Theory of Stochastic Processes
Vol.13 (29), no.1-2, 2007, pp.35-43

ANNA BONDARENKO

A LIMIT THEOREM FOR SEMI-MARKOV PROCESS

A limit theorem for the strongly regular semi-Markov process is proved under conditions C1 – C3.

1. Introduction

This article deals with the asymptotic behavior of the strongly regular semi-Markov process $\xi(t)$ as $t \to \infty$. It may be considered as continuation of the article [1] motivated by the book by A. N. Korlat, V. N. Kuznetsov, M. M. Novikov and A. F. Turbin (1991). Let us introduce basic notations and necessary results from [1], [2].

Let $\xi(t)$ be a strongly regular semi-Markov process with the phase space $\{X, \mathcal{B}\}$ and semi-Markov kernel $Q(t, x, B),\ t \geq 0,\ x \in X,\ B \in \mathcal{B}$. Let $H(t, x, B),\ t \geq 0,\ x \in X,\ B \in \mathcal{B}$ be the Markov renewal function of $\xi(t)$. Define $\mathcal{D}(X)$ as Banach space of $\mathcal{B}$ - measurable bounded functions with values in $\mathbb{R}$ with the norm $\|f\| = \sup_{x \in X} |f(x)|$. Consider two operator family $Q(t)$ and $H(t), t \geq 0$, in $\mathcal{D}(X)$, defined for all $f \in \mathcal{D}(X)$:

$$[Q(t)f](x) = \int_X Q(t, x, dy) f(y),$$

$$[H(t)f](x) = \int_X H(t, x, dy) f(y).$$

Suppose that $\xi(t)$ satisfies the following conditions:

C1. Markov chain $\xi_n,\ n \geq 0$, embedded in the $\xi(t)$, is uniformly recurring;

C2. $\|M_l\| < \infty$ for $l = \overline{1, k+2},\ \ k \geq 1$, where $M_l = \int\limits_0^\infty t^l Q(dt)$;

C3. Semi-Markov kernel of the process $\xi(t)$ is absolutely continuous in t:

$$Q(t, x, B) = \int_0^t q(s, x, B) ds, \ \ t \geq 0,\ x \in X,\ B \in \mathcal{B},$$

2000 *Mathematics Subject Classifications*: 60K15, 60K20

Key words and phrases. Strongly regular semi-Markov process, embedded Markov chain, uniformly recurring, Markov renewal theorem.

or in the operator form:

$$Q(t) = \int_0^t q(s)ds,\ t \geq 0.$$

Condition C3 guarantees existence of the density of the Markov renewal function $h(t, x, B)$:

$$H(t, x, B) = I_B(x) + \int_0^t h(s, x, B)ds,\ \ t \geq 0,\ x \in X,\ B \in \mathcal{B},$$

or in the operator form

$$H(t) = \mathbb{I} + \int_0^t h(s)ds,\ \ t \geq 0,$$

where $\mathbb{I}$ is the identity operator, $I_B(x)$ is the indicator function.

Let Π_0 be the stationary projector of the embedded Markov chain ξ_n defined under condition C1 as follows:

$$[\Pi_0 f](x) = \int_X \rho(dy) f(y) \mathbb{I}(x),\ \forall f \in \mathcal{D}(X)$$

where $\rho(x)$ is the stationary distribution of the Markov chain ξ_n, $\mathbb{I}(x) \equiv 1$ $\forall x \in X$. Denote

$$h_*(t) = h(t) - \frac{1}{\widehat{m}_1} \Pi_0, \tag{1}$$

where

$$\widehat{m}_1 = \int_X \rho(dx) m_1(x),\ \ m_1(x) = \int_0^\infty t Q(dt, x, X).$$

Let $T_n,\ n = \overline{0, k}$ be bounded operators in $\mathcal{D}(X)$, introduced in the book [2, p. 1.4], and let $P = Q(\infty)$ be the operator of transient probabilities of Markov chain ξ_n. The following result was proved for $n = 0$ in [2] and for $n = \overline{1, k}$ in [1]:

Theorem 1. *Let a strongly regular semi-Markov process satisfies conditions C1 – C3. Then there exists the limit*

$$U_n = \lim_{p \to 0} \frac{(-1)^n}{n!} \int_0^\infty e^{-pt} t^n h_*(t) dt, \quad n = \overline{0, k} \tag{2}$$

and the following relations hold:

$$U_n = \sum_{r=0}^{n} \frac{(-1)^r}{(r)!} M_r U_{n-r} + \frac{(-1)^n}{n!} M_n + \frac{(-1)^{n+1}}{(n+1)!\widehat{m}_1} M_{n+1} \Pi_0, \quad n = \overline{0,k}, \quad (3)$$

$$U_n = \begin{cases} T_0 - I, & \text{for} \quad n = 0; \\ T_n, & \text{for} \quad n = \overline{1,k}, \end{cases} \quad (4)$$

where $M_0 = P$.

2. Basic results.

In this paper we present a theorem, which is proved by means of the above mentioned results and the Markov renewal theorem.

Let's introduce a family of operators

$$U_0(t) = \int_0^t h_*(s)ds, \qquad U_n(t) = \int_0^t (U_{n-1}(s) - U_{n-1})ds, \quad t \geq 0, \quad n = \overline{1,k}. \quad (5)$$

The following result holds true:

Theorem 2. *Let a strongly regular semi-Markov process satisfies conditions C1 – C3. Then there exists the limit*

$$\lim_{t\to\infty} U_n(t) = U_n, \quad t \geq 0, \quad n = \overline{0,k}. \quad (6)$$

Proof. **1.** Consider the case $n = 0$. Under condition C3 the operator renewal equation holds true [3]:

$$h(t) = q(t) + \int_0^t q(s)h(t-s)ds.$$

Hence, subject to (1)

$$h_*(t) = q(t) - \frac{1}{\widehat{m}_1}(I - Q(t))\Pi_0 + \int_0^t q(s)h_*(t-s)ds. \quad (7)$$

Taking integral of (7) and using the Fubbini theorem [4] we will get

$$\int_0^t h_*(s)ds = Q(t) - \frac{1}{\widehat{m}_1} \int_0^t (I - Q(s))ds\,\Pi_0 + \int_0^t ds\,q(s) \int_0^{t-s} h_*(l)dl,$$

or

$$U_0(t) = Q(t) - \frac{1}{\widehat{m}_1}\int_0^t (I - Q(s))ds\,\Pi_0 + \int_0^t q(s)U_0(t-s)ds. \qquad (8)$$

In the case $n = 0$ from (3) we get

$$U_0 = P + PU_0 - \frac{1}{\widehat{m}_1}M_1\Pi_0. \qquad (9)$$

Taking into account the property of stationary projector Π_0 :

$$P\Pi_0 = \Pi_0 = \Pi_0 P, \qquad (10)$$

consider the difference between (8) and (9):

$$U_0(t) - U_0 = V_0(t) + \int_0^t q(s)(U_0(t-s) - U_0)ds, \qquad (11)$$

where

$$V_0(t) = \int_t^\infty (P - Q(s))ds\frac{\Pi_0}{\widehat{m}_1} + Q(t) - P - (P - Q(t))U_0. \qquad (12)$$

Lemma 1. *Let conditions of Theorem 2 be satisfied. Then there exists the limit*

$$\lim_{t\to\infty}(U_0(t) - U_0) = \frac{\Pi_0}{\widehat{m}_1}\int_0^\infty V_0(s)ds. \qquad (13)$$

Proof. To prove the operator equation (13) it is sufficient to verify it for functions $I_B(x),\ x \in X,\ B \in \mathcal{B}$, generating $D(X)$. Define $V_0(t,x,B)$, $U_0(t,x,B)$, $U_0(x,B)$ as action of operators $V_0(t),\ U_0(t),\ U_0$ on function $I_B(x)$. Consider positive and negative parts of the function $V_0(t,x,B)$:

$$V_0^1(t,x,B) := max\{V_0(t,x,B),0\}, \quad V_0^2(t,x,B) := -min\{V_0(t,x,B),0\}.$$

Similarly $U_0^1(x,B)$ and $U_0^2(x,B)$ are defined as positive and negative parts of function $U_0(x,B)$. From (12) it follows that for $t \geq 0,\ x \in X,\ B \in \mathcal{B}$

$$V_0^1(t,x,B) = \frac{\rho(B)}{\widehat{m}_1}\int_t^\infty dt_0\int_{t_0}^\infty q(s,x,X)ds + \int_t^\infty ds\int_X q(s,x,dy)U_0^2(y,B),$$

$$V_0^2(t,x,B) = \int_t^\infty q(s,x,B)ds + \int_t^\infty ds\int_X q(s,x,dy)U_0^1(y,B).$$

Functions $V_0^1(t,x,B)$ and $V_0^2(t,x,B)$ are bounded. It follows from condition C2 for $l = 1$ and boundedness of the operator $T_0 = U_0$. Besides, for any $x \in X,\ B \in \mathcal{B}$ functions $V_0^1(t,x,B)$, $V_0^2(t,x,B)$ are non-negative, monotone decreasing and integrable in t functions on $[0,\infty)$. Thus for any $B \in \mathcal{B}$ they are directly Riemann integrable [5], so that $\int\limits_X \rho(dx) \int\limits_0^\infty dt V_0^j(t,x,B) < \infty,\ j = 1,2$. So for a fixed $B \in \mathcal{B}$ the above point and conditions C1 – C3 give a possibility to apply the Markov renewal theorem ([5, p. 107], [6, p. 31]) to the following Markov renewal equation:

$$Z^j(t,x,B) = V_0^j(t,x,B) + \int\limits_0^t ds \int\limits_X q(s,x,dy) Z^j(t-s,y,B),\ j = 1,2. \quad (14)$$

By the Markov renewal theorem there exists

$$\lim_{t\to\infty} Z^j(t,x,B) = \frac{1}{\widehat{m}_1} \int\limits_X \rho(dx) \int\limits_0^\infty dt V_0^j(t,x,B),\ x \in X,\ B \in \mathcal{B}. \quad (15)$$

As by definition $V_0^1(t,x,B) - V_0^2(t,x,B) = V_0(t,x,B)$, then from (11) and (14) it follows that $U_0(t,x,B) - U_0(x,B) = Z^1(t,x,B) - Z^2(t,x,B)$. Hence, from (15) follows statement of the lemma. □

Since

$$\int_0^\infty V_0(s)ds = -M_1 - M_1 U_0 + \frac{M_2 \Pi_0}{2\widehat{m}_1},$$

then from (3) for $n = 1$ and Lemma 1 we get

$$\lim_{t\to\infty} (U_0(t) - U_0) = \frac{\Pi_0}{\widehat{m}_1}(I - P)U_1 = 0.$$

Theorem 2 for $n = 0$ is proved.

2. Consider the case $n = \overline{1,k}$. Define

$$E_0(t) = \int\limits_t^\infty dt_0 \int\limits_{t_0}^\infty q(s)ds, \quad E_1(t) = \int\limits_t^\infty dt_1 \int\limits_{t_1}^\infty dt_0 \int\limits_{t_0}^\infty q(s)ds,$$

$$E_n(t) = \int\limits_t^\infty dt_n \int\limits_{t_n}^\infty dt_{n-1} ... \int\limits_{t_1}^\infty dt_0 \int\limits_{t_0}^\infty q(s)ds,\ n = \overline{2,k}.$$

It is easy to see that for $t = 0$ the following equalities hold true:

$$E_0(0) = \int\limits_0^\infty (P - Q(t))dt = M_1,$$

$$E_n(0) = \int\limits_0^\infty E_{n-1}(t)dt = \frac{M_{n+1}}{(n+1)!},\ n = \overline{1, k+1.} \qquad (16)$$

Transform (3) to the form

$$U_n = \sum_{r=0}^{n-1} (-1)^{(r+1)} E_r(0) U_{n-r-1} +$$

$$+(-1)^{(n+1)} E_n(0) \frac{\Pi_0}{\widehat{m}_1} + PU_n + (-1)^n E_{n-1}(0). \qquad (17)$$

Lemma 2. *Let conditions of Theorem 2 be satisfied. Then the following relations hold true:*

$$U_n(t) - U_n = V_n(t) + \int\limits_0^t q(s)(U_n(t-s) - U_n)ds, \quad t \geq 0, \quad n = \overline{1, k,} \quad (18)$$

where

$$V_n(t) = \sum_{r=0}^{n-1} (-1)^r E_r(t) U_{n-r-1} +$$

$$+(-1)^n E_n(t) \frac{\Pi_0}{\widehat{m}_1} + (-1)^{n-1} E_{n-1}(t) - \int\limits_t^\infty q(s)ds U_n. \qquad (19)$$

Proof. The lemma is proved by means of mathematical induction method. From (5) and (11) we have

$$U_1(t) = -E_0(t)U_0 + E_1(t)\frac{\Pi_0}{\widehat{m}_1} - E_0(t) + \int\limits_0^t q(s)U_1(t-s)ds. \qquad (20)$$

From (17) for $n = 1$ and (20) we obtain statement of the lemma for the case $n = 1$. So we have the base of induction. Suppose that statement of the lemma is true for some $n,\ n = \overline{1, k-1}$ and show that it is also true for $n+1$. Indeed, let us integrate (18) and apply the Fubbini theorem. We get

$$U_{n+1}(t) = \int\limits_0^t V_n(s)ds + \int\limits_0^t ds\, q(s) \int\limits_0^{t-s} (U_n(l) - U_n)dl \pm \int\limits_0^\infty V_n(s)ds,$$

or

$$U_{n+1}(t) = -\int\limits_t^\infty V_n(s)ds + \int\limits_0^\infty V_n(s)ds + \int\limits_0^t q(s)U_{n+1}(t-s)ds. \qquad (21)$$

Since by definition $\int\limits_t^\infty E_n(s)ds = E_{n+1}(t)$, we have

$$\int\limits_t^\infty V_n(s)ds = \sum_{r=0}^{n-1}(-1)^r E_{r+1}(t)U_{n-r-1}+$$

$$+(-1)^n E_{n+1}(t)\frac{\Pi_0}{\widehat{m}_1} + (-1)^{n-1}E_n(t) - E_0(t)U_n =$$

$$= \sum_{r=0}^{n}(-1)^{(r+1)}E_r(t)U_{n-r} + (-1)^n E_{n+1}(t)\frac{\Pi_0}{\widehat{m}_1} + (-1)^{n-1}E_n(t). \qquad (22)$$

It follows from (22) for $t = 0$ and (17) that

$$\int\limits_0^\infty V_n(s)ds = U_{n+1} - PU_{n+1}. \qquad (23)$$

So if relation (18) is true for some n, $n = \overline{1, k-1}$, then, as follows from (21), (22) and (23), it is also true for $n+1$. □

Prove the next lemma in the way similar to one of Lemma 1.

Lemma 3. *Let conditions of Theorem 2 are satisfied. Then there exists the limit*

$$\lim_{t\to\infty}(U_n(t) - U_n) = \frac{\Pi_0}{\widehat{m}_1}\int\limits_0^\infty V_n(s)ds, \quad n = \overline{1,k}. \qquad (24)$$

Proof. To prove the operator equation (24) it is sufficient to verify it for the indicator functions $I_B(x)$, $x \in X$, $B \in \mathcal{B}$, generating $D(X)$. Define $V_n(t,x,B)$ and $U_r(x,B)$, $r = \overline{0,n}$ as action of operators $V_n(t)$, U_r on function $I_B(x)$. From (19) it follows

$$V_n(t,x,B) = (-1)^n\frac{\rho(B)}{\widehat{m}_1}\int\limits_t^\infty dt_n \int\limits_{t_n}^\infty dt_{n-1}... \int\limits_{t_1}^\infty dt_0 \int\limits_{t_0}^\infty q(s,x,X)ds+ \qquad (25)$$

$$+\sum_{r=0}^{n-1}(-1)^r\int\limits_t^\infty dt_r\int\limits_{t_r}^\infty dt_{r-1}...\int\limits_{t_1}^\infty dt_0\int\limits_{t_0}^\infty ds\int\limits_X q(s,x,dy)U_{n-r-1}(y,B)+$$

$$+(-1)^{n-1}\int\limits_t^\infty dt_{n-1}...\int\limits_{t_1}^\infty dt_0\int\limits_{t_0}^\infty q(s,x,B)ds - \int\limits_t^\infty ds\int\limits_X q(s,x,dy)U_n(y,B).$$

Consider positive and negative parts of the function $V_n(t,x,B)$:

$$V_n^1(t,x,B) := max\{V_n(t,x,B),0\},\ V_n^2(t,x,B) := -min\{V_n(t,x,B),0\}.$$

Represent functions $U_r(x,B), r=\overline{0,n}$ in (25) as $U_r^1(x,B)-U_r^2(x,B)$ where $U_r^1(x,B)$ and $U_r^2(x,B)$ are its positive and negative parts. Then from (25) it follows that $V_n(t,x,B)$ is a sum of functions of constant signs. It is easy to see from (25) the structure of functions V_n^+ and V_n^- and make a conclusion, that for any fixed $x \in X$, $B \in \mathcal{B}$ functions $V_0^+(t,x,B)$, $V_0^-(t,x,B)$ are non-negative, monotone decreasing and integrable in t functions on $[0,\infty)$. Boundedness of this functions follows from the condition C2, (4) and boundedness of operators T_i, $i=\overline{0,n}$. Thus V_n^1 and V_n^2 are directly Riemann integrable, so that $\int\limits_X \rho(dx)\int\limits_0^\infty dtV_n^j(t,x,B)<\infty$, $j=1,2$. Hence the above point and conditions C1 – C3 give a possibility to apply the Markov renewal theorem to the next equation:

$$Z^j(t,x,B) = V_n^j(t,x,B) + \int\limits_0^t ds \int\limits_X q(s,x,dy)Z^j(t-s,y,B),\ j=1,2. \quad (26)$$

By the Markov renewal theorem there exists

$$\lim_{t\to\infty} Z^j(t,x,B) = \frac{1}{\widehat{m}_1}\int\limits_X \rho(dx)\int\limits_0^\infty dtV_n^j(t,x,B), \quad x\in X,\ B\in\mathcal{B}. \quad (27)$$

Note that by definition $V_n^1(t,x,B) - V_n^2(t,x,B) = V_n(t,x,B)$. So from (26) and Lemma 2 it follows that $U_n(t,x,B) - U_n(x,B) = Z^1(t,x,B) - Z^2(t,x,B)$. From (27) statement of the lemma follows. □

From (23) and (10) it follows that

$$\Pi_0\int\limits_0^\infty V_n(s)ds = \Pi_0(U_{n+1} - PU_{n+1}) = 0.$$

Using Lemma 3, we get statement of the theorem for $n=\overline{1,k}$. □

Conclusion.

In Theorem 2 the asymptotic equality (6) is proved for a strongly regular semi-Markov process which satisfies conditions C1 – C3. In the case $n=0$ this asymptotic equality follows from results of [2] but under two additional conditions. Note, that (6) is more weak result than existence of $\int_0^\infty t^n h_*(t)dt$, $n = \overline{0,k}$. Indeed, if such integral exists, then

$$U_n = \frac{(-1)^n}{n!} \int\limits_0^\infty t^n h_*(t)dt, \quad n = \overline{0,k},$$

and according to formula of integration by parts, we get

$$\int\limits_0^\infty t^n h_*(t)dt = n! \int\limits_0^\infty dt_n \int\limits_{t_n}^\infty dt_{n-1} ... \int\limits_{t_2}^\infty dt_1 \int\limits_{t_1}^\infty h_*(t)dt, \quad n = \overline{1,k}.$$

from which asymptotic equality (6) follows. However, as far as the author knows, at the present moment existence of $\int_0^\infty t^n h_*(t)dt$ for the general semi-Markov process is not proved. It is known that such integral is convergent for the renewal process under conditions that are a particular case of the conditions C1-C3 for the renewal process ([7], [8]).

Bibliography

1. Bondarenko H. I., *On some consequences of the equation for the Markov renewal function of a semi-Markov process,* Ukrain. Mat. Zh., **56**, No. 12, (2004), 1684 – 1690.
2. Korlat A. N., Kuznetsov V. N., Novikov M. M. and Turbin, A. F., *Semi-Markovm models of renewal and queuing systems,* Shtiintsa, Kishinev, (1991).
3. Korolyuk V. S. and Turbin A. F., *Mathematical foundations for phase consolidation of complex systems,* Naukova Dumka, Kyiv, (1978).
4. Kolmogorov A. N., Fomin S. V., *Elements of function theory and functional analysis*, Nauka, Moscow, (1972).
5. Shurenkov V. M., *Ergodic Markov processes,* Nauka, Moscow, (1989).
6. Korolyuk V. S., Swishchuk V. V., *Evolution stochastic systems. Algorithms of averaging and diffusive approximation,* Ins. of mat. of Ukraine, Kyiv, (2000).
7. Stone C. J., *On characteristic functions and renewal theory,* Trans. Amer. Math. Soc., **120**, No. 2., (1965), 327 – 347.
8. Sevastianov B. A., *Renewal type equations and moment coefficients of branching processes,* Mat. Notes., **3**, No. 1., (1968), 3 - 14.

Department of Probability Theory and Mathematical Statistics, Kyiv National Taras Shevchenko University, Kyiv, Ukraine
E-mail address: gannucia@ukr.net