n}} dt\int_\theta^\tau d\eta Q^{X^\partial_\eta}(2^{-n}-t)\right) \\ = \; & 2\int_0^\tau d\theta E^{\alpha_0}\left(2^n\int_0^{2^{-n}} dt Q^{X^\partial_\theta}(2^{-n}-t)\sum_{k\geq 1} P^0_{\delta_0}(\theta_k\leq\theta)\right) \\ = \; & 2H\left(2^{-n}\right)\int_0^\tau \theta\left(2^n\int_0^{2^{-n}} dt E^{\alpha_0} Q^{X^\partial_\theta}(2^{-n}-t)\right)d\theta + \sum_{i=1}^3 R_i(\infty). \end{aligned}$$

The treatment of both the main term and the remainders $R_i(\infty)$ in the second moment above pass by appealing to Theorem 2 as in Lemma 17. The Lemma follows by straightforward calculations. □

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