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TOPOLOGICAL, METRIC AND FRACTAL PROPERTIES OF PROBABILITY DISTRIBUTIONS ON THE SET OF INCOMPLETE SUMS OF POSITIVE SERIES

We study the structure, topological, metric and fractal properties of the distribution of random incomplete sum of the convergent positive series with independent terms under certain conditions on the rate of convergence of series and on the distributions of its terms. We also study the behaviour of the absolute value of the characteristic function of this random variable at infinity and the fractal dimension preservation by its distribution function.

1. Introduction

Let

$$a_1 + a_2 + \ldots + a_k + \ldots = \sum_{k=1}^{\infty} a_k = r \tag{1}$$

be a convergent positive series with $a_{k+1} < a_k$ for any $k \in N$, let $S_m = \sum\limits_{n=1}^{m} a_n$ be a sequence of its partial sums, and let $r_m = \sum\limits_{n=m+1}^{\infty} a_n$ be a sequence of its remainders, $m = 1, 2, \ldots$.

Expression $\sum\limits_{n\in M} a_n$, where M is a finite or an infinite subset of set N of positive integers, is called the *subseries of series (1)*. Sum of any subseries of series (1) is called the *incomplete sum of series (1)*. In another words, the sum of any series

$$\sum_{n=1}^{\infty} \varepsilon_n a_n, \quad \text{where} \quad \varepsilon_n \in \{0, 1\},$$

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is called the *incomplete sum of series (1).* We denote the set of all incomplete sums of series (1) by A. It is clear that all partial sums S_m and remainders r_m of series (1) belong to set A. Moreover, set A' of incomplete sums of any subseries of series (1) belongs to A.

For example, the set of incomplete sums of the series $\sum\limits_{k=1}^{\infty} 2 \cdot 3^{-k}$ is a classical Cantor set, and the set of incomplete sums of the series $\sum\limits_{k=1}^{\infty} 3 \cdot 4^{-k}$ is a self-similar set of the Cantor type.

The set of incomplete sums of given series is a Lebesgue measurable set. As far as we know, necessary and sufficient conditions for this set to be of zero Lebesgue measure in terms of sequences $\{a_n\}$, $\{r_n\}$ are not known today. Some results and interesting examples are found in the paper [9]. The fractal properties of sets of incomplete sums of some series were studied in [26,20]. However, in general case the problem does not have solution yet.

In this paper we study properties of the distribution of the random variable

$$\xi = \sum_{k=1}^{\infty} \eta_k a_k, \tag{2}$$

where η_k is a sequence of independent random variables taking the values 0 and 1 with probabilities p_{0k} and p_{1k} correspondingly ($p_{ik} \geq 0$, $p_{0k}+p_{1k} = 1$).

Generally speaking, properties of distribution of ξ depend both on series $\sum\limits_{n=1}^{\infty} a_n$ and infinite "matrix" $||p_{ik}||$.

We are interested in the structure (i.e., the content of discrete, singular and absolutely continuous component) of the distribution of ξ, and topological, metric and fractal properties of essential sets of the distribution. We also study the behaviour of the absolute value of the characteristic function $f_\xi(t)$ of the random variable ξ, namely we study $L_\xi = \limsup\limits_{|t|\to\infty} |f_\xi(t)|$. The problem of the Hausdorff-Besicovitch dimension preservation by the distribution function F_ξ of the random variable ξ is also studied [3,21].

According to the Jessen-Wintner Theorem [11] (see also [10,20]) the random variable ξ has a pure distribution, i.e., it is either pure discrete or pure absolutely continuous or pure singular (with respect to the Lebesgue measure). So, the problem about structure of ξ is a problem about type of distribution of ξ. Criterion for the discreteness (and so, for the continuity) of ξ follows from P. Lévy Theorem [12] (see also [10,20]). Further we consider only continuous distributions. Hence, the problem about type of distribution of ξ is a problem about necessary and sufficient conditions for singularity (and so, for absolute continuity) of ξ.

In this paper we understand topological, metric and fractal properties of distribution of the random variable ξ as topological, metric and fractal properties of its spectrum.

Note that the random variable ξ is a generalisation of infinite symmetric Bernoulli convolution [14,20,19,27] that has received attention of many mathematicians from the thirties of last century [22,23,17,18,25]. But even in simplest case $a_n = \lambda^n$, $p_{0n} = \frac{1}{2} = p_{1n}$ the problem about type of distribution (the structure) is not solved yet [17].

Distributions in cases $a_n = 2^{-n}$ (see [24,16,5,27]) and $a_n = c\cdot\lambda^{-n}$, where $\lambda < \frac{1}{2}$ (see [7,20,27]), are studied in details. Note that a generalisation of such distributions were studied in [2,1,9].

The random variable ξ if $a_n \geq r_n$ for any $n \in N$, were studied in [20]. In this paper we continue to study its properties. The case if $a_n < r_n$ for infinitely many n, is more complicated. Let us consider the case $a_n = r_{n+1}$ for any $n \in N$ here.

2. Cylindrical sets and their properties

The set $\Delta'_{c_1\ldots c_m}$ of all incomplete sums

$$\sum_{n=1}^{m} c_n a_n + \sum_{n=m+1}^{\infty} \varepsilon_n a_n, \quad \text{where} \quad \varepsilon_n \in \{0,1\},$$

of series (1) is called the *cylinder of rank m with base $c_1 \ldots c_m$* ($c_i \in \{0,1\}$).

The segment

$$\Delta_{c_1\ldots c_m} = \left[\inf \Delta'_{c_1\ldots c_m},\ \sup \Delta'_{c_1\ldots c_m}\right] = \left[\sum_{n=1}^{m} c_n a_n,\ r_m + \sum_{n=1}^{m} c_n a_n\right]$$

is called the *cylindrical segment of rank m with base $c_1 \ldots c_m$* ($c_i \in \{0,1\}$). We denote the interval with the same ends as in $\Delta_{c_1\ldots c_m}$ by $\nabla_{c_1\ldots c_m}$ and call it the *cylindrical interval of rank m with base $c_1 \ldots c_m$*.

Depending on $\{a_n\}$ and sequence $(c_1, \ldots, c_m)$ it is possible that $\Delta'_{c_1\ldots c_m}$ and $\Delta_{c_1\ldots c_m}$ coincide or not, but always

$$\Delta'_{c_1\ldots c_m} \subset \Delta_{c_1\ldots c_m}.$$

Definitions directly imply the following properties of cylindrical sets:

1. $\inf \Delta_{c_1\ldots c_m} = \inf \Delta'_{c_1\ldots c_m}$, $\sup \Delta_{c_1\ldots c_m} = \sup \Delta'_{c_1\ldots c_m}$.
2. $\Delta_{c_1\ldots c_m} = \Delta_{c_1\ldots c_m 0} \cup \Delta_{c_1\ldots c_m 1}$, $\Delta'_{c_1\ldots c_m} = \Delta'_{c_1\ldots c_m 0} \cup \Delta'_{c_1\ldots c_m 1}$.
3. $\inf \Delta_{c_1\ldots c_m} = \inf \Delta_{c_1\ldots c_m 0} < \inf \Delta_{c_1\ldots c_m 1}$,
   $\sup \Delta_{c_1\ldots c_m} = \sup \Delta_{c_1\ldots c_m 1}$.
4. $|\Delta_{c_1\ldots c_m}| = r_m \to 0$ ($m \to \infty$).

5. $\bigcap\limits_{m=1}^{\infty} \Delta_{c_1\ldots c_m} = \bigcap\limits_{m=1}^{\infty} \Delta'_{c_1\ldots c_m} \equiv \Delta_{c_1\ldots c_m\ldots} = x \in [0, r]$.

   It is easy to see that

6. $\frac{|\Delta_{c_1\ldots c_m c}|}{|\Delta_{c_1\ldots c_m}|} = \frac{r_{m+1}}{a_{m+1}+r_{m+1}} = \frac{1}{\delta_{m+1}+1}$, where $\delta_{m+1} = \frac{a_{m+1}}{r_{m+1}}$.

7. $\Delta_{c_1\ldots c_m} \neq \Delta_{s_1\ldots s_k}$ if $m \neq k$.

8. $\Delta_{c_1\ldots c_m} = \Delta_{s_1\ldots s_k} \quad \Leftrightarrow \quad \begin{cases} m = k, \\ \sum\limits_{n=1}^{m} (c_n - s_n)a_n = 0. \end{cases}$

9. $\Delta_{c_1\ldots c_m 0} \cap \Delta_{c_1\ldots c_m 1} = \begin{cases} [b + a_{m+1}, b + r_{m+1}] & \text{if } a_{m+1} < r_{m+1}; \\ \Delta_{c_1\ldots c_m 10\ldots0\ldots} & \text{if } a_{m+1} = r_{m+1}; \\ \emptyset & \text{if } a_{m+1} > r_{m+1}; \end{cases}$ where $b = \sum\limits_{n=1}^{m} c_n a_n$.

   Corollary. $|\Delta_{c_1\ldots c_m 0} \cap \Delta_{c_1\ldots c_m 1}| = r_{m+1} - a_{m+1}$.

10. The equality $|\Delta_{c_1\ldots c_m 0} \cap \Delta_{c_1\ldots c_m 1}| = \frac{1}{2}|\Delta_{c_1\ldots c_m 0}|$ is equivalent to

$$\delta_{m+1} = \frac{a_{m+1}}{r_{m+1}} = \frac{1}{2}.$$

11. $|\Delta_{c_1\ldots c_m 0} \cap \Delta_{c_1\ldots c_m 1}| < \frac{1}{2}|\Delta_{c_1\ldots c_m 0}| < \frac{1}{2}r_{m+1}$.

12. Let $\inf \Delta_{c_1\ldots c_m} < \inf \Delta_{s_1\ldots s_k}$. Then

$$\Delta_{c_1\ldots c_m} \cap \Delta_{s_1\ldots s_k} = \begin{cases} \left[\sum\limits_{i=1}^{k} s_i a_i, \ \sum\limits_{i=1}^{m} c_i a_i + r_m\right] & \text{if } \sum\limits_{i=1}^{k} s_i a_i \leq \sum\limits_{i=1}^{m} c_i a_i + r_m, \\ \emptyset & \text{if } \sum\limits_{i=1}^{k} s_i a_i > \sum\limits_{i=1}^{m} c_i a_i + r_m. \end{cases}$$

13. Sets $\Delta'_{c_1\ldots c_m}$ and $\Delta'_{(1-c_1)\ldots(1-c_m)}$ are symmetrical with respect to the middle of the segment $[0, r]$, since $x \in \Delta'_{c_1\ldots c_m}$ implies

$$\begin{aligned} x' = r - x &= \sum_{k=1}^{\infty} a_n - \sum_{n=1}^{m} c_n a_n - \sum_{n=m+1}^{\infty} \varepsilon_n a_n = \\ &= \sum_{n=1}^{m} (1 - c_n)a_n + \sum_{n=m+1}^{\infty} (1 - \varepsilon_n)a_n \in \Delta'_{(1-c_1)\ldots(1-c_m)}. \end{aligned}$$

14. Sets $\Delta'_{c_1\ldots c_m 0}$ and $\Delta'_{c_1\ldots c_m 1}$ are symmetrical with respect to the middle of cylindrical segment $\Delta_{c_1\ldots c_m}$.

15. If $\nabla_{c_1\ldots c_m 0} \cap \nabla_{c_1\ldots c_m 1} \neq \emptyset$, i.e., $r_{m+1} > a_{m+1}$, then

$$\Delta_{c_1\ldots c_m 0} \cap \Delta_{c_1\ldots c_m 1} \subset \Delta_{c_1\ldots c_m 01}$$

and

$$\Delta_{c_1\ldots c_m 0} \cap \Delta_{c_1\ldots c_m 1} \subset \Delta_{c_1\ldots c_m 10},$$

hence,

$$\Delta_{c_1\ldots c_m 0} \cap \Delta_{c_1\ldots c_m 1} = \Delta_{c_1\ldots c_m 01} \cap \Delta_{c_1\ldots c_m 10}.$$

16. The following equalities

$$\Delta_{c_1\ldots c_m 0} \cap \Delta_{c_1\ldots c_m 1} = \Delta_{c_1\ldots c_m 011} = \Delta_{c_1\ldots c_m 100}$$

hold.

3. Cylindrical representation of points of the set of incomplete sums of given series

Definitions of cylindrical sets (cylinders and segments), properties 2, 4 and 5 imply that

$$\Delta_{c_1} \supset \Delta_{c_1 c_2} \supset \ldots \supset \Delta_{c_1\ldots c_k} \supset \ldots \quad \text{and} \quad \Delta'_{c_1} \supset \Delta'_{c_1 c_2} \supset \ldots \supset \Delta'_{c_1\ldots c_k} \supset \ldots$$

for any sequence $\{c_k\}$, $c_k \in \{0,1\}$. Moreover, there exists unique number $x \in [0,r]$ such that

$$x = \bigcap_{m=1}^{\infty} \Delta_{c_1\ldots c_m} = \bigcap_{m=1}^{\infty} \Delta'_{c_1\ldots c_m} = \sum_{k=1}^{\infty} c_k a_k. \tag{3}$$

We denote expression (3) symbolically by $x = \Delta_{c_1\ldots c_m\ldots}$ and call the *cylindrical representation of number (point) x.*

Set of all points $x \in [0,r]$ having the cylindrical representation coincides with the set of incomplete sums of series (1).

Directly from the cylindrical representation definition we obtain that *numbers $u = \Delta_{c_1\ldots c_m\ldots}$ and $v = \Delta_{s_1\ldots s_m\ldots}$ coincide if and only if*

$$\sum_{i=1}^{\infty} (c_i - s_i) a_i = 0.$$

Lemma 1 ([20]). *If $a_k \leq r_k$ that is equivalent to $r_k \geq 2r_{k+1}$ then any point of segment $[0,r]$ has not more than two cylindrical representations.*

4. Topological, metric and fractal properties of the set of incomplete sums of the series

The following proposition describes the structure and topological properties of the set of incomplete sums of series (1).

Theorem 1 ([9]). *The set of incomplete sums A has the following properties*

1. it is a perfect set (closed set without isolated points);

2. $A = \bigcup\limits_{(c_1 \dots c_m)} \Delta'_{c_1 \dots c_m}$ for any $m \in N$ and all $\Delta'_{c_1 \dots c_m}$ are isometric;

3. it is a union of segments if inequality $r_j < a_j$ holds only for finitely many j;

4. it is a nowhere dense set otherwise.

If the set of incomplete sums of series (1) is of zero Lebesgue measure, then the Hausdorff measure and the Hausdorff-Besicovitch dimension give us more detailed information about its "massivity". Let us recall these notions.

Let E be a bounded subset of the space R^1. The *α-dimensional Hausdorff measure* of E is defined as follows

$$H_\alpha(E) = \lim_{\varepsilon \to 0} \left[\inf_{|u_i| \le \varepsilon} \left\{ \sum_i |u_i|^\alpha : \quad \bigcup_i u_i \supset E \right\} \right],$$

where the infimum is taken over all coverings $\{u_i\}$ of the set E by segments u_i with $|u_i| \le \varepsilon$, where $|u_i|$ is a diameter of u_i. Generally speaking, the measure $H_\alpha(E)$ may be equal to zero, infinity or positive integer. The number

$$\alpha_0(E) = \sup \{\alpha : \quad H_\alpha(E) \ne 0\} = \inf \{\alpha : \quad H_\alpha(E) = 0\}$$

is called the *Hausdorff-Besicovitch dimension* of the set E. This notion characterise the massivity of a set and "compactness" of its points, since it has the following properties: 1. If $E_1 \subset E_2$, then $\alpha_0(E_1) \le \alpha_0(E_2)$. 2. $\alpha_0 \left(\bigcup\limits_i E_i \right) = \sup\limits_i \alpha_0(E_i)$.

Theorem 2 ([20]). *If series (1) satisfies the condition $a_k \le r_k$ for any $k \in N$ (it is equivalent to $\delta_k = \frac{a_k}{r_k} \ge 1$), then the Hausdorff-Besicovitch dimension of the set of its incomplete sums is equal to*

$$\alpha_0(A) = \left[\varlimsup_{k \to \infty} \left(\frac{1}{k} \sum_{i=1}^{k} \log_2(\delta_i + 1) \right) \right]^{-1}. \tag{4}$$

Corollary 1. *If conditions of Theorem 2 hold and* $\lim_{k\to\infty}\delta_k=\delta$, *the the Hausdorff-Besicovitch dimension of the set of incomplete sums of series (1) is equal to*

$$\alpha_0(A)=\log_2^{-1}(\delta+1).$$

Corollary 2. *If conditions of Theorem 2 hold and* $\lim_{k\to\infty}\delta_k=1$, *then* $\alpha_0(A)=1$.

Corollary 3. *If conditions of Theorem 2 hold and* $\lim_{k\to\infty}\delta_k=\infty$, *then* $\alpha_0(A)=0$.

5. About one special series

There exists the unique positive series $a_1+a_2+\ldots+a_n+\ldots=1$ such that $a_n=r_n$ for any $n\in N$. This is the series $\sum_{k=1}^{\infty}2^{-k}$. The set of its incomplete sums is $[0,1]$. Any irrational point from $[0,1]$ has the unique cylindrical representation corresponding to this series, and some rational numbers have two cylindrical representations.

Let us consider the series with $a_n=r_{n+1}$ for any positive integer n.

Lemma 2. *If series* $a_1+a_2+\ldots+a_n+\ldots=1$ *has property*

$$a_n=r_{n+1}\quad\Leftrightarrow\quad(a_{n+2}=a_n-a_{n+1}) \tag{5}$$

for any $n\in N$, *then*

$$a_n=(-1)^{n-1}(u_na_1-u_{n-2})\quad\textit{for}\quad n\geq 2, \tag{6}$$

where $\{u_n\}$ *is a classical Fibonacci sequence, i.e.,*

$$u_0=1,\quad u_1=1,\quad u_{n+1}=u_n+u_{n-1}.$$

Proof. 1. Since

$$\begin{cases} a_1+a_2+r_2=1,\\ r_2=a_1,\end{cases}$$

we have

$$\begin{cases} a_2=1-2a_1,\\ s_2=a_1+a_2=1-a_1.\end{cases}$$

Analogously, since

$$\begin{cases} a_1+a_2+a_3+r_3=1,\\ r_3=a_2,\end{cases}$$

we have

$$\begin{cases} a_3 = 1 - s_2 - a_2 = a_1 - a_2 = 3a_1 - 1, \\ s_3 = s_2 + a_3 = 2a_1. \end{cases}$$

So, equality (6) holds for $n = 2$ and 3.

2. Suppose that (6) holds for $n = k$.

3. Prove that 1 and 2 implies that equality (6) holds for $n = k + 1$. From equality $a_n = r_{n+1}$ we obtain

$$a_n = a_{n+2} + r_{n+2}, \quad a_n = a_{n+2} + a_{n+1}.$$

Hence, $a_{n+2} = a_n - a_{n+1}$. This implies

$$s_n = a_1 + (1 - 2a_1) + (a_1 - a_2) + \ldots + (a_{n-2} - a_{n-1}) = 1 - a_{n-1}.$$

From $s_n + a_{n+1} + r_{n+1} = 1$ and $r_{n+1} = a_n$ it follows that

$$\begin{aligned} a_{n+1} &= 1 - s_n - a_n = 1 - (1 - a_{n-1}) - (-1)^{n-1}(u_n a_1 - u_{n-2}) = \\ &= (-1)^n[(u_n + u_{n-1})a_1 - (u_{n-2} + u_{n-3})] = \\ &= (-1)^n(u_{n+1}a_1 - u_{n-1}). \end{aligned}$$

By induction, equality (6) holds for any positive integer n.

Lemma 3. *Let $\{u_n\}$ be a classical Fibonacci sequence. Then sequence $x_m = \frac{u_{2m-1}}{u_{2m+1}}$ is increasing and sequence $y_m = \frac{u_{2m}}{u_{2m+2}}$ is decreasing. Moreover,*

$$\lim_{m\to\infty} x_m = \lim_{m\to\infty} y_m = \frac{1}{\varphi^2}, \quad \textit{where} \quad \varphi = \frac{1+\sqrt{5}}{2}.$$

Proof. It is known that $u_{2m+1}^2 - u_{2m-1}u_{2m+3} = 1$. Hence,

$$x_{m+1} - x_m = \frac{u_{2m+1}}{u_{2m+3}} - \frac{u_{2m-1}}{u_{2m+1}} = \frac{1}{u_{2m+3}u_{2m+1}} > 0,$$

and sequence $\{x_m\}$ is increasing.

Taking into account that $u_{2m+2}^2 - u_{2m}u_{2m+4} = -1$, we have

$$y_{m+1} - y_m = \frac{u_{2m+2}}{u_{2m+4}} - \frac{u_{2m}}{u_{2m+2}} = \frac{-1}{u_{2m+4}u_{2m+2}} < 0.$$

So, sequence $\{y_m\}$ is decreasing.

It is known that

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \frac{1}{\varphi}.$$

Therefore,

$$\lim_{m\to\infty} x_m = \lim_{m\to\infty} y_m = \lim_{n\to\infty} \left(\frac{u_n}{u_{n+1}} \cdot \frac{u_{n+1}}{u_{n+2}} \right) = \frac{1}{\varphi^2},$$

which proves the Lemma.

Theorem 3. *There exists the unique positive series with property (5). It has* $a_1 = \frac{1}{\varphi^2}$ *and*

$$a_n = r_{n+1} = (-1)^{n-1} \left(\frac{u_n}{\varphi^2} - u_{n-2} \right) = -\beta a_{n-1} = (-1)^{n-1} \frac{\beta^{n-1}}{\varphi^2}, \tag{7}$$

where $\varphi = \frac{1+\sqrt{5}}{2}$, $\beta = \frac{1-\sqrt{5}}{2}$, *and* $\{u_n\}$ *is a classical Fibonacci sequence.*

Proof. If series has property (5), then according to Lemma 2 the n-th term of the series has a form (6). Then condition $a_n > 0$ is equivalent to the following conditions

$$\begin{cases} a_{2m+1} = u_{2m+1} a_1 - u_{2m-1} > 0, \\ a_{2m} = u_{2m-2} - u_{2m} a_1 > 0, \end{cases}$$

i.e.,

$$\frac{u_{2m-1}}{u_{2m+1}} < a_1 < \frac{u_{2m-2}}{u_{2m}}.$$

According to Lemma 3, there exists unique a_1 such that the latter double inequality holds for any positive integer m, namely $a_1 = \frac{1}{\varphi^2}$.

By using the known Binet formula

$$u_n = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}},$$

one can to express the ratio

$$\frac{a_{n+1}}{a_n} = -\frac{u_{n+1} - \varphi^2 u_{n-1}}{u_n - \varphi^2 u_{n-2}} = -\beta = \frac{\sqrt{5}-1}{2}.$$

Hence, equality (7) holds.

6. Type and properties of distribution of random variable ξ

As it is known, the random variable ξ has a pure distribution. The following statement follows from P. Lévy Theorem [12].

Theorem 4. *The random variable ξ has a discrete distribution if and only if*

$$M=\prod_{k=1}^{\infty}\max\{p_{0k},p_{1k}\}>0.$$

Corollary. *The random variable ξ has a continuous distribution if and only if $M=0$.*

The *spectrum* S_ζ of the distribution of random variable ζ is the minimal closed support of ζ, i.e.,

$$\begin{aligned} S_\zeta &= \{x: P\{\zeta\in(x-\varepsilon,x+\varepsilon)\}>0\ \forall\,\varepsilon>0\}= \\ &= \{x: F_\zeta(x+\varepsilon)-F_\zeta(x-\varepsilon)>0\ \forall\,\varepsilon>0\}. \end{aligned}$$

It is easy to prove the following proposition.

Lemma 4. *The spectrum of ξ is the set*

$$S_\xi=\{x:\ x=\Delta_{c_1c_2\ldots c_k\ldots},\quad p_{c_kk}>0\quad \forall k\in N\},$$

which is a subset of the set of incomplete sums of series (1).

Let us focus our attention on the distribution of the random variable ξ if the corresponding series has property $a_n=r_{n+1}$.

Theorem 5. *Let a_n is given by (7), let $M=0$ and let*

$$p_{0(2m)}p_{1(2m)}=0,\quad p_{0(2m-1)}p_{1(2m-1)}\neq 0.$$

Then the random variable ξ has a singular distribution of the Cantor type with a self-similar spectrum. The Hausdorff-Besicovitch dimension of the spectrum of ξ is equal to

$$\alpha_0(S_\xi)=-\log_{\beta^2}2.$$

Proof. Taking into account that $a_n=r_{n+1}$, we have

$$\sup\Delta_{c_1\ldots c_m00}=\inf\Delta_{c_1\ldots c_m10},$$

$$\sup\Delta_{c_1\ldots c_m01}=\inf\Delta_{c_1\ldots c_m11}.$$

Let $\Delta^*_{c_1\ldots c_m}=\Delta_{c_1\ldots c_m}\cap S_\xi$. From $p_{01}p_{11}\neq 0$, $p_{0(2m)}=1$ it follows that $p_{1(2m)}=0$ and

$$S_\xi=\Delta^*_{00}\cup\Delta^*_{10},$$

where the sets Δ^*_{00} and Δ^*_{10} are isometric (congruent) and self-similar sets. That is

$$\Delta^*_{c0} = \Delta^*_{c000} \cup \Delta^*_{c010}, \quad c = 0, 1,$$

$$S_\xi \overset{k}{\sim} \Delta^*_{c010} = r_4 \oplus \Delta^*_{c000},$$

where

$$k = \frac{|\Delta_{c000}|}{|\Delta_{c0}|} = \frac{r_4}{r_2} = \frac{a_3}{a_1} = \beta^2.$$

The set Δ^*_{c0} is a perfect set. Its self-similar dimension α_s coincides with its Hausdorff-Besicovitch dimension [6] and it is a solution of the Moran equation

$$\beta^{2x} + \beta^{2x} = 1, \quad \text{i.e.,} \quad x = -\log_{\beta^2} 2.$$

By the Corollary after Theorem 4, the random variable ξ has a continuous distribution, since $M = 0$. Since $\alpha_s(S_\xi) < 1$, the Lebesgue measure $\lambda(S_\xi) = 0$. So, the random variable ξ has a singular distribution of the Cantor type by definition.

Theorem 6. *Let a_n is given by (7), let $M = 0$ and let $p_{0(2m)}p_{1(2m)} = 0$. Then the random variable ξ has a singular distribution of the Cantor type. The Hausdorff-Besicovitch dimension of the spectrum of ξ is equal to*

$$\alpha_0(S_\xi) = -\varliminf_{j\to\infty} \log_{d_j} A_j,$$

where

$$A_j = 2^{\sum\limits_{k=1}^{j}[b_k-1]}, \qquad b_k = \#\{i: \ p_{i(2k+1)} \neq 0\}, \qquad d_j = \frac{\beta^{2j+1}}{-\varphi^2}.$$

Proof. The random variable ξ has a singular distribution, since $M = 0$. The spectrum of ξ is a subset of the set

$$B = \{x: \ x = \Delta_{c_1\ldots c_m\ldots}, \ \text{ where } \ c_{2m-1} \in \{0;1\}, \ c_{2m} \text{ are fixed}\}.$$

The Lebesgue measure of B is equal to 0 (see the proof of the previous theorem). So, ξ has a singular distribution of the Cantor type.

If $p_{01}p_{11} \neq 0$, then sets $\Delta^*_{00} = \Delta_{00} \cap S_\xi$ and $\Delta^*_{10} = \Delta_{10} \cap S_\xi$ are isometric. If $p_{(1-c)1} = 0$, then $S_\xi = \Delta^*_{c1}$. Therefore, the Hausdorff-Besicovitch dimension of the spectrum S_ξ coincides with the Hausdorff-Besicovitch dimension of the set Δ^*_{c0}, where $p_{c1} \neq 0$.

The set Δ^*_{c0} is a union of A_j isometric closed sets of diameter d_j. The α-volume of such a covering is given by

$$l^\alpha_j \equiv A_j d^\alpha_j,$$

where

$$d_j = r_{2j+2} - \sum_{i=j+1}^{\infty} a_{2i} = a_{2j+1} - \frac{a_{2j+4}}{1-\beta^2} = a_{2j+1}\left(1 + \frac{\beta^3}{1-\beta^2}\right) =$$

$$= \frac{a_{2j+1}}{1-\beta^2}(1-\beta^2+\beta^3) = -\beta a_{2j+1} = -\beta^{2j+1} a_1 = \frac{\beta^{2j+1}}{-\varphi^2}.$$

Therefore [13],

$$\alpha_0(\Delta^*_{c0}) = \alpha_0(S_\xi) = -\varliminf_{j\to\infty} \log_{d_j} A_j,$$

which proves the Theorem.

Let us return to the case $a_n \geq r_n$ for any $n \in N$. It is sufficient to define the distribution function $F_\xi(x)$ of the random variable ξ at the points of the spectrum S_ξ, since it is defined by the continuity and monotonicity in remaining points.

Lemma 5. *At the point $x = \Delta_{c_1c_2\ldots c_k\ldots} = x \in S_\xi$ the distribution function F_ξ of the random incomplete sum ξ is of the following form*

$$F_\xi(x) = \beta_{c_1 1} + \sum_{k=2}^{\infty}\left(\beta_{c_k k}\prod_{j=1}^{k-1} p_{c_j j}\right), \tag{8}$$

where

$$\beta_{c_k k} = \begin{cases} 0, & \text{if } \; c_k = 0, \\ p_{0k}, & \text{if } \; c_k = 1. \end{cases}$$

Proof. The event $\{\xi < x\}$ can be represented in the following form

$$\{\xi < x\} = \{\eta_1 < c_1\} \cup \{\eta_1 = c_1, \eta_2 < c_2\} \cup \{\eta_1 = c_1, \eta_2 = c_2, \eta_3 < c_3\} \cup \ldots$$

$$\ldots \cup \{\eta_1 = c_1, \eta_2 = c_2, \ldots, \eta_{k-1} = c_{k-1}, \eta_k < c_k\} \cup \ldots .$$

Then $P\{\xi < x\} =$

$$= P\{\eta_1 < c_1\} + P\{\eta_1 = c_1, \eta_2 < c_2\} + P\{\eta_1 = c_1, \eta_2 = c_2, \eta_3 < c_3\} +$$

$$\ldots + P\{\eta_1 = c_1, \eta_2 = c_2, \ldots, \eta_{k-1} = c_{k-1}, \eta_k < c_k\} + \ldots$$

and

$$P\{\eta_1 = c_1, \eta_2 = c_2, \ldots, \eta_{k-1} = c_{k-1}, \eta_k < c_k\} =$$

$$= \left(\prod_{j=1}^{k-1} P\{\eta_j = c_j\}\right) \cdot P\{\eta_k < c_k\} = \beta_{c_k k}\prod_{j=1}^{k-1} p_{c_j j}.$$

Hence, $F_\xi(x) = P\{\xi < x\}$ is expressed as (8).

Lemma 6 ([20]). *If $\delta_k = \frac{a_k}{r_k} \geq 1$ (it is equivalent to $r_k \geq 2r_{k+1}$) and $p_{0k}p_{1k} \neq 0$ for any $k \in N$, then the spectrum S_ξ of the random variable ξ is a perfect nowhere dense set. Moreover, its Lebesgue measure is equal to*

$$\lambda(S_\xi) = 2 \lim_{k\to\infty} 2^k r_k = 2r \prod_{k=1}^{\infty} \frac{1}{\delta_k + 1} = \begin{cases} 0 & \text{if } \sum\limits_{k=1}^{\infty} (\delta_k - 1) = \infty, \\ A > 0 & \text{if } \sum\limits_{k=1}^{\infty} (\delta_k - 1) < \infty. \end{cases}$$

Theorem 7 ([20]). *Let $\delta_k = \frac{a_k}{r_k}$ for any $k \in N$ and let the series $\sum\limits_{k=1}^{\infty} (\delta_k - 1)$ is convergent. Then the random variable ξ has an absolutely continuous distribution if and only if*

$$\sum_{k=1}^{\infty} \ln[\sqrt{p_{0k}p_{1k}}(\delta_k + 1)] < \infty.$$

Theorem 8 ([20]). *Let $\delta_k \geq 1$ (it is equivalent to $r_k \geq 2r_{k+1}$) and $p_{0k}p_{1k} \neq 0$ for any $k \in N$, let $M = 0$, and let the series $\sum\limits_{k=1}^{\infty} (\delta_k - 1)$ is divergent. Then the random variable ξ has a singular distribution of the Cantor type. Moreover, the Hausdorff-Besicovitch dimension of its spectrum is equal to (4).*

7. Fractal dimension preservation

Let us recall that the *α-dimensional Hausdorff-Billingsley measure* of a set E is defined as follows

$$\widehat{H}^\alpha(E) = \lim_{\varepsilon \downarrow 0} \left(\inf_{\mu(u_i) \leq \varepsilon} \sum_i \mu^\alpha(u_i), \quad \bigcup_i u_i \supset E \right),$$

where the infimum is taken over all coverings $\{u_i\}$ of the set $E \subset A$ by segments u_i with $\mu(u_i) \leq \varepsilon$. The number

$$\alpha_\mu = \alpha_\mu(E) = \inf\{\alpha : \ \widehat{H}^\alpha(E) = 0\} = \sup\{\alpha : \ \widehat{H}^\alpha(E) \neq 0\}$$

is called the *Hausdorff-Billingsley dimension* of the set E with respect to measure μ.

We say that a distribution function $F_\xi(x)$ *preserves the fractal dimension* if the Hausdorff-Billingsley dimension $\alpha_{\overline{\mu}}(\cdot)$ of any subset $E \subset S_\xi$ is equal to the Hausdorff-Besicovitch dimension of its image $E' = F_\xi(E)$, i.e., $\alpha_{\overline{\mu}}(E) = \alpha_{\mu_\xi}(E) \equiv \alpha_0(E')$, where $\overline{\mu}$ is a probability measure which gives uniform distribution on S_ξ.

Theorem 9. *If $a_n \geq r_n$ for any $n \in N$ and*

$$\lim_{k\to\infty} p_{0k} = \frac{1}{2}, \tag{9}$$

then the distribution function F_ξ of the random variable ξ defined by (2) preserves the fractal dimension.

Proof. Without loss of generality, let us assume that $p_{ik} > 0$. Then the expression of the distribution function $F_\xi(x)$ is a Q^*-representation [20] of the number $F_\xi(x)$ and

$$\overline{\mu}(\Delta_{\alpha_1(x)\ldots\alpha_k(x)}) = 2^{-k},$$

$$\mu_\xi(\Delta_{\alpha_1(x)\ldots\alpha_k(x)}) = P\{\xi \in \Delta_{\alpha_1(x)\ldots\alpha_k(x)}\} = \prod_{j=1}^{k} p_{\alpha_j(x)j} = \prod_{j=1}^{k}\left(\frac{1}{2}\tau_{\alpha_j(x)j}\right),$$

where $\tau_{\alpha_j(x)j} = 2p_{\alpha_j(x)j}$. Taking into account (9), we have

$$\tau_{\alpha_j(x)j} \to 1 \quad (j\to\infty) \quad \text{and} \quad \lim_{j\to\infty} \ln(\tau_{\alpha_j(x)j}) = 0. \tag{10}$$

Then

$$\lim_{k\to\infty} \frac{\ln \mu_\xi(\Delta_{\alpha_1(x)\ldots\alpha_k(x)})}{\ln \overline{\mu}(\Delta_{\alpha_1(x)\ldots\alpha_k(x)})} = \lim_{k\to\infty} \frac{\ln \prod\limits_{i=1}^{k} p_{\alpha_j(x)j}}{\ln 2^{-k}} =$$

$$= \lim_{k\to\infty} \frac{\sum\limits_{j=1}^{k} \ln p_{\alpha_j(x)j}}{-k\ln 2} = \lim_{k\to\infty} \frac{k\ln 2^{-1} + \sum\limits_{j=1}^{k} \ln \tau_{\alpha_j(x)j}}{-k\ln 2} = 1 + \lim_{k\to\infty} \frac{\frac{1}{k}\sum\limits_{j=1}^{k} \ln \tau_{\alpha_j(x)j}}{-\ln 2}.$$

Taking into account (10), we have

$$\lim_{k\to\infty} \frac{1}{k}\sum_{j=1}^{k} \ln \tau_{\alpha_j(x)j} = 0.$$

Therefore,

$$\lim_{k\to\infty} \frac{\ln \mu_\xi(\Delta_{\alpha_1(x)\ldots\alpha_k(x)})}{\ln \overline{\mu}(\Delta_{\alpha_1(x)\ldots\alpha_k(x)})} = 1 \tag{11}$$

for any $x \in S_\xi$.

For cylindrical representation and the Hausdorff-Billingsley dimension the analogue of Billingsley Theorem for s-adic intervals [4] holds: if ν_1 and ν_2 are continuous probability measures and

$$E \subset E_0 = \left\{x: \ \lim_{k\to\infty} \frac{\ln \nu_1(\Delta_{\alpha_1(x)\ldots\alpha_k(x)})}{\ln \nu_2(\Delta_{\alpha_1(x)\ldots\alpha_k(x)})} = \delta\right\},$$

then $\alpha_{\nu_2}(E) = \delta\alpha_{\nu_1}(E)$. Hence, from equality (11) it follows that $\alpha_{\overline{\mu}}(E) = 1\cdot\alpha_{\mu_\xi}(E)$. So, F_ξ preserves the fractal dimension.

8. Characteristic function of the random incomplete sum with independent terms

Characteristic function $f_\xi(t)$ *of random variable* ξ is a mathematical expectation of random variable $e^{it\xi}$, i.e.

$$f_\xi(t) = Me^{it\xi}.$$

Characteristic functions provide suitable tools for the investigation of the structure and properties of real-valued random variables. In particular, it is known that

$$L_\xi = \limsup_{|t|\to\infty} |f_\xi(t)|$$

is equal to

1) 1, if ξ has a discrete distribution;

2) 0, if ξ has an absolutely continuous distribution.

For singular distributions L_ξ can be equal to any number from $[0, 1]$. Singular distributions with $L_\xi = 1$ are close to discrete distributions, and distributions with $L_\xi = 0$ are close to absolutely continuous ones. Hence, the value L_ξ characterise how close are the properties of singular distribution to the properties of discrete and absolutely continuous ones. Note that measure μ_ξ such that $L_\xi = 0$ is called the Rajchman measure. Some important for probability theory problems are related with such measures [15].

Lemma 7. *The characteristic function of random variable* ξ *defined by (2) is of the following form*

$$f_\xi(t) = \prod_{k=1}^{\infty} \left(p_{0k} + p_{1k}e^{ita_k}\right) = \prod_{k=1}^{\infty} \left(p_{0k} + p_{1k}\cos(a_k t) + ip_{1k}\sin(a_k t)\right),$$

and its absolute value is f the following form

$$|f_\xi(t)| = \prod_{k=1}^{\infty} |f_k(t)|, \quad \text{where} \quad |f_k(t)| = \sqrt{1 - 4p_{0k}p_{1k}\sin^2\frac{ta_k}{2}}.$$

Proof. From properties of characteristic functions and mathematical expectation we obtain

$$\begin{aligned}
f_\xi(t) &= Me^{it\xi} = Me^{it\sum\limits_{k=1}^{\infty} a_k\eta_k} = M\prod_{k=1}^{\infty} e^{ita_k\eta_k} = \\
&= \prod_{k=1}^{\infty} Me^{ita_k\eta_k} = \prod_{k=1}^{\infty}(p_{0k} + p_{1k}e^{ita_k}) = \\
&= \prod_{k=1}^{\infty}(p_{0k} + p_{1k}\cos(ta_k) + ip_{1k}\sin(ta_k)) = \prod_{k=1}^{\infty} f_k(t)
\end{aligned}$$

and

$$|f_k(t)| = \sqrt{p_{0k}^2 + 2p_{0k}p_{1k}\cos(ta_k) + p_{1k}^2} = \sqrt{1 - 4p_{0k}p_{1k}\sin^2\frac{ta_k}{2}},$$

which proves the Lemma.

Let us study the behaviour of the absolute value of the characteristic function of random variable ξ at infinity under extra conditions.

Theorem 10. *If*

$$\frac{1}{g_n} = \frac{a_{n+1}}{a_n} \to 0 \quad (n \to \infty), \quad \textit{where} \quad 2 \le g_n \in N, \tag{12}$$

then

$$L_\xi = \limsup_{|t|\to\infty} |f_\xi(t)| = 1.$$

Proof. Let us consider the sequence $t_n = \frac{2\pi}{a_n}$. And let us estimate

$$|f_\xi(t)| = \prod_{k=1}^{\infty}\sqrt{1 - 4p_{0k}p_{1k}\sin^2\frac{ta_k}{2}} \ge \prod_{k=1}^{\infty}\sqrt{1 - \sin^2\frac{ta_k}{2}} = \prod_{k=1}^{\infty}\left|\cos\frac{ta_k}{2}\right|.$$

Hence,

$$L_\xi \ge \lim_{n\to\infty}|f_\xi(t_n)| = \lim_{n\to\infty}\prod_{k=1}^{\infty}|f_k(t_n)| \ge \lim_{n\to\infty}\prod_{k=1}^{\infty}\left|\cos\frac{t_na_k}{2}\right|.$$

Since

$$\frac{t_na_k}{2} = \begin{cases} \pi g_kg_{k+1}\dots g_{n-1} & \text{if} \quad k \le n, \\ \frac{\pi}{g_{n+1}g_{n+2}\cdots g_k} & \text{if} \quad k > n, \end{cases}$$

we have

$$\left|\cos\frac{t_na_k}{2}\right| = \begin{cases} 1 & \text{if} \quad k \le n, \\ \cos\frac{\pi}{g_{n+1}g_{n+2}\cdots g_k} & \text{if} \quad k > n. \end{cases}$$

So,

$$\prod_{k=1}^{\infty}|f_k(t_n)| \ge \prod_{k=n+1}^{\infty}\cos\frac{\pi}{g_{n+1}g_{n+2}\dots g_k}. \tag{13}$$

Since $g_n^{-1} \to 0$ $(n \to \infty)$, there exists n_0 such that $g_n \ge 4$ for all $n > n_0$. Then for $k > n > n_0$ we have

$$\cos\frac{\pi}{g_{n+1}g_{n+2}\dots g_k} \ge \cos\frac{\pi}{4^{k-n}} = 1 - 2\sin^2\frac{\pi}{2\cdot 4^{k-n}} > 1 - \frac{2\pi^2}{4\cdot 16^{k-n}} =$$

$$= 1 - \frac{\pi^2}{2^{4k-4n+1}}.$$

Hence,

$$\prod_{k=n+1}^{\infty} \cos \frac{\pi}{g_{n+1}g_{n+2}\cdots g_k} \geq \prod_{k=n+1}^{\infty} \left(1 - \frac{\pi^2}{2^{4k-4n+1}}\right).$$

Since series $\sum\limits_{k=n+1}^{\infty} \frac{\pi^2}{2^{4k-4n+1}}$ converges, infinite product (13) also converges, hence,

$$\lim_{n\to\infty} \prod_{k=n+1}^{\infty} \cos \frac{\pi}{g_{n+1}g_{n+2}\cdots g_k} = 1$$

and $L_\xi \geq 1$. Since always $L_\xi \leq 1$, we have $L_\xi = 1$.

Corollary. *If sequence $\{a_n\}$ satisfies the condition (12) and condition $M = 0$ holds for matrix $||p_{ik}||$, then the random variable ξ has an anomalously fractal singular distribution.*

Lemma 8. *If for any positive integer n condition (12) holds and $L_\xi = 0$, then*

$$p_{0k} \to \frac{1}{2} \quad (k \to \infty)$$

and $g_n = 2$ for any $n > n_0$ for some n_0.

Proof. Since $L_\xi = 0$, the equality

$$\lim_{n\to\infty} |f_\xi(t_n)| = 0 \tag{14}$$

holds for any sequence $\{t_n\}$ such that $t_n \to \infty$. Let us consider $t_n = 2g_1 \ldots g_n \pi$. Then

$$\sin^2 \frac{t_n a_k}{2} = \sin^2 \frac{\pi g_1 \ldots g_k}{g_1 \ldots g_n} = \begin{cases} \sin^2 \pi g_{k+1} \ldots g_n & \text{if} \quad k < n, \\ \sin^2 \pi & \text{if} \quad k = n, \\ \sin^2 \frac{\pi}{g_{n+1}\cdots g_k} & \text{if} \quad k > n, \end{cases}$$

and $f_k(t_n) = 1$ if $k \leq n$. Hence,

$$|f_\xi(t_n)| = |f_{n+1}(t_n)| \cdot B_n,$$

where

$$|f_{n+1}(t_n)| = \sqrt{1 - 4p_{0(n+1)}p_{1(n+1)} \sin^2 \frac{\pi}{g_{n+1}}}$$

and

$$B_n = \prod_{k=n+2}^{\infty} |f_k(t_n)| = \prod_{k=n+2}^{\infty} \sqrt{1 - 4p_{0k}p_{1k} \sin^2 \frac{\pi}{g_{n+1}\cdots g_k}}.$$

The sequence $\{B_n\}$ is convergent, moreover,

$$B_n \geq \prod_{k=n+2}^{\infty} \sqrt{1 - \sin^2 \frac{\pi}{g_{n+1}\cdots g_k}} = \prod_{k=n+2}^{\infty} \cos \frac{\pi}{g_{n+1}\cdots g_k} = b > 0.$$

Therefore, equality (14) holds if and only if $|f_{n+1}(t_n)| \to 0$ $(n \to \infty)$. It is possible if $p_{0n} \to \frac{1}{2}$ $(n \to \infty)$ and $g_n = 2$ for $n > n_0$.

Corollary. *Let $M = 0$, let condition (12) holds, and let $\lim\limits_{k\to\infty} p_{0k} \neq \frac{1}{2}$. Then the random variable ξ has a singular distribution.*

Lemma 9. *Let condition (12) holds, let $g_n = 2$ for $n > n_0$, and let $p_{0k} \to \frac{1}{2}$ $(k \to \infty)$. Then $L_\xi = 0$.*

Proof. Let us consider the random variable

$$\zeta = \sum_{m=1}^{\infty} 2^{-m} \eta_{n_0+m}$$

with independent binary digits η_{n_0+m}. The behaviour of L_ζ was studied in the paper [8]. It is known [8] that

$$L_\zeta = 0 \quad \Leftrightarrow \quad p_{0k} \to \frac{1}{2} \quad (k \to \infty).$$

This fact remains valid for the random variable $\hat{\zeta} = \frac{1}{g_1 \cdots g_{n_0}} \zeta$. Since

$$\xi = \xi_1 + \hat{\zeta}, \quad \text{where} \quad \xi_1 = \sum_{n=1}^{n_0} a_n \eta_n,$$

and

$$f_\xi(t) = f_{\xi_1}(t) \cdot f_{\hat{\zeta}}(t),$$

we have that $L_\xi = 0$ if and only if $L_{\hat{\zeta}} = 0$, i.e., if condition $p_{0k} \to 0$ $(k \to \infty)$ holds.

Lemmas 8 and 9 imply the following proposition.

Theorem 11. *Let $M = 0$, and let for any $k \in N$*

$$a_k = \frac{1}{g_1 g_2 \cdots g_k}, \quad 2 \leq g_k \in N.$$

Then $L_\xi = 0$ if and only if

$$\begin{cases} p_{0k} \to \frac{1}{2} & (k \to \infty), \\ g_n = 2 & \text{for } n > n_0. \end{cases}$$

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