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REMARK ON OPTIMAL INVESTMENT IN A MARKET WITH MEMORY

We consider a financial market model driven by a Gaussian semimartingale with stationary increments. This driving noise process consists of n independent components and each component has memory described by two parameters. We extend results of the authors on optimal investment in this market.

1. Introduction

In this paper, we extend results of Inoue and Nakano [12] on optimal investment in a financial market model with memory. This market model $\mathcal{M}$ consists of n risky and one riskless assets. The price of the riskless asset is denoted by $S_0(t)$ and that of the ith risky asset by $S_i(t)$. We put $S(t) = (S_1(t), \ldots, S_n(t))'$, where A' denotes the transpose of a matrix A. The dynamics of the $\mathbf{R}^n$-valued process $S(t)$ are described by the stochastic differential equation

$$dS_i(t) = S_i(t) \left[\mu_i(t)dt + \sum_{j=1}^n \sigma_{ij}(t)dY_j(t)\right], \quad t \geq 0, \quad S_i(0) = s_i, \quad i = 1, \ldots, n, \tag{1}$$

while those of $S_0(t)$ by the ordinary differential equation

$$dS_0(t) = r(t)S_0(t)dt, \quad t \geq 0, \quad S_0(0) = 1,$$

where the coefficients $r(t) \geq 0$, $\mu_i(t)$, and $\sigma_{ij}(t)$ are continuous deterministic functions on $[0, \infty)$ and the initial prices s_i are positive constants. We assume that the $n \times n$ volatility matrix $\sigma(t) = (\sigma_{ij}(t))_{1\leq i,j\leq n}$ is nonsingular for $t \geq 0$.

Invited lecture.

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We define the jth component $Y_j(t)$ of the $\mathbf{R}^n$-valued driving noise process $Y(t) = (Y_1(t), \ldots, Y_n(t))'$ of (1) by the autoregressive type equation

$$\frac{dY_j(t)}{dt} = -\int_{-\infty}^{t} p_j e^{-q_j(t-s)} \frac{dY_j(s)}{ds} ds + \frac{dW_j(t)}{dt}, \quad t \in \mathbf{R}, \quad Y_j(0) = 0,$$

where $W(t) = (W_1(t), \ldots, W_n(t))'$, $t \in \mathbf{R}$, is an $\mathbf{R}^n$-valued standard Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, P)$, the derivatives $dY_j(t)/dt$ and $dW_j(t)/dt$ are in the random distribution sense, and p_j's and q_j's are constants such that

$$0 < q_j < \infty, \quad -q_j < p_j < \infty, \quad j = 1, \ldots, n \tag{2}$$

(see Anh–Inoue [1]). Equivalently, we may define $Y_j(t)$ by the moving-average type representation

$$Y_j(t) = W_j(t) - \int_0^t \left[\int_{-\infty}^{s} p_j e^{-(q_j+p_j)(s-u)} dW_j(u) \right] ds, \quad t \in \mathbf{R}$$

(see [1], Examples 2.12 and 2.14). The components $Y_j(t)$, $j = 1, \ldots, n$, are Gaussian processes with stationary increments that are independent of each other. Each $Y_j(t)$ has short memory that is described by the two parameters p_j and q_j. Notice that, in the special case $p_j = 0$, $Y_j(t)$ reduces to the Brownian motion $W_j(t)$.

The underlying information structure of the market model $\mathcal{M}$ is the filtration $(\mathcal{F}_t)_{t\geq 0}$ defined by

$$\mathcal{F}_t := \sigma\left(\sigma(Y(s) : 0 \leq s \leq t) \cup \mathcal{N}\right), \quad t \geq 0,$$

where $\mathcal{N}$ is the P-null subsets of $\mathcal{F}$. With respect to this filtration, $Y(t)$ is a semimartingale. In fact, we have the following two kinds of semimartingale representations of $Y(t)$ (see Anh et al. [2], Example 5.3, and Inoue et al. [13], Theorem 2.1):

$$Y_j(t) = B_j(t) - \int_0^t \left[\int_0^s k_j(s,u) dY_j(u) \right] ds, \quad t \geq 0, \quad j = 1, \ldots, n, \tag{3}$$

$$Y_j(t) = B_j(t) - \int_0^t \left[\int_0^s l_j(s,u) dB_j(u) \right] ds, \quad t \geq 0, \quad j = 1, \ldots, n, \tag{4}$$

where, for $j = 1, \ldots, n$, $(B_j(t))_{t\geq 0}$ is the innovation process, i.e., an $\mathbf{R}$-valued standard Brownian motion such that

$$\sigma(Y_j(s) : 0 \leq s \leq t) = \sigma(B_j(s) : 0 \leq s \leq t), \quad t \geq 0,$$

and B_j's are independent of each other. The point of (3) and (4) is that the deterministic kernels $k_j(t,s)$ and $l_j(t,s)$ are given explicitly by

$$k_j(t,s) = p_j(2q_j + p_j) \frac{(2q_j + p_j)e^{q_j s} - p_j e^{-q_j s}}{(2q_j + p_j)^2 e^{q_j t} - p_j^2 e^{-q_j t}}, \quad 0 \leq s \leq t, \tag{5}$$

$$l_j(t,s) = e^{-(p_j+q_j)(t-s)} l_j(s), \quad 0 \leq s \leq t, \tag{6}$$

with

$$l_j(s) := p_j \left[1 - \frac{2p_jq_j}{(2q_j+p_j)^2e^{2q_js} - p_j^2}\right], \quad s \geq 0. \tag{7}$$

There already exist many references in which the standard driving noise, that is, Brownian motion, is replaced by a different one, such as fractional Brownian motion, so that the market model can capture *memory effect*. See, e.g., Barndorff-Nielsen and Shephard [3], Hu et al. [11], Mishura [15] and Heyde and Leonenko [10]. Among such models, the above model $\mathcal{M}$ driven by $Y(t)$ which is a Gaussian semimartingale with *stationary increments* is possibly the simplest one. One advantage of $\mathcal{M}$ is that, assuming $\sigma_{ij}(t) = \sigma_{ij}$, real constants, we can easily estimate the characteristic parameters p_j, q_j and σ_{ij} from stock price data. See [12], Appendix C, for this parameter estimation from real market data.

In the market $\mathcal{M}$, an agent with initial endowment $x \in (0,\infty)$ invests, at each time t, $\pi_i(t)X^{x,\pi}(t)$ dollars in the ith risky asset for $i = 1,\ldots,n$ and $[1 - \sum_{i=1}^n \pi_i(t)]X^{x,\pi}(t)$ dollars in the riskless asset, where $X^{x,\pi}(t)$ denotes the agent's wealth at time t. The wealth process $X^{x,\pi}(t)$ is governed by the stochastic differential equation

$$\frac{dX^{x,\pi}(t)}{X^{x,\pi}(t)} = \left[1 - \sum\nolimits_{i=1}^n \pi_i(t)\right]\frac{dS_0(t)}{S_0(t)} + \sum\nolimits_{i=1}^n \pi_i(t)\frac{dS_i(t)}{S_i(t)}, \quad X^{x,\pi}(0) = x.$$

Here, the self-financing strategy $\pi(t) = (\pi_1(t),\ldots,\pi_n(t))'$ is chosen from the admissible class

$$\mathcal{A}_T := \left\{\pi = (\pi(t))_{0\leq t\leq T} : \begin{array}{r}\pi \text{ is an } \mathbf{R}^n\text{-valued, progressively measurable} \\ \text{process satisfying } \int_0^T \|\pi(t)\|^2dt < \infty \text{ a.s.}\end{array}\right\}$$

for the finite time horizon of length $T \in (0,\infty)$, where $\|\cdot\|$ denotes the Euclidean norm of $\mathbf{R}^n$. If the time horizon is infinite, $\pi(t)$ is chosen from

$$\mathcal{A} := \{(\pi(t))_{t\geq 0} : (\pi(t))_{0\leq t\leq T} \in \mathcal{A}_T \text{ for every } T \in (0,\infty)\}.$$

Let $\alpha \in (-\infty,1)\setminus\{0\}$ and $c \in \mathbf{R}$. In [12], the following three optimal investment problems for the model $\mathcal{M}$ are considered:

$$V(T,\alpha) := \sup_{\pi\in\mathcal{A}_T} \frac{1}{\alpha}E\left[(X^{x,\pi}(T))^\alpha\right], \tag{8}$$

$$J(\alpha) := \sup_{\pi\in\mathcal{A}} \limsup_{T\to\infty} \frac{1}{\alpha T}\log E\left[(X^{x,\pi}(T))^\alpha\right], \tag{9}$$

$$I(c) := \sup_{\pi\in\mathcal{A}} \limsup_{T\to\infty} \frac{1}{T}\log P\left[X^{x,\pi}(T) \geq e^{cT}\right]. \tag{10}$$

Problem (8) is the classical optimal investment problem that dates back to Merton (cf. Karatzas and Shreve [14]). Hu et al. [11] studied this problem for a Black–Scholes type model driven by fractional Brownian motion. Problem (9) is a kind of long term optimal investment problem which is studied by Bielecki and Pliska [4], and also by other authors under various settings, including Fleming and Sheu [5,6], Nagai and Peng [16], Pham [17,18], Hata and Iida [7], and Hata and Sekine [8,9]. Problem (10) is another type of long term optimal investment problem, the aim of which is to maximize the large deviation probability that the wealth grows at a higher rate than the given benchmark c. Pham [17,18] studied this problem and established a duality relation between Problems (9) and (10). Subsequently, this problem is studied by Hata and Iida [7] and Hata and Sekine [8,9] under different settings.

In [12], Problems (8)–(10) are studied for the market model $\mathcal{M}$ which has memory. There, the following condition, rather than (2), is assumed in solving (8)–(10):

$$0 < q_j < \infty, \quad 0 \le p_j < \infty, \qquad j = 1, \dots, n. \tag{11}$$

Thus, in [12], $p_j \ge 0$ rather than $p_j > -q_j$ for $j = 1, \dots, n$. In this paper, we focus on Problems (8) and (9), and extends the results of [12] so that p_j's may take negative values. The key to this extension is to show the existence of solution for a relevant Riccati type equation.

In Sections 2 and 3, we review the results of [12] on Problems (8) and (9), respectively, and, in Section 4, we extend these results.

2. Optimal investment over a finite horizon

In this section, we review the result of [12] on the finite horizon optimization problem (8) for the market model $\mathcal{M}$. We assume $\alpha \in (-\infty, 1) \setminus \{0\}$ and (11).

Let $Y(t) = (Y_1(t), \dots, Y_n(t))'$ and $B(t) = (B_1(t), \dots, B_n(t))'$ be the driving noise and innovation processes, respectively, described in the previous section. We define an $\mathbf{R}^n$-valued deterministic function $\lambda(t) = (\lambda_1(t), \dots, \lambda_n(t))'$ by

$$\lambda(t) := \sigma^{-1}(t)\left[\mu(t) - r(t)\mathbf{1}\right], \qquad t \ge 0, \tag{12}$$

where $\mathbf{1} := (1, \dots, 1)' \in \mathbf{R}^n$. For $k_j(t,s)$'s in (5), we put

$$k(t,s) := \mathrm{diag}(k_1(t,s), \dots, k_n(t,s)), \qquad 0 \le s \le t.$$

Let $\xi(t) = (\xi_1(t), \dots, \xi_n(t))'$ be the $\mathbf{R}^n$-valued process $\int_0^t k(t,s)dY(s)$, i.e.,

$$\xi_j(t) := \int_0^t k_j(t,s)dY_j(s), \qquad t \ge 0, \quad j = 1, \dots, n. \tag{13}$$

Let β be the conjugate exponent of α, i.e.,

$$(1/\alpha) + (1/\beta) = 1.$$

Notice that $0 < \beta < 1$ (resp. $-\infty < \beta < 0$) if $-\infty < \alpha < 0$ (resp. $0 < \alpha < 1$).

We put $l(t) := \mathrm{diag}(l_1(t), \dots, l_n(t))$, $p := \mathrm{diag}(p_1, \dots, p_n)$, and $q := \mathrm{diag}(q_1, \dots, q_n)$ with $l_j(t)$'s as in (7). We also put

$$\rho(t) = (\rho_1(t), \dots, \rho_n(t))', \quad b(t) = \mathrm{diag}(b_1(t), \dots, b_n(t))$$

with

$$\rho_j(t) := -\beta l_j(t)\lambda_j(t), \qquad t \geq 0, \quad j = 1, \dots, n, \tag{14}$$

$$b_j(t) := -(p_j + q_j) + \beta l_j(t), \qquad t \geq 0, \quad j = 1, \dots, n. \tag{15}$$

We consider the following one-dimensional backward Riccati equations: for $j = 1, \dots, n$

$$\begin{cases} \dot{R}_j(t) - l_j^2(t)R_j^2(t) + 2b_j(t)R_j(t) + \beta(1-\beta) = 0, \quad 0 \leq t \leq T, \\ R_j(T) = 0. \end{cases} \tag{16}$$

We have the following result on the existence of solution to (16).

Lemma 1 ([12], Lemma 2.1)**.** *Let $j \in \{1, \dots, n\}$.*

1. *If $p_j = 0$, then (16) has a unique solution $R_j(t) \equiv R_j(t;T)$.*
2. *If $p_j > 0$ and $-\infty < \alpha < 0$, then (16) has a unique nonnegative solution $R_j(t) \equiv R_j(t;T)$.*
3. *If $p_j > 0$ and $0 < \alpha < 1$, then (16) has a unique solution $R_j(t) \equiv R_j(t;T)$ such that $R_j(t) \geq b_j(t)/l_j^2(t)$ for $t \in [0,T]$.*

In what follows, we write $R_j(t) \equiv R_j(t;T)$ for the unique solution to (16) in the sense of Lemma 1, and we put $R(t) := \mathrm{diag}(R_1(t), \dots, R_n(t))$.

For $j = 1, \dots, n$, let $v_j(t) \equiv v_j(t;T)$ be the solution to the following one-dimensional linear equation:

$$\begin{gathered} \dot{v}_j(t) + [b_j(t) - l_j^2(t)R_j(t;T)]v_j(t) + \beta(1-\beta)\lambda_j(t) - R_j(t;T)\rho_j(t) = 0, \\ 0 \leq t \leq T, \qquad v_j(T) = 0. \end{gathered} \tag{17}$$

We put $v(t) \equiv v(t;T) := (v_1(t;T), \dots, v_n(t;T))'$.

For $j = 1, \dots, n$ and $(t,T) \in \Delta$, write

$$g_j(t;T) := v_j^2(t;T)l_j^2(t) + 2\rho_j(t)v_j(t;T) - l_j^2(t)R_j(t;T) - \beta(1-\beta)\lambda_j^2(t),$$

where

$$\Delta := \{(t,T) : 0 < T < \infty,\ 0 \leq t \leq T\}.$$

Recall that we have assumed $\alpha \in (-\infty, 1) \setminus \{0\}$ and (11). Here is the solution to Problem (8) under the condition (11).

Theorem 2 ([12], Theorem 2.3)**.** *For $T \in (0, \infty)$, the strategy $(\hat{\pi}_T(t))_{0 \le t \le T} \in \mathcal{A}_T$ defined by*

$$\hat{\pi}_T(t) := (\sigma')^{-1}(t)\,[(1-\beta)\lambda(t) - \{1 - \beta + l(t)R(t;T)\}\xi(t) + l(t)v(t;T)] \quad (18)$$

is the unique optimal strategy for Problem (8). The value function $V(T) \equiv V(T, \alpha)$ in (8) is given by

$$V(T) = \frac{1}{\alpha}[xS_0(T)]^{\alpha} \exp\left[\frac{(1-\alpha)}{2} \sum\nolimits_{j=1}^{n} \int_0^T g_j(t;T)dt\right].$$

3. Optimal investment over an infinite horizon

In this section, we review the result of [12] on the infinite horizon optimization problem (9) for the financial market model $\mathcal{M}$. Throughout this section, we assume (11) and the following two conditions:

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T r(t)dt = \bar{r} \quad \text{with } \bar{r} \in \mathbf{R}, \quad (19)$$

$$\lim_{T\to\infty} \lambda(t) = \bar{\lambda} \quad \text{with } \bar{\lambda} = (\bar{\lambda}_1, \ldots, \bar{\lambda}_n)' \in \mathbf{R}^n. \quad (20)$$

Here recall $\lambda(t) = (\lambda_1(t), \ldots, \lambda_n(t))'$ from (12).

Let $\alpha \in (-\infty, 1) \setminus \{0\}$ and β be its conjugate exponent. Let $j \in \{1, \ldots, n\}$. For $b_j(t)$ in (15), we have $\lim_{t\to\infty} b_j(t) = \bar{b}_j$, where

$$\bar{b}_j := -(1-\beta)p_j - q_j.$$

Notice that $\bar{b}_j < 0$. We consider the equation

$$p_j^2 x^2 - 2\bar{b}_j x - \beta(1-\beta) = 0. \quad (21)$$

When $p_j = 0$, we write $\bar{R}_j$ for the unique solution $\beta(1-\beta)/(2q_j)$ of this linear equation. If $p_j > 0$, then

$$\bar{b}_j^2 + \beta(1-\beta)p_j^2 = (1-\beta)[(p_j + q_j)^2 - q_j^2] + q_j^2 \ge q_j^2 > 0,$$

so that we may write $\bar{R}_j$ for the larger solution to the quadratic equation (21). Let $j \in \{1, \ldots, n\}$. For $\rho_j(t)$ in (14), we have $\lim_{t\to\infty} \rho_j(t) = \bar{\rho}_j$, where

$$\bar{\rho}_j := -\beta p_j \bar{\lambda}_j.$$

Define $\bar{v}_j$ by

$$\left(\bar{b}_j - p_j^2 \bar{R}_j\right)\bar{v}_j + \beta(1-\beta)\bar{\lambda}_j - \bar{R}_j\bar{\rho}_j = 0.$$

For $j = 1, \dots, n$ and $-\infty < \alpha < 1$, $\alpha \neq 0$, we put

$$F_j(\alpha) := \frac{(p_j + q_j)^2 \bar{\lambda}_j^2 \alpha}{[(1-\alpha)(p_j + q_j)^2 + \alpha p_j (p_j + 2q_j)]},$$

and

$$\begin{aligned} G_j(\alpha) &:= (p_j + q_j) - q_j \alpha \\ &\quad - (1-\alpha)^{1/2} \left[(1-\alpha)(p_j + q_j)^2 + \alpha p_j (p_j + 2q_j) \right]^{1/2}. \end{aligned}$$

Recall $\xi(t)$ from (13). Taking into account (18), we consider $\hat{\pi} = (\hat{\pi}(t))_{t \geq 0} \in \mathcal{A}$ defined by

$$\hat{\pi}(t) := (\sigma')^{-1}(t) \left[(1-\beta)\lambda(t) - (1 - \beta + p\bar{R})\xi(t) + p\bar{v} \right], \qquad t \geq 0,$$

where $\bar{R} := \mathrm{diag}(\bar{R}_1, \dots, \bar{R}_n)$, $\bar{v} := (\bar{v}_1, \dots, \bar{v}_n)'$, and $p := \mathrm{diag}(p_1, \dots, p_n)$.

We define

$$\alpha_* := \max(\alpha_{1*}, \dots, \alpha_{n*}) \tag{22}$$

with

$$\alpha_{j*} := \begin{cases} -\infty & \text{if } \; -\infty < p_j \leq 2q_j, \\ -3 - \frac{8q_j}{p_j - 2q_j} & \text{if } \; 2q_j < p_j < \infty. \end{cases} \tag{23}$$

Notice that $\alpha_* \in [-\infty, -3)$.

Recall that we have assumed (11), (19) and (20). Here is the solution to Problem (9) under the condition (11).

Theorem 3 ([12], Theorem 3.4)**.** *Let $\alpha_* < \alpha < 1$, $\alpha \neq 0$. Then $\hat{\pi}$ is an optimal strategy for Problem (9) with limit rather than limsup in (9). The optimal growth rate $J(\alpha)$ in (9) is given by*

$$J(\alpha) = \bar{r} + \frac{1}{2\alpha} \sum_{j=1}^{n} F_j(\alpha) + \frac{1}{2\alpha} \sum_{j=1}^{n} G_j(\alpha).$$

4. Extensions

In this section, we extend Theorems 2 and 3 so that p_j's may take negative values. The key is to extend Lemma 1 properly. We assume $\alpha \in (-\infty, 1) \setminus \{0\}$ and (2).

We put, for $j = 1, \dots, n$,

$$\begin{aligned} a_{1j}(t) &= l_j(t)^2, \\ a_{2j}(t) &= \beta l_j(t) - (p_j + q_j), \\ a_3 &= \beta(1-\beta). \end{aligned}$$

Then the Riccati equation (16) becomes

$$\begin{aligned}&\dot{R}_j(t) - a_{1j}(t)R_j^2(t) + 2a_{2j}(t)R_j(t) + a_3 = 0, \quad 0 \le t \le T,\\ &R_j(T) = 0.\end{aligned}$$

Note that $l_j(t)$ is increasing and satisfies

$$l_j(0) = \frac{p_j(p_j + 2q_j)}{2(p_j + q_j)} \le l_j(t) \le p_j, \qquad t \ge 0.$$

Proposition 4. *Let $j \in \{1, \dots, n\}$. We assume $-q_j < p_j < 0$ and*

$$0 < \alpha \le \left(\frac{p_j + q_j}{p_j + q_j - l_j(0)}\right)^2. \tag{24}$$

1. *It holds that $a_{2j}(t) \le 0$ for $t \ge 0$.*
2. *It holds that $a_{2j}(t)^2 + a_{1j}(t)a_3 \ge 0$ for $t \ge 0$.*

Proof. We have

$$a_{2j}(t) = \beta l_j(t) - (p_j + q_j) \le \beta l_j(0) - (p_j + q_j),$$

whence $a_{2j}(t) \le 0$ if $\beta \ge (p_j + q_j)/l_j(0)$ or

$$\alpha \le \frac{(p_j + q_j)/l_j(0)}{[(p_j + q_j)/l_j(0)] - 1} = \frac{p_j + q_j}{p_j + q_j - l_j(0)}.$$

However, $0 < (p_j + q_j)/[p_j + q_j - l_j(0)] < 1$, whence

$$\frac{p_j + q_j}{p_j + q_j - l_j(0)} > \left(\frac{p_j + q_j}{p_j + q_j - l_j(0)}\right)^2.$$

Thus the first assertion follows.

We have

$$\begin{aligned}a_{2j}(t)^2 + a_{1j}(t)a_3 &= \beta l_j(t)^2 - 2(p_j + q_j)l_j(t)\beta + (p_j + q_j)^2\\ &\ge \beta l_j(0)^2 - 2(p_j + q_j)l_j(0)\beta + (p_j + q_j)^2\\ &= \beta[\{(p_j + q_j) - l_j(0)\}^2 - (p_j + q_j)^2] + (p_j + q_j)^2,\end{aligned}$$

whence $a_{2j}(t)^2 + a_{1j}(t)a_3 \ge 0$ if

$$\beta \ge \frac{(p_j + q_j)^2}{(p_j + q_j)^2 - [(p_j + q_j) - l_j(0)]^2}.$$

However, this is equivalent to $\alpha \le [(p_j + q_j)/\{p_j + q_j - l_j(0)\}]^2$. Thus the second assertion follows.

Lemma 5. *Let $j \in \{1, \dots, n\}$.*

1. *We assume* $-q_j < p_j < 0$ *and* $-\infty < \alpha < 0$. *Then (16) has a unique nonnegative solution* $R_j(t) \equiv R_j(t;T)$.

2. *We assume* $-q_j < p_j < 0$ *and (24). Then (16) has a unique solution* $R_j(t) \equiv R_j(t;T)$ *such that* $R_j(t) \geq R_j^*(t)$ *for* $t \in [0,T]$, *where*

$$R_j^*(t) := \frac{a_{2j}(t) + \sqrt{a_{2j}(t)^2 + a_{1j}(t)a_3}}{a_{1j}(t)}.$$

Proof. The first assertion follows in the same way as in the proof of [12], Lemma 2.1 (ii). Thus we prove the second assertion.

Notice that $R_j^*(t)$ is the larger solution to the quadratic equation $a_{1j}(t)x^2 - 2a_{2j}(t)x - a_3 = 0$. Thus

$$a_{1j}(t)R_j^*(t)^2 - 2a_{2j}(t)R_j^*(t) - a_3 = 0. \tag{25}$$

Since $a_{1j}(t) > 0$, $a_{2j}(t) \leq 0$ and $a_3 < 0$, we see that $R_j^*(t) \leq 0$. The equation for $V(t) := R_j(t) - R_j^*(t)$ becomes

$$\dot{V}(t) - a_{1j}(t)V(t)^2 + 2[a_{2j}(t) - a_{1j}(t)R_j^*(t)]V(t) + \dot{R}_j^*(t) = 0. \tag{26}$$

By differenciating (25), we get

$$\dot{a}_{1j}(t)R_j^*(t)^2 + 2a_{1j}(t)R_j^*(t)\dot{R}_j^*(t) - 2\dot{a}_{2j}R_j^*(t) - 2a_{2j}(t)\dot{R}_j^*(t) = 0,$$

whence

$$\dot{R}_j^*(t) = \frac{2\dot{a}_{2j}(t)R_j^*(t) - \dot{a}_{1j}(t)R^*(t)^2}{2\sqrt{a_{2j}(t)^2 + a_{1j}(t)a_3}}.$$

Now

$$2\dot{a}_{2j}(t)R_j^*(t) - \dot{a}_{1j}R_j^*(t)^2 = -2\dot{l}_j(t)R_j^*(t)\{l_j(t)R_j^*(t) - \beta\}.$$

Since

$$a_{2j}(t)^2 + a_{1j}(t)a_3 = \beta l_j(t)^2 - 2(p_j + q_j)l_j(t)\beta + (p_j + q_j)^2 < (p_j + q_j)^2,$$

we see that

$$l_j(t)R_j^*(t) - \beta = \frac{1}{l_j(t)}\left[-(p_j + q_j) + \sqrt{a_{2j}(t)^2 + a_{1j}(t)a_3}\right] > 0.$$

Thus $\dot{R}_j^*(t) \geq 0$. This and $a_{1j}(t) > 0$ imply that (26) has a unique nonnegative solution. The second assertion follows from this.

We define

$$\alpha^* := \min(\alpha_1^*, \ldots, \alpha_n^*)$$

with

$$\alpha_j^* := \begin{cases} \left(\frac{p_j+q_j}{p_j+q_j-l_j(0)}\right)^2 & \text{if} \quad -q_j < p_j < 0, \\ 1 \quad \text{if} \quad 0 \leq p_j < \infty. \end{cases}$$

Notice that $\alpha^* \in (0,1]$. Recall α_* from (22).

Taking the solution $R_j(t) \equiv R_j(t;T)$ of (16) in the sense of Lemma 1 or 5 and running through the same arguments as those in [12], Sections 2 and 3, we obtain the following extensions to Theorems 2 and 3.

Theorem 6. *We assume (2) and $-\infty < \alpha < \alpha^*$, $\alpha \neq 0$. Then the same conclusions as those of Theorem 2 hold.*

Theorem 7. *We assume (2), (19), (20) and $\alpha_* < \alpha < \alpha^*$, $\alpha \neq 0$. Then the same conclusions as those of Theorem 3 hold.*

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