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A NOTE ON THE KOLMOGOROV–MARCINKIEWICZ–ZYGMUND TYPE STRONG LAW OF LARGE NUMBERS FOR ELEMENTS OF AUTOREGRESSION SEQUENCES

In the paper we consider the Kolmogorov–Marcinkiewicz–Zygmund type strong law of large numbers for sums whose terms are elements of regression sequences of random variables. Some necessary and sufficient conditions providing SLLN are obtained in terms of coefficients of the regression sequence. Several special cases of regression sequences are considered as well.

1. Introduction

Consider a zero-mean linear regression sequence of random variables $(\xi_k) = (\xi_k, \ \ k \geq 1)$ defined on a probability space $(\Omega, \mathfrak{F}, \mathbb{P})$ by the system of the following recurrence equations:

$$\xi_1 = \beta_1 \theta_1, \quad \xi_k = \alpha_k \xi_{k-1} + \beta_k \theta_k, \quad k \geq 2, \tag{1}$$

where (α_k) and (β_k) are nonrandom real sequences, and (θ_k) is a sequence of independent symmetric random variables such that $\mathbb{P}\{\theta_k = 0\} < 1$ for any $k \geq 1$. Let

$$S_n = \sum_{k=1}^{n} \xi_k, \quad n \geq 1.$$

In this paper we are mainly interested in necessary and sufficient conditions for the convergence almost surely (a.s.) of the series

$$\sum_{n=1}^{\infty} \frac{S_n}{n^{1+1/p}}, \tag{2}$$

for $p > 0$.

To a certain extent the subject of our investigation is connected to results which provide Kolmogorov-Marcinkiewicz-Zygmund type strong law of large numbers (SLLN). The celebrated Kolmogorov-Marcinkiewicz-Zygmund type SLLN deals with sums whose terms are independent random variables or independent random elements. Recall that at first A.N. Kolmogorov [11] for $p = 1$, later J. Marcinkiewicz and A. Zygmund [12] for $0 < p < 2$ proved that if (X_n) is a sequence of independent copies of random variable X defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $S_n = \sum_{k=1}^{n} X_k$, $n \geq 1$, then $\lim\limits_{n\to\infty} \frac{S_n}{n^{1/p}} = 0$, a.s., if and only if $E|X|^p < \infty$, where $EX = 0$ in case $p \geq 1$. Further extensions of these results to sums of independent Banach space valued random elements were made by E. Mourier [13] and later by T.A. Azlarov and N.A. Volodin [2], and A. de Acosta [1].

In recent works by F. Hechner and B. Heinkel [8], and Deli Li, Yongcheng Qi and A. Rosalsky [6], the Kolmogorov-Marcinkiewicz-Zygmund type SLLN is considered for sums of independent Banach space valued random elements in a sense that authors obtain sets

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of necessary and sufficient conditions providing convergence of the series $\sum_{n=1}^{\infty} \frac{E\|S_n\|}{n^{1+1/p}}$ and convergence a.s. of the series $\sum_{n=1}^{\infty} \frac{\|S_n\|}{n^{1+1/p}}$ (for more details, see [8, 6]).

Encouraged by the paper by Deli Li, Yongcheng Qi and A. Rosalsky [6] we intend to obtain necessary and sufficient conditions providing almost sure convergence of the series (2) for sums whose terms are elements of regression sequence (1). In particular, special cases such as the sequence of independent random variables (i.e. $\alpha_k = 0$) and an autoregression sequence (i.e. $\alpha_k = const$) are considered.

Techniques applied to prove main results differ completely from those developed in cited works. We use results due to V. Buldygin and M. Runovska [5] on the almost sure convergence of series whose terms are elements of regression sequences and the Kahane's contraction principle [10]. In order to make the text more self-contained we recall all required statements in section 2.

2. Preliminaries

First of all let us remind the well-known Kahane's contraction principle [10, 3] in order to feel free using it throughout the paper.

Let $\mathfrak{X}$ be a separable Banach space endowed with the norm $\|\cdot\|$, and (X_k) be a sequence of independent symmetric $\mathfrak{X}$-valued random elements, $S_n = X_1 + X_2 + \ldots + X_n$, $n \geq 1$. Let also (c_k) be a non-random sequence such that $\sup_{k\geq 1}|c_k| < \infty$, and $\widetilde{S}_n = c_1X_1 + c_2X_2 + \ldots + c_nX_n$, $n \geq 1$.

Proposition 2.1. *[Kahane's contraction principle]*

If the sequence (S_n) converges a.s. in the norm of the space $\mathfrak{X}$, then the sequence $(\widetilde{S}_n)$ converges a.s. in the norm of the space $\mathfrak{X}$.

In the main part of the paper we will also need a criterion for the convergence a.s. of the series whose terms are elements of linear second order regression sequence of random variables. Namely, consider a zero-mean second order regression sequence of random variables (ζ_k), i.e. the sequence which obeys the system of following recurrence equations:

$$\zeta_{-1} = \zeta_0 = 0, \quad \zeta_k = b_k^{(1)}\zeta_{k-1} + b_k^{(2)}\zeta_{k-2} + \widetilde{\beta}_k\theta_k, \quad k \geq 1,$$

where $(\widetilde{\beta}_k)$ is a nonrandom real sequence, $(b_k^{(j)};\ \ j = 1, 2,\ \ k \geq 1)$ is a nonrandom real-valued set, and (θ_k) is a sequence of independent symmetric random variables such that $\mathbb{P}\{\theta_k = 0\} < 1$, $k \geq 1$.

Let $\mathfrak{N}^\infty$ be the class of all sequences of positive integers increasing to infinity. The following proposition gives necessary and sufficient conditions for the almost sure convergence of the series $\sum_{k=1}^{\infty} \zeta_k$. For more details, see [9, 5] where this result is presented for any linear m-regression.

Proposition 2.2. *The series $\sum_{k=1}^{\infty} \zeta_k$ converges a.s. if and only if the following three conditions hold*:

1) *for any $k \geq 1$ the nonrandom series $\left(\sum_{l=0}^{\infty} \widetilde{\beta}_k u_{k+l}^{(k+1)}\right)$ is convergent, where $(u_n^{(k+1)}, n \geq k-1)$ is a nonrandom sequence which obeys the system of recurrence equations*

$$u_n^{(k+1)} = b_n^{(1)} u_{n-1}^{(k+1)} + b_n^{(2)} u_{n-2}^{(k+1)}, \quad n \geq k+1, \quad u_{k-1}^{(k+1)} = 0, \quad u_k^{(k+1)} = 1;$$

2) *the series $\sum_{k=1}^{\infty} U_k\theta_k$ converges a.s., with $U_k = \sum_{l=0}^{\infty} \widetilde{\beta}_k u_{k+l}^{(k+1)}$, $k \geq 1$;*

3) *for all the sequences (m_j) from the class $\mathfrak{N}^\infty$*

$$\lim_{j\to\infty} \Big| \sum_{k=m_j+1}^{m_{j+1}} \Big(\sum_{l=0}^{m_{j+1}-k} u_{k+l}^{(k+1)} \Big) \widetilde{\beta}_k\theta_k \Big| = 0, \quad a.s.$$

Remark that the result of Proposition 2.2 was first proved in [5] under assumption that $b_k^{(2)} \neq 0$ for any $k \geq 1$. In [9] author succeeded to get rid of the mentioned assumption. Moreover, it was shown that the series $\sum_{k=1}^{\infty} \zeta_k$ converges a.s. if and only if the assumptions 1) and 3) of Proposition 2.2 hold true, i.e. the assumption 2) follows directly from assumption 3). Taking into account that the assumption 2) is of a more simple form than the assumption 3) and in a view of the following statement below it is convenient for us to have formulated Proposition 2.2 in that way.

Corollary 2.1. *Let $b_k^{(j)} \geq 0$, for all $j = 1, 2$; $k \geq 1$. Then the series $\sum_{k=1}^{\infty} \zeta_k$ converges a.s. if and only if the first two assumptions of Proposition 2.2 hold true.*

For Gaussian 2-Markov sequence of random variables, that is if (θ_k) in (1) is a sequence of independent standard Gaussian random variables, Corollary 2.1 was proved in [4]. Its proof in the general case does not present any difficulty when applying the Kahane's contraction principle. We omit this proof, since a similar idea will be presented in the main part of the paper.

3. Main results

Let

$$a(n,k) = \begin{cases} 0, & 1 \leq n < k; \\ 1, & n = k; \\ 1 + \sum_{l=1}^{n-k} \left(\prod_{j=k+1}^{k+l} \alpha_j \right), & n > k. \end{cases}$$

For $k \geq 1$ consider the nonrandom series

$$\sum_{l=0}^{\infty} \beta_k \frac{a(k+l,k)}{(k+l)^{1+1/p}}, \tag{3}$$

and set

$$C(k) = \sum_{l=0}^{\infty} \beta_k \frac{a(k+l,k)}{(k+l)^{1+1/p}},$$

if the series (3) is convergent.

The following theorem gives necessary and sufficient conditions providing almost sure convergence of the series (2).

Theorem 3.1. *The series* (2) *converges a.s. if and only if the following three conditions hold*:
1) the series (3) converges for any $k \geq 1$;
2) the series $\sum_{k=1}^{\infty} C(k)\theta_k$ converges a.s.;
3) for all the sequences $(m_j) \in \mathfrak{R}^{\infty}$ one has

$$\lim_{j\to\infty} \Big| \sum_{k=m_j+1}^{m_{j+1}} \Big(\sum_{l=0}^{m_{j+1}-k} \frac{a(k+l,k)}{(k+l)^{1+1/p}} \Big) \beta_k \theta_k \Big| = 0, \quad \text{a.s.}$$

Proof. For the sequence (ξ_k) consider the sequence of its partial sums (S_n), $S_n = \sum_{k=1}^{n} \xi_k$, $n \geq 1$, and the series

$$\sum_{n=1}^{\infty} c_n S_n,$$

where (c_n) is some nonrandom sequence. Without loss of generality, we assume that $c_n \neq 0$, $n \geq 1$.

Note that sequence of partial sums (S_n) obeys the system of following linear recurrence equations:

$$S_{-1} = S_0 = 0, \quad S_k = (\alpha_k + 1)S_{k-1} - \alpha_k S_{k-2} + \beta_k \theta_k, \quad k \geq 1.$$

Therefore, the sequence (c_nS_n) obeys the system of following linear recurrence equations:

$$c_{-1}S_{-1} = c_0S_0 = 0,$$

$$c_kS_k = \frac{c_k(\alpha_k+1)}{c_{k-1}}c_{k-1}S_{k-1} - \frac{c_k\alpha_k}{c_{k-2}}c_{k-2}S_{k-2} + c_k\beta_k\theta_k, \quad k \geq 1.$$

For $k \geq 1$, set

$$\zeta_k = c_kS_k, \qquad \widetilde{\beta}_k = c_k\beta_k,$$

and

$$b_k^{(1)} = \frac{c_k(\alpha_k+1)}{c_{k-1}}, \quad b_k^{(2)} = -\frac{c_k\alpha_k}{c_{k-2}}.$$

Then the sequence (c_nS_n) is a second-order linear regression sequence:

$$\zeta_{-1} = \zeta_0 = 0, \quad \zeta_k = b_k^{(1)}\zeta_{k-1} + b_k^{(2)}\zeta_{k-2} + \widetilde{\beta}_k\theta_k, \quad k \geq 1.$$

Therefore the series $\sum_{n=1}^{\infty} c_nS_n$ converges a.s. if and only if the series $\sum_{n=1}^{\infty}\zeta_n$ converges a.s. Thus, in order to complete the proof one should apply Proposition 2.2 with

$$u_{k+l}^{(k+1)} = \frac{c_{k+l}}{c_k}a(k+l,k), \quad l \geq 0, \quad k \geq 1,$$

and

$$c_n = \frac{1}{n^{1+1/p}}, \quad n \geq 1.$$

Proof of Theorem 3.1 is complete. □

Let now proceed to some partial cases.

Corollary 3.1. *Let $-1 \leq \alpha_k \leq 0$, $k \geq 2$. The series* (2) *converges a.s. if and only if the following two conditions hold*:
1) the series (3) converges for any $k \geq 1$;
2) the series $\sum_{k=1}^{\infty} C(k)\theta_k$ converges a.s.

Proof. Let us reduce this case to the result of Corollary 2.1. Indeed, in our terms

$$b_k^{(1)} = \frac{c_k(\alpha_k+1)}{c_{k-1}}, \quad b_k^{(2)} = -\frac{c_k\alpha_k}{c_{k-2}}, \quad k \geq 1.$$

Since $c_k = \frac{1}{k^{1+1/p}} > 0$, $k \geq 1$, and $-1 \leq \alpha_k \leq 0$, $k \geq 2$, then both $b_k^{(1)} \geq 0$ and $b_k^{(2)} \geq 0$, $k \geq 1$.

Finally, by applying Corollary 2.1 the proof of Corollary 3.1 is complete. □

Independent case. Let us check whether the result of Theorem 3.1 distinguishes the independent case.

Corollary 3.2. *Let $\alpha_k = 0$ for any $k \geq 2$, i.e. (ξ_k) is a sequence of independent symmetric random variables: $\xi_k = \beta_k\theta_k$, $k \geq 1$. Then the series* (2) *converges a.s. if and only if the series*

$$\sum_{n=1}^{\infty}\frac{\beta_n\theta_n}{n^{1/p}}$$

converges a.s.

Proof. Assume that $\alpha_k = 0$ for any $k \geq 2$. Then according to Theorem 3.1 the series (2) converges a.s. if and only if the following three assumptions hold true:

1) the series $\sum_{l=0}^{\infty}\frac{\beta_k}{(k+l)^{1+1/p}}$ converges for any $k \geq 1$;

2) the random series $\sum_{k=1}^{\infty}\left(\sum_{l=0}^{\infty}\frac{1}{(k+l)^{1+1/p}}\right)\beta_k\theta_k$ converges a.s.;

3) for all the sequences $(m_j) \in \mathfrak{R}^\infty$ one has

$$\lim_{j\to\infty}\Big|\sum_{k=m_j+1}^{m_{j+1}}\Big(\sum_{l=0}^{m_{j+1}-k}\frac{1}{(k+l)^{1+1/p}}\Big)\beta_k\theta_k\Big| = 0, \quad \text{a.s.}$$

Since $p>0$, the series $\sum_{l=0}^{\infty}\frac{\beta_k}{(k+l)^{1+1/p}}$ converges for any $k\ge 1$, that is the assumption 1) holds true. Therefore we need to show that the assumptions 2) and 3) hold true if and only if the series $\sum_{n=1}^{\infty}\frac{\beta_n\theta_n}{n^{1/p}}$ converges a.s.

Indeed, since for any $k\ge 1$

$$\sum_{l=1}^{\infty}\frac{1}{(k+l)^{1+1/p}} \le \int_0^\infty \frac{dx}{(k+x)^{1+1/p}} \le \sum_{l=0}^{\infty}\frac{1}{(k+l)^{1+1/p}},$$

then

$$\sum_{l=0}^{\infty}\frac{1}{(k+l)^{1+1/p}} \underset{k\to\infty}{\sim} \frac{p}{k^{1/p}}. \tag{4}$$

Set

$$q_k = k^{1/p}\sum_{l=0}^{\infty}\frac{1}{(k+l)^{1+1/p}}, \quad k\ge 1.$$

Obviously, (4) implies that $\lim\limits_{k\to\infty} q_k = p$, i.e. the sequences $\big(q_k\big)$ and $\big(q_k^{-1}\big)$ are bounded.

Since

$$\sum_{k=1}^{\infty}\Big(\sum_{l=0}^{\infty}\frac{1}{(k+l)^{1+1/p}}\Big)\beta_k\theta_k = \sum_{k=1}^{\infty} q_k\frac{\beta_k\theta_k}{k^{1/p}},$$

and

$$\sum_{k=1}^{\infty}\frac{\beta_k\theta_k}{k^{1/p}} = \sum_{k=1}^{\infty} q_k^{-1}\Big(\sum_{l=0}^{\infty}\frac{1}{(k+l)^{1+1/p}}\Big)\beta_k\theta_k,$$

then according to Proposition 2.1 the random series $\sum_{k=1}^{\infty}\Big(\sum_{l=0}^{\infty}\frac{1}{(k+l)^{1+1/p}}\Big)\beta_k\theta_k$ converges a.s. if and only if the series $\sum_{k=1}^{\infty}\frac{\beta_k\theta_k}{k^{1/p}}$ converges a.s.

Finally, the assumption 3) follows from the almost sure convergence of the series $\sum_{k=1}^{\infty}\frac{\beta_k\theta_k}{k^{1/p}}$ according to the Kahane's contraction principle again, since

$$\sum_{l=0}^{m_{j+1}-k}\frac{1}{(k+l)^{1+1/p}} \le \sum_{l=0}^{\infty}\frac{1}{(k+l)^{1+1/p}} \underset{k\to\infty}{\sim} \frac{1}{k^{1/p}}.$$

The proof of Corollary 3.2 is complete. □

Before summarizing the independent case results let us formulate a simple lemma which refines Corollary 3.2.

Lemma 3.1. *Let $0<p<2$ and (X_n) be a sequence of independent copies of random variable X with $EX=0$. The series*

$$\sum_{n=1}^{\infty}\frac{X_n}{n^{1/p}}$$

converges a.s. if and only if $E|X|^p<\infty$.

Proof. According to Kolmogorov's three-series theorem we have to show that for some $c > 0$ series

$$\sum_{n=1}^{\infty} \mathbb{P}\Big\{\Big|\frac{X_n}{n^{1/p}}\Big| \geq c\Big\} \quad \text{and} \quad \sum_{n=1}^{\infty} E\frac{(X_n^c)^2}{n^{2/p}}$$

converge if and only if $E|X|^p < \infty$, where $X^c = \begin{cases} X, & |X| \leq c, \\ 0, & |X| > c. \end{cases}$

Indeed, let $c = 1$ and $F(x)$ be a distribution function of $|X|$. Then

$$\sum_{n=1}^{\infty} \mathbb{P}\Big\{\Big|\frac{X_n}{n^{1/p}}\Big| \geq c\Big\} = \sum_{n=1}^{\infty} \mathbb{P}\Big\{\Big|X_n\Big| \geq n^{1/p}\Big\} = \sum_{n=1}^{\infty} \int_{n^{1/p}}^{\infty} dF(x) =$$

$$= \int_1^{\infty} \Big(\sum_{n=1}^{\lfloor x^p \rfloor} 1\Big) dF(x) = \int_1^{\infty} \lfloor x^p \rfloor dF(x) \sim \int_1^{\infty} x^p dF(x) = E|X|^p,$$

where $\lfloor \cdot \rfloor$ indicates the floor function. Writing $\sim$ sign we mean that both integrals are convergent or divergent simultaneously. By analogue,

$$\sum_{n=1}^{\infty} E\frac{(X_n^c)^2}{n^{2/p}} = \sum_{n=1}^{\infty} E\frac{X^2}{n^{2/p}} \mathbb{I}\Big\{\frac{|X|}{n^{1/p}} \leq 1\Big\} = \sum_{n=1}^{\infty} \int_0^{n^{1/p}} \frac{x^2}{n^{2/p}} dF(x) =$$

$$= \int_0^{\infty} x^2 \Big(\sum_{n=\lceil x^p \rceil}^{\infty} \frac{1}{n^{2/p}}\Big) dF(x) \sim \int_0^{\infty} x^2 \cdot \lceil x^p \rceil^{1-2/p} dF(x) \sim$$

$$\sim \int_0^{\infty} x^2 (x^p)^{1-2/p} dF(x) = \int_0^{\infty} x^p dF(x) = E|X|^p,$$

where $\mathbb{I}$ stands for the indicator function and $\lceil \cdot \rceil$ for the ceiling function.

The proof of Lemma 3.1 is complete. □

Summarizing all results concerning the Kolmogorov-Marcinkiewicz-Zygmund type SLLN for sums of independent random variables, one has the following chain of implications. If (ξ_n) is a sequence of independent copies of symmetric random variable ξ, and $S_n = \sum_{k=1}^{n} \xi_k$, $n \geq 1$, then for $0 < p < 2$

$$\sum_{n=1}^{\infty} \frac{S_n}{n^{1+1/p}} \text{ converges a.s.} \iff \sum_{n=1}^{\infty} \frac{\xi_n}{n^{1/p}} \text{ converges a.s.} \iff$$

$$\iff E|\xi|^p < \infty \iff \lim_{n\to\infty} \frac{S_n}{n^{1/p}} = 0, \quad \text{a.s.}$$

and whenever $0 < p < 1$

$$\iff \sum_{n=1}^{\infty} \frac{|S_n|}{n^{1+1/p}} < \infty, \quad \text{a.s.}$$

Here the penultimate implication sign is due to Kolmogorov-Marcinkiewicz-Zygmund type SLLN and the last implication sign is due to Deli Li, Yongcheng Qi and A. Rosalsky [6].

Autoregression. Now assume that $\alpha_k = \alpha = const$, i.e. the sequence (1) is represented as follows

$$\xi_1 = \beta_1 \theta_1, \quad \xi_k = \alpha \xi_{k-1} + \beta_k \theta_k, \quad k \geq 2,$$

where (β_k) is a nonrandom real sequence, and (θ_k) is a sequence of independent symmetric random variables such that $\mathbb{P}\{\theta_k = 0\} < 1$, for any $k \geq 1$.

For such an autoregression sequence the result of Theorem 3.1 is as follows.

Corollary 3.3. *The series* (2) *converges a.s. if and only if one of the following two cases holds true*:
a) *for* $-1 \le \alpha < 1$ *and any* $p > 0$ *the series* $\sum_{n=1}^{\infty} \frac{\beta_n \theta_n}{n^{1/p}}$ *converges a.s.;*
b) *for* $\alpha = 1$ *and any* $0 < p < 1$ *the series* $\sum_{n=1}^{\infty} \frac{\beta_n \theta_n}{n^{1/p-1}}$ *converges a.s.*

Proof. Two cases should be considered in their own right.
a) For $-1 \le \alpha < 1$ one has

$$a(n,k) = \begin{cases} 0, & 1 \le n < k; \\ \frac{1-\alpha^{n-k+1}}{1-\alpha}, & n \ge k, \end{cases}$$

and the series (3), i.e. the series $\frac{\beta_k}{1-\alpha} \sum_{l=0}^{\infty} \frac{1-\alpha^{l+1}}{(k+l)^{1+1/p}}$, is convergent for any $k \ge 1$, which implies that values $C(k)$, $k \ge 1$, are well-defined. Taking into account (4) one has also

$$\sum_{l=0}^{\infty} \frac{1-\alpha^{l+1}}{(k+l)^{1+1/p}} \underset{k\to\infty}{\sim} \frac{p}{k^{1/p}}.$$

Therefore, the further proof literary repeats the steps of the proof of Corollary 3.2 when applying the Kahane's contraction principle.
b) Let $\alpha = 1$ and $0 < p < 1$. In this case

$$a(n,k) = \begin{cases} 0, & 1 \le n < k; \\ n-k+1, & n \ge k, \end{cases}$$

and the series (3), i.e. the series $\beta_k \sum_{l=0}^{\infty} \frac{l+1}{(k+l)^{1+1/p}}$, is convergent for any $k \ge 1$, which implies that values $C(k)$, $k \ge 1$ are well-defined. Moreover,

$$\sum_{l=0}^{\infty} \frac{l+1}{(k+l)^{1+1/p}} \underset{k\to\infty}{\sim} \frac{p}{1-p} \cdot \frac{1}{k^{1/p-1}}.$$

Also here the proof may be repeated from that one of Corollary 3.2. □

Finally, let us consider a specific situation, when the autoregression is generated by i.i.d. sequence of random variables, that is we consider the following autoregression:

$$\xi_1 = \eta_1, \quad \xi_k = \alpha \xi_{k-1} + \eta_k, \quad k \ge 2, \tag{5}$$

where (η_k) is a sequence of independent copies of symmetric random variable η. For such an autoregression the result of Corollary 3.3 is specified as follows.

Corollary 3.4. *For the sequence* (5) *the series* (2) *converges a.s. if and only if one of the following two cases is satisfied*:
a) *for* $-1 \le \alpha < 1$ *and* $p > 0$ *one has* $E|\eta|^p < \infty$;
b) *for* $\alpha = 1$ *and* $0 < p < 1$ *one has* $E|\eta|^{\frac{p}{1-p}} < \infty$.

Proof. Assertion of Corollary 3.4 immediately follows from Corollary 3.3 and 3.1. □

Example 3.1. Let in (5) η be a standard Gaussian random variable. Then Corollary 3.4 immediately implies that in order for the series (2) to converge a.s. it is necessary an sufficient that $-1 \le \alpha \le 1$.

Remark 3.1. The assumption of symmetry, imposed throughout the paper on the generating sequence (θ_n), is technically important and cannot be omitted as long as we apply results by V. Buldygin and M. Runovska on the almost sure convergence of series whose terms are elements of regression sequences of random variables. But one may try to use the symmetrization principle for non-symmetric sequence (θ_n), where possible.

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