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SOME RANDOM INTEGRAL OPERATORS RELATED TO A POINT PROCESSES

We study some properties of a random integral operator in $L_2(\mathbb{R})$ whose kernel is defined as a convolution of a Gaussian density and a stationary point process.

1. Introduction

Let Θ be a stationary point process on the real line [1]. In this paper we consider integral operators in $L_2(\mathbb{R})$ with the kernel

$$k(u,v) = \sum_{\theta\in\Theta} p(u-\theta)p(v-\theta), \tag{1}$$

where p is some square-integrable function. The problem of investigating of a random kernel of the kind arises in the theory of stochastic flows. For example, in articles [2, 3] the strong random operators related to an Arratia flow [4] were introduced. If $\{x(u,t), u\in\mathbb{R}, t\geq 0\}$ is an Arratia flow then for every $f\in L_2(\mathbb{R})$ and $t>0$ a random element $T_t f$ in $L_2(\mathbb{R})$ is defined by the equality

$$T_t f(u) = f(x(u,t)),\ u\in\mathbb{R}.$$

It was proved in [2] that T_t is a strong random operator in the Skorokhod sense [5] and it is not a bounded random operator, though [3]. As is known, the map $x(\cdot,t):\mathbb{R}\to\mathbb{R}$ is a step function with probability one, so one can assert that for any boundedly supported function $f\in L_2(\mathbb{R})$ the $L_2(\mathbb{R})$-norm of $T_t f$ equals zero with positive probability. To avoid such a situation one can consider $f*p_\varepsilon$, where p_ε is the density of a normal distribution with zero mean and variance ε. Then, due to the formula of change of variables for an Arratia flow [3], one can write down

$$\int_{\mathbb{R}} T_t(f*p_\varepsilon)(u)^2du = \sum_{\theta:\ \Delta y(\theta,t)>0} \Delta y(\theta,t)\int_{\mathbb{R}}\int_{\mathbb{R}} p_\varepsilon(v_1-\theta)p_\varepsilon(v_2-\theta)f(v_1)f(v_2)dv_1dv_2, \tag{2}$$

where $\{y(u,s), u\in\mathbb{R}, s\in[0;t]\}$ is the conjugated Arratia flow [4]. On the right-hand side of (2) one may see the quadratic form of the operator similar to (1). Hence, the knowledge of the properties of (1) can be helpful for investigating the random operators generated by the stochastic flows. The article continues studying the characteristics of random operators from [6, 7].

2. Shifts of a Gaussian density along a point process

We will start with the following statement.

Theorem 2.1. *Let Θ be a stationary ergodic point process on $\mathbb{R}$ [1] and $0 < E|\Theta\cap[0;1)\,| < +\infty$. Then there exists an event Ω_0 of probability one such that for any $\omega\in\Omega_0$ and $a<b$ the linear span of the functions $\{p_\varepsilon(u-\theta(\omega)),\ \theta(\omega)\in\Theta(\omega)\}$ is dense in $L_2([a;b])$.*

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Proof. Let us break the proof into several steps.

Lemma 2.1. *Let* Θ *be a stationary ergodic point process on* $\mathbb{R}$ *with* $0 < E|\Theta \cap [0;1)\,| < +\infty$. *Then with probability one*

$$\sum_{\theta\in\Theta} \frac{1}{|\theta|} = +\infty.$$

Proof. It is sufficient to prove that

$$\sum_{\theta\in\Theta\cap[1;+\infty)} \frac{1}{\theta} = +\infty \quad \text{a.s.} \tag{3}$$

Since $\sum_{\theta\in\Theta\cap[1;+\infty)} \frac{1}{\theta} \geq \sum_{n=1}^{\infty} \frac{1}{n+1}|\Theta \cap [n;n+1)\,|$, it suffices to show that for a sequence $\xi_n = |\Theta \cap [n;n+1)\,|$ the series $\sum_{n=1}^{\infty} \frac{1}{n}\xi_n$ diverges almost surely. One may note that $\{\xi_n\}_{n=0}^{\infty}$ is a stationary, ergodic, and $E|\xi_0| < \infty$. Hence, due to the ergodic theorem, for $S_n = \sum_{k=0}^{n} \xi_n$ the following convergence holds

$$\frac{1}{n}S_n \to E\xi_0,\ n \to \infty \quad \text{a.s.} \tag{4}$$

Thus, with probability one $\widetilde{C} = \sup_{n\in\mathbb{N}} \frac{1}{n}S_n < +\infty$, and there exists $N \in \mathbb{N}$ such that for any $n \geq N$

$$\frac{1}{n}S_n \geq \frac{E\xi_0}{2}. \tag{5}$$

One can check that for any $m \in \mathbb{N}$

$$\sum_{k=2}^{m} \frac{1}{k}\xi_k = \sum_{k=2}^{m}\left(\frac{S_k}{k} - \frac{S_{k-1}}{k-1}\right) + \sum_{k=2}^{m} \frac{S_{k-1}}{k(k-1)}. \tag{6}$$

Due to (4)

$$\sum_{k=2}^{\infty}\left(\frac{S_k}{k} - \frac{S_{k-1}}{k-1}\right) \leq 2\widetilde{C}. \tag{7}$$

By (5), the series $\sum_{k=2}^{+\infty} \frac{S_{k-1}}{k(k-1)}$ diverges, which, by (6) and (7), proves the statement. □

Corollary 2.1. *Using Lemma 2.1 and Muntz theorem one may verify that there exists* Ω_0 *of probability one such that for any* $\omega \in \Omega_0$ *and* $0 < a < b$ *the linear span of the functions* $\left\{u^{\theta(\omega)},\ \theta(\omega) \in \Theta(\omega)\right\}$ *is dense in* $L_2([a;b])$.

Corollary 2.2. *There exists* Ω_0 *of probability one such that for any* $\omega \in \Omega_0$ *and* $a < b$ *the linear span of the functions* $\left\{e^{\theta(\omega)u},\ \theta(\omega) \in \Theta(\omega)\right\}$ *is dense in* $L_2([a;b])$.

Proof. Denote by $LS\left\{f_k, k = \overline{1,n}\right\}$ the linear span of $f_1, \ldots, f_n$. Notice that for any $a < b$ and $f \in L_2([a;b])$ the following relations hold

$$d\left(f, LS\left\{e^{\theta u},\ \theta\in\Theta\right\}\right)^2_{L_2([a;b])} = \inf_{c_\theta}\int_a^b \left(f(u) - \sum_{\theta\in\Theta} c_\theta e^{\theta u}\right)^2 du =$$

$$= \inf_{c_\theta}\int_{e^a}^{e^b}\left(f(\ln u) - \sum_{\theta\in\Theta} c_\theta u^\theta\right)^2 \frac{du}{u} \leq e^{-a} d\left(\tilde{f}, LS\left\{v^\theta, \theta\in\Theta\right\}\right)^2_{L_2([e^a;e^b])},$$

where the function $\tilde{f}(u) = f(\ln u)$ belongs to $L_2([e^a;e^b])$.

Thus, due to Corollary 2.1, with probability one for any $a < b$ and $f \in L_2([a;b])$

$$d\left(f, LS\left\{e^{\theta u},\ \theta\in\Theta\right\}\right)_{L_2([a;b])} = 0.$$

□

To complete the proof of the theorem let us consider a fixed point $\tilde{\theta} \in \Theta$, and a linear bounded operator B on $L_2([a;b])$ such that $(Bf)(u) = f(u)h(u)$, where

$$h(u) = \frac{1}{\sqrt{2\pi\varepsilon}} e^{-\frac{u^2}{2\varepsilon}} e^{-\frac{\tilde{\theta}^2}{2\varepsilon}}.$$

Then for any $a < b$ and $f \in L_2([a;b])$

$$d\left(f, LS\left\{p_\varepsilon(u-\theta), \theta \in \Theta\right\}\right)^2_{L_2([a;b])} =$$

$$= d\left(B\left(f(u)\sqrt{2\pi\varepsilon} e^{\frac{u^2}{2\varepsilon}} e^{\frac{\tilde{\theta}^2}{2\varepsilon}}\right), LS\left\{B\left(e^{-\frac{\tilde{\theta}^2-\theta^2}{2\varepsilon}} e^{\frac{\theta u}{\varepsilon}}\right), \theta \in \Theta\right\}\right)^2_{L_2([a;b])} =$$

$$= d\left(B\tilde{f}(u), LS\left\{Be^{\frac{\theta u}{\varepsilon}}, \theta \in \Theta\right\}\right)^2_{L_2([a;b])},$$

where $\tilde{f}(u) = f(u)\sqrt{2\pi\varepsilon} e^{\frac{u^2}{2\varepsilon}} e^{\frac{\tilde{\theta}^2}{2\varepsilon}}$. Since B is a bounded linear operator on $L_2([a;b])$, the inequality

$$d\left(B\tilde{f}(u), LS\left\{Be^{\frac{\theta u}{\varepsilon}}, \theta \in \Theta\right\}\right)^2_{L_2([a;b])} \le \|B\|^2 d\left(\tilde{f}(u), LS\left\{e^{\frac{\theta u}{\varepsilon}}, \theta \in \Theta\right\}\right)^2_{L_2([a;b])}$$

holds true, and its right-hand side equals 0, due to Corollary 2.2. Consequently, with probability one the linear span of the functions $\{p_\varepsilon(\cdot - \theta);\ \theta \in \Theta\}$ is dense in $L_2([a;b])$. The theorem is proved. □

3. Properties of the integral random operator

Now let us turn to the integral operator with kernel (1). Let p_ε be the same as before.

Lemma 3.1. *For any $f \in L_2(\mathbb{R})$ and a stationary point process Θ with $E|\Theta \cap [0;1)\,| < +\infty$*

$$\sum_{\theta \in \Theta} \left(\int_{\mathbb{R}} f(u) p_\varepsilon(u-\theta) du\right)^2 < +\infty \quad a.s.$$

Proof. Since Θ is a stationary point process, its intensity measure is $C\lambda$, where λ is the Lebesgue measure on $\mathbb{R}$, and $C = E|\Theta \cap [0;1)\,|$. Thus, by Campbell's formula [1], for every $f \in L_2(\mathbb{R})$

$$E\sum_{\theta \in \Theta} \left(\int_{\mathbb{R}} f(u) p_\varepsilon(u-\theta) du\right)^2 \le E\sum_{\theta \in \Theta} \int_{\mathbb{R}}\int_{\mathbb{R}} |f(u)||f(v)| p_\varepsilon(u-\theta) p_\varepsilon(v-\theta) du dv =$$

$$= \int_{\mathbb{R}}\int_{\mathbb{R}} |f(u)||f(v)| E\sum_{\theta \in \Theta} p_\varepsilon(u-\theta) p_\varepsilon(v-\theta) du dv =$$

$$= C\int_{\mathbb{R}}\int_{\mathbb{R}} |f(u)||f(v)| \int_{\mathbb{R}} p_\varepsilon(u-t) p_\varepsilon(v-t) dt du dv =$$

$$= C\int_{\mathbb{R}}\int_{\mathbb{R}} |f(u)||f(v)| p_{2\varepsilon}(u-v) du dv = C\int_{\mathbb{R}} h^2(\lambda) e^{-\varepsilon\lambda^2} d\lambda \le C\int_{\mathbb{R}} |f(u)|^2 du,$$

where h is the Fourier transform of $f \in L_2(\mathbb{R})$. □

Remark 3.1. *It follows from the proof of Lemma 3.1 that the following integral operator*

$$Af(v) = \sum_{\theta \in \Theta} \int_{\mathbb{R}} f(u) p_\varepsilon(u-\theta) du \cdot p_\varepsilon(v-\theta)$$

is well-defined and is a strong random operator in Skorokhod sense [5].

The next lemma shows that A is not a bounded random operator in the most interesting cases.

Lemma 3.2. *Let Θ be an ergodic stationary point process such that* $\operatorname{essup}|\Theta \cap [0;1]| = +\infty$*. Then A is not a bounded random operator.*

Proof. It can be checked that under the assumption on the process Θ with probability one there exists an increasing sequence of natural numbers $\{n_k;\ k \geq 1\}$ such that

$$\sup_{k\geq 1} |\Theta \cap [\,n_k; n_k+1)\,| = +\infty.$$

Consider the following sequence of functions from $L_2(\mathbb{R})$

$$f_k = 1\!\!1_{[\,n_k;n_k+1)}\,, \quad k \geq 1.$$

Then

$$\|Af_k\|^2 \geq \sum_{\theta\in\Theta\cap[\,n_k;n_k+1)} \left(\int_0^1 p_\varepsilon(v)dv\right)^2 p_\varepsilon(1)^2.$$

Hence, $\sup_k \|Af_k\| = +\infty$, and the lemma is proved. □

For a fixed interval $[a;b]$ denote by $Q_{a,b}$ the orthogonal projection of $L_2(\mathbb{R})$ onto $L_2([a;b])$, which we identify with the subspace of $L_2(\mathbb{R})$ of functions supported on $[a;b]$.

Remark 3.2. *For any $a, b \in \mathbb{R}$ the random operators $AQ_{a,b}$, $Q_{a,b}A$ are bounded.*

Proof. One can check, by Hölder's inequality, that for any $f, g \in L_2(\mathbb{R})$

$$(AQ_{a,b}f, g) = \sum_{\theta\in\Theta} \int_{\mathbb{R}} g(v)p_\varepsilon(v-\theta) \int_a^b f(u)p_\varepsilon(u-\theta)du \leq$$

$$\leq 2^{-\frac{1}{4}}(b-a)^{\frac{1}{2}}\|g\|_{L_2(\mathbb{R})}\|f\|_{L_2(\mathbb{R})} \sum_{\theta\in\Theta} \max_{u\in[a;b]} p_\varepsilon(u-\theta).$$

By Campbell's formula [1],

$$E\sum_{\theta\in\Theta} \max_{u\in[a;b]} p_\varepsilon(u-\theta) = \int_{\mathbb{R}} \max_{u\in[a;b]} p_\varepsilon(u-r)dr < +\infty.$$

Thus, $\sum_{\theta\in\Theta} \max_{u\in[a;b]} p_\varepsilon(u-\theta) < +\infty$ a.s., which proves the statement. □

Lemma 3.3. *For any $a, b \in \mathbb{R}$ with probability one the random operator $A_{Q_{a,b}} = Q_{a,b}AQ_{a,b}$ is nuclear.*

Proof. To prove the statement, let us estimate the nuclear norm of $Q_{a,b}AQ_{a,b}$. For any $\theta \in \Theta$ denote by e_θ the function

$$e_\theta = Q_{a,b}p_\varepsilon(\cdot - \theta).$$

Evidently, the operator $e_\theta \otimes e_\theta$ is nuclear, and its nuclear norm equals $\|e_\theta\|^2$. Notice that

$$E\sum_{\theta\in\Theta}\|e_\theta\|^2 = E\sum_{\theta\in\Theta}\int_a^b p_\varepsilon(u-\theta)^2du =$$

$$= C\int_a^b\int_{\mathbb{R}} p_\varepsilon(u-v)^2dvdu < +\infty,$$

where, as before, $C = E|\Theta \cap [0;1]|$. Its enough to note that

$$Q_{a,b}AQ_{a,b} = \sum_{\theta\in\Theta} e_\theta \otimes e_\theta. \tag{8}$$

The lemma is proved. □

Due to the previous statement, the image K of the unit ball in $L_2([a;b])$ under the operator $A_{Q_{a,b}}$ is a compact set with probability one. We obtain the following statement about asymptotic behavior of the Kolmogorov width for the compact set K.

Theorem 3.1. *Let Θ be an ergodic stationary point process. Then with probability one there exists $C > 0$ such that*

$$d_n(K) = O\left(e^{-\frac{(Cn-b)^2}{\varepsilon}} \vee e^{-\frac{(Cn+a)^2}{\varepsilon}}\right),\ n \to \infty.$$

Proof. Representation (8) allows us to estimate Kolmogorov widths of K. Let us denote by $N_x,\ x > 0$, the number of elements of the set $\Theta \cap [-x; x]$, and by d_n the n-th Kolmogorov width of K. It follows from (8) that

$$d_{N_x} \leq \sum_{\theta \in \Theta \setminus [-x;x]} \|e_\theta\|^2. \tag{9}$$

Due to the ergodic theorem, $N_x \sim 2Cx$ when $x \to +\infty$.

To estimate the right-hand side of (9), suppose that $x > \max\{-a, b\}$, and consider the sum

$$\sum_{\theta \in \Theta, \theta > x} \|e_\theta\|^2 \leq \sum_{\theta \in \Theta, \theta > x} (b-a) p_\varepsilon(\theta - b)^2.$$

Denote by $\xi_n = |\Theta \cap [n; n+1)\ |$. Then $\{\xi_n; n \geq 1\}$ is a stationary ergodic sequence. For a natural x

$$\sum_{\theta \in \Theta, \theta > x} p_\varepsilon(\theta - b)^2 \leq \sum_{k=x}^{+\infty} p_\varepsilon(x-b)^2 \xi_k.$$

For any $k \geq 1$ let $S_k = \sum_{j=1}^k \xi_j$. Since $S_k \sim Ck,\ k \to \infty$ a.s., then, by Abel transform, one can check that

$$\sum_{k=x}^{+\infty} p_\varepsilon(k-b)^2 \xi_k = -p_\varepsilon(x-b)^2 S_{x-1} + \sum_{k=x}^{+\infty} S_k(p_\varepsilon(k-b)^2 - p_\varepsilon(k+1-b)^2) \sim$$

$$\sim C \sum_{k=x}^{+\infty} (p_\varepsilon(k-b)^2 - p_\varepsilon(k+1-b)^2)k,\ \ x \to +\infty.$$

Observe that

$$\sum_{k=x}^{+\infty} (p_\varepsilon(k-b)^2 - p_\varepsilon(k+1-b)^2)k = \frac{1}{2\pi\varepsilon} \sum_{k=x}^{+\infty} (1 - e^{-\frac{(2k+1-2b)}{\varepsilon}}) e^{-\frac{(k-b)^2}{\varepsilon}} k \sim \frac{1}{4\pi} e^{-\frac{(x-b)^2}{\varepsilon}},$$

and the statement is proved. □

For any interval $[a; b]$, $A_{Q_{a,b}}$ is a bounded (nuclear) random operator. Despite this, as $[a; b]$ increases to $\mathbb{R}$, $A_{Q_{a,b}}$ must converge to unbounded random operator A. Consequently, one can expect that the operator norm $\|A_{Q_{a,b}}\|$ will increase to infinity as $[a; b]$ increases to $\mathbb{R}$. Using the arguments from the proof of Lemma 3.2 one can prove the following statement.

Theorem 3.2. *Let Θ be a Poisson point process with intensity 1. Then*

$$\frac{\ln \ln n}{\ln n} \|A_{Q_{-n,n}}\| \to +\infty,\ n \to \infty \text{ a.s.}$$

Proof. Using the arguments from the proof of Lemma 3.2, one can show that

$$\|A_{Q_{-n,n}}\| \geq C \max_{\overline{1,n}} \xi_k,$$

where the random variables $\{\xi_n; n \geq 1\}$ were introduced above. Now $\{\xi_n; n \geq 1\}$ are independent poissonian random variables with intensity 1. Consequently,

$$P\{\xi_1 \geq m\} \sim \frac{e^{-1}}{m!},\ \ m \to +\infty.$$

For any $R > 0$ $P\{\max_{k=\overline{1,n}} \xi_k \le m_n R\} = (1 - P\{\xi_1 > m_n R\})^n$. Thus, for $m_n = \frac{\ln n}{\ln \ln n}$

$$\frac{\max_{k=\overline{1,n}} \xi_k}{m_n} \to +\infty,\ n \to \infty \quad \text{a.s.},$$

and the theorem is proved. □

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