Theory of Stochastic Processes
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ROBUST FILTERING OF STOCHASTIC PROCESSES

The considered problem is estimation of the unknown value of the functional $A\vec{\xi} = \int_0^\infty \vec{a}(t)\vec{\xi}(-t)dt$ which depends on the unknown values of a multidimensional stationary stochastic process $\vec{\xi}(t)$ based on observations of the process $\vec{\xi}(t) + \vec{\eta}(t)$ for $t \leq 0$. Formulas are obtained for calculation the mean square error and the spectral characteristic of the optimal estimate of the functional under the condition that the spectral density matrix $F(\lambda)$ of the signal process $\vec{\xi}(t)$ and the spectral density matrix $G(\lambda)$ of the noise process $\vec{\eta}(t)$ are known. The least favorable spectral densities and the minimax-robust spectral characteristic of the optimal estimate of the functional $A\vec{\xi}$ are found for concrete classes $D = D_F \times D_G$ of spectral densities under the condition that spectral density matrices $F(\lambda)$ and $G(\lambda)$ are not known, but classes $D = D_F \times D_G$ of admissible spectral densities are given.

1. Introduction

Traditional methods of solution of the linear extrapolation, interpolation and filtering problems for stationary stochastic processes may be employed under the condition that spectral densities of processes are known exactly (see, for example, selected works of A. N. Kolmogorov (1992), survey by T. Kailath (1974), Yu. A. Rozanov (1990), N. Wiener (1966); A. M. Yaglom (1987)). In practice, however, complete information on the spectral densities is impossible in most cases. To solve the problem one finds parametric or nonparametric estimates of the unknown spectral densities or selects these densities by other reasoning. Then applies the classical estimation method provided that the estimated or selected density is the true

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one. This procedure can result in a significant increasing of the value of error as K. S. Vastola and H. V. Poor (1983) have demonstrated with the help of some examples. This is a reason to search estimates which are optimal for all densities from a certain class of the admissible spectral densities. These estimates are called minimax since they minimize the maximal value of the error. Many investigators have been interested in minimax extrapolation, interpolation and filtering problems for stationary stochastic sequences. A survey of results in minimax (robust) methods of data processing can be found in the paper by S. A. Kassam and H. V. Poor (1985). The paper by Ulf Grenander (1957) should be marked as the first one where the minimax approach to extrapolation problem for stationary processes was proposed. J. Franke (1984, 1985, 1991), J. Franke and H. V. Poor (1984) investigated the minimax extrapolation and filtering problems for stationary sequences with the help of convex optimization methods. This approach makes it possible to find equations that determine the least favorable spectral densities for various classes of densities. In the papers by Mikhail Moklyachuk (1994, 1997, 1998, 2000, 2001), Mikhail Moklyachuk and Aleksandr Masyutka (2005, 2006) the minimax approach to extrapolation, interpolation and filtering problems are investigated for functionals which depend on the unknown values of stationary processes and sequences.

In this article we deal with the problem of estimation of the unknown value of the functional $A\vec{\xi} = \int_0^\infty \vec{a}(t)\vec{\xi}(-t)dt$ which depends on the unknown values of a multidimensional stationary stochastic process $\vec{\xi}(t) = \{\xi_k(t)\}_{k=1}^T$, $E\vec{\xi}(t) = 0$, with the spectral density matrix $F(\lambda) = \{f_{ij}(\lambda)\}_{i,j=1}^T$ based on observations of the process $\vec{\xi}(t)+\vec{\eta}(t)$ for $t \leq 0$, where $\vec{\eta}(t) = \{\eta_k(t)\}_{k=1}^T$ is an uncorrelated with $\vec{\xi}(t)$ multidimensional stationary stochastic process with the spectral density matrix $G(\lambda) = \{g_{ij}(\lambda)\}_{i,j=1}^T$. Formulas are proposed that determine the least favorable spectral densities and the minimax-robust spectral characteristic of the optimal estimate of the functional for concrete classes $D = D_F \times D_G$ of spectral densities under the condition that spectral density matrices $F(\lambda)$, $G(\lambda)$ are not known, but classes $D = D_F \times D_G$ of admissible spectral densities are given.

2. Hilbert space projection method of filtering

Let the vector function $\vec{a}(t)$ which determines the functional $A\vec{\xi}$ satisfies the following technical conditions:

$$\int_0^\infty \sum_{k=1}^T |a_k(t)|\, dt < \infty, \qquad \int_0^\infty t \sum_{k=1}^T |a_k(t)|^2 dt < \infty. \tag{1}$$

The process $\vec{\xi}(t) + \vec{\eta}(t)$ admits the canonical moving average representation

$$\vec{\xi}(t)+\vec{\eta}(t)=\int_{-\infty}^{t} d(t-u)\,d\vec{\varepsilon}(u), \qquad (2)$$

if the spectral density matrix $F(\lambda)+G(\lambda)=\{f_{ij}(\lambda)+g_{ij}(\lambda)\}_{i,j=1}^{T}$ of the stationary stochastic process $\vec{\xi}(t)+\vec{\eta}(t)$ admits the canonical factorization

$$F(\lambda)+G(\lambda)=d(\lambda)d^{*}(\lambda), \quad d(\lambda)=\int_{0}^{\infty} d(u)e^{-iu\lambda}du \qquad (3)$$

where $d(u)=\{d_{ij}(u)\}_{i=\overline{1,T}}^{j=\overline{1,m}}$, $\vec{\varepsilon}(u)=\{\varepsilon_k(u)\}_{k=1}^{m}$ is a multidimensional stationary stochastic process with uncorrelated increments (see, for example, Yu. A. Rozanov (1990)).

The spectral density matrices $F(\lambda)$ and $G(\lambda)$ admit the canonical factorizations if

$$F(\lambda)=\varphi(\lambda)\varphi^{*}(\lambda), \quad \varphi(\lambda)=\int_{0}^{\infty}\varphi(u)e^{-iu\lambda}du, \qquad (4)$$

$$G(\lambda)=\psi(\lambda)\psi^{*}(\lambda), \quad \psi(\lambda)=\int_{0}^{\infty}\psi(u)e^{-iu\lambda}du, \qquad (5)$$

where $\varphi(u)=\{\varphi_{ij}(u)\}_{i=\overline{1,T}}^{j=\overline{1,m}}$ and $\psi(u)=\{\psi_{ij}(u)\}_{i=\overline{1,T}}^{j=\overline{1,m}}$ are matrix functions. The value of the mean square error $\Delta(h,F,G)$ of a linear estimate $\hat{A}\vec{\xi}$ of the functional $A\vec{\xi}$ with the spectral characteristic $h(\lambda)=\int_0^\infty \vec{h}(t)e^{-it\lambda}dt$ is determined by the formula

$$\Delta(h;F,G)=E\left|A\vec{\xi}-\hat{A}\vec{\xi}\right|^{2}=$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}\Big\{(A(\lambda)-h(\lambda))(F(\lambda)+G(\lambda))(A(\lambda)-h(\lambda))^{*}-$$

$$-(A(\lambda)-h(\lambda))G(\lambda)A^{*}(\lambda)-A(\lambda)G(\lambda)(A(\lambda)-h(\lambda))^{*}+A(\lambda)G(\lambda)A^{*}(\lambda)\Big\}d\lambda=$$

$$=\int_{0}^{\infty}\int_{0}^{\infty}\int_{-\infty}^{\min(t,u)}\Big\{(\vec{a}(t)-\vec{h}(t))d(t-x)d^{*}(u-x)(\vec{a}(u)-\vec{h}(u))^{*}-$$

$$-(\vec{a}(t)-\vec{h}(t))\psi(t-x)\psi^{*}(u-x)\vec{a}^{*}(u)-\vec{a}(t)\psi(t-x)\psi^{*}(u-x)(\vec{a}(u)-\vec{h}(u))^{*}+$$

$$+\vec{a}(t)\psi(t-x)\psi^{*}(u-x)\vec{a}^{*}(u)\Big\}dx\,du\,dt,$$

where $A(\lambda)=\int_0^\infty \vec{a}(t)e^{-it\lambda}dt$.

The spectral characteristic $h(F,G)$ and the mean square error $\Delta(F,G) = \Delta(h(F,G);F,G)$ of the optimal estimate of the functional $A\vec{\xi}$ in the case of given spectral density matrices $F(\lambda)$ and $G(\lambda)$ are determined as solutions to the extremum problem

$$\Delta(F,G) = \Delta(h(F,G);F,G) = \min_{h\in L_2^-(F+G)} \Delta(h;F,G), \tag{6}$$

where $L_2^-(F+G)$ is the subspace of the space $L_2(F+G)$ generated by the functions $e^{it\lambda}\delta_k$, $\delta_k = \{\delta_{kl}\}_{l=1}^T$, $k = 1,\dots,T$, $t < 0$, $\delta_{kk} = 1$, $\delta_{kl} = 0$ for $k \neq l$.

If the spectral density matrices $F(\lambda)$ and $G(\lambda)$ admit the canonical factorization (3), (5), then, as follows from the preceding formulas, the mean square error of the optimal linear estimate can be calculated by the formula

$$\Delta(F,G) = \langle c_G, a\rangle - \|C_G b^*\|^2, \tag{7}$$

where

$$c_G(t) = \int_0^{\infty}\int_{-\infty}^{\min(s,t)} \vec{a}(s)\psi(t-u)\psi^*(s-u)du ds,$$

$$(C_G b^*)(t) = \int_0^{\infty} c_G(t+u)b^*(u)du,$$

$$\langle c_G, a\rangle = \int_0^{\infty}\sum_{k=1}^{T}(c_G)_k(t)\overline{a_k(t)}\,dt,$$

$$\|C_G b^*\|^2 = \int_0^{\infty}\sum_{k=1}^{m}|(C_G b^*)_k(t)|^2 dt.$$

Here $b(\lambda) = \{b_{ij}(\lambda)\}_{i=\overline{1,m}}^{j=\overline{1,T}}$ is such a matrix-valued function that $b(\lambda) = \int_0^{\infty} b(t)e^{-it\lambda}dt$, $b(\lambda)d(\lambda) = I_m$, where I_m is the identity matrix of order m.

The spectral characteristic $h(F,G)$ of the optimal estimate of the functional $A\vec{\xi}$ in this case can be calculated by the formula

$$h(F,G) = A(\lambda) - r_G(\lambda)b(\lambda), \quad r_G(\lambda) = \int_0^{\infty}(C_G b^*)(t)e^{-it\lambda}dt. \tag{8}$$

If the spectral density matrices $F(\lambda)$ and $G(\lambda)$ admit the canonical factorization (3), (4), then the mean square error and the spectral characteristic of the optimal linear estimate can be calculated by the formulas

$$\Delta(F,G) = \langle c_F, a\rangle - \|C_F b^*\|^2, \tag{9}$$

$$h(F,G)=r_F(\lambda)b(\lambda),\quad r_F(\lambda)=\int_0^\infty (C_Fb^*)(t)e^{-it\lambda}dt, \tag{10}$$

where

$$c_F(t)=\int_0^\infty \int_{-\infty}^{\min(s,t)} \vec{a}(s)\varphi(t-u)\varphi^*(s-u)du\,ds,$$

$$(C_Fb^*)(t)=\int_0^\infty c_F(t+u)b^*(u)\,du.$$

The preceding reasonings show us that the following theorem holds true.

Theorem 1.1. *Let $\vec{\xi}(t)=\{\xi_k(t)\}_{k=1}^T$, $E\vec{\xi}(t)=0$ and $\vec{\eta}(t)=\{\eta_k(t)\}_{k=1}^T$, $E\vec{\eta}(t)=0$, be uncorrelated multidimensional stationary stochastic processes with the spectral density matrices $F(\lambda)=\{f_{ij}(\lambda)\}_{i,j=1}^T$, $G(\lambda)=\{g_{ij}(\lambda)\}_{i,j=1}^T$ and let condition (1) be satisfied. The mean-square error $\Delta(h(F),F,G)$ of the optimal linear estimate of the functional $A\vec{\xi}=\int_0^\infty \vec{a}(t)\vec{\xi}(-t)dt$ which depends on the unknown values of the process $\vec{\xi}(t)$ based on observations of the process $\vec{\xi}(t)+\vec{\eta}(t)$ for $t\leq 0$ can be calculated by formula (7) if the spectral density matrices $F(\lambda)$ and $G(\lambda)$ admit the canonical factorization (3), (5) (by formula (9) if the spectral density matrices $F(\lambda)$ and $G(\lambda)$ admit the canonical factorization (3), (4)). The spectral characteristic $h(F,G)$ of the optimal linear estimate can be calculated by formula (8) (by formula (10)).*

Remark. In the case where the process $\vec{\xi}(t)+\vec{\eta}(t)$ is of the maximal rank ($m=T$), the matrix function $b(\lambda)$ is the inverse matrix to the matrix $d(\lambda): b(\lambda)=d^{-1}(\lambda)$. In the case where the process $\vec{\xi}(t)+\vec{\eta}(t)$ is of the rank 1 ($m=1$), the matrix function $b(\lambda)$ is a row matrix $b(\lambda)=\{b_k(\lambda)\}_{k=1}^T$ determined from the equation $\sum_{k=1}^T b_k(\lambda)d_k(\lambda)=1$.

Example 1. Consider the problem of estimation of the value $\vec{a}(0)\vec{\xi}(0)=c\xi_1(0)+d\xi_2(0)$ based on observations of the process $\vec{\xi}(t)+\vec{\eta}(t)$, $t\leq 0$, in the case where

$$F(\lambda)=\begin{pmatrix} f(\lambda) & f(\lambda)\\ f(\lambda) & f(\lambda)+f_1(\lambda)\end{pmatrix},\ f(\lambda)=\frac{P_1^2}{|\alpha_1+i\lambda|^2},\quad f_1(\lambda)=\frac{P_2^2}{|\alpha_2+i\lambda|^2};$$

$$G(\lambda)=\begin{pmatrix} g(\lambda) & g(\lambda)\\ g(\lambda) & g(\lambda)+g_1(\lambda)\end{pmatrix},\quad g(\lambda)=0,\quad g_1(\lambda)=\frac{P_3^2}{|\alpha_3+i\lambda|^2}.$$

In this case $F(\lambda)+G(\lambda)=d(\lambda)d^*(\lambda)$, $F(\lambda)=\varphi(\lambda)\varphi^*(\lambda)$,

$$d(\lambda)=\begin{pmatrix} \frac{P_1}{\alpha_1+i\lambda} & 0\\ \frac{P_1}{\alpha_1+i\lambda} & A\frac{\beta+i\lambda}{(\alpha_2+i\lambda)(\alpha_3+i\lambda)}\end{pmatrix},\quad A^2=P_2^2+P_3^2,\quad \beta=\frac{P_2^2\alpha_3^2+P_3^2\alpha_2^2}{A^2};$$

$$\varphi(\lambda) = \int_0^\infty \varphi(u) e^{-iu\lambda} du, \quad \varphi(u) = \begin{pmatrix} P_1 e^{-\alpha_1 u} & 0 \\ P_1 e^{-\alpha_1 u} & P_2 e^{-\alpha_2 u} \end{pmatrix}.$$

The inverse matrix

$$b(\lambda) = d(\lambda)^{-1} = \begin{pmatrix} \frac{\alpha_1 + i\lambda}{P_1} & 0 \\ -\frac{1}{A}\frac{(\alpha_2+i\lambda)(\alpha_3+i\lambda)}{\beta+i\lambda} & \frac{1}{A}\frac{(\alpha_2+i\lambda)(\alpha_3+i\lambda)}{\beta+i\lambda} \end{pmatrix}.$$

The spectral characteristic of the optimal estimate is of the form

$$h(\lambda) = (h_1(\lambda), h_2(\lambda)) = r_F(\lambda) b(\lambda),$$

where

$$r_F(\lambda) = \int_0^\infty (C_F b^*)(t) e^{-it\lambda} dt, \quad (C_F b^*)(t) = \int_0^\infty c_F(t+u) b^*(u) du.$$

Since

$$c_F(t) = \vec{a}(0) \int_0^\infty \varphi(t+u)\varphi^*(u)\, du = (c_1(t), c_2(t)),$$

where

$$c_1(t) = \left[\frac{(c+d)P_1^2}{|\alpha_1 + i\lambda|^2}\right]_t, \quad c_2(t) = c_1(t) + \left[\frac{dP_2^2}{|\alpha_2 + i\lambda|^2}\right]_t$$

are the Fourier transforms, we have

$$h_1(\lambda) = (c+d) - \frac{dP_2^2}{A^2} \frac{(\alpha_2+i\lambda)(\alpha_3+i\lambda)}{\beta+i\lambda} \left[\frac{\alpha_3 - i\lambda}{(\beta - i\lambda)(\alpha_2 + i\lambda)}\right]_-,$$

where $[f(\lambda)]_-$ is a representation of the function as the Fourier integral transform with respect to the negative powers of $e^{-it\lambda}$, $t \geq 0$. Taking into account that

$$\left[\frac{\alpha_3 - i\lambda}{(\beta - i\lambda)(\alpha_2 + i\lambda)}\right]_- = \frac{\alpha_2 + \alpha_3}{(\alpha_2 + \beta)(\alpha_2 + i\lambda)},$$

we will have

$$h_1(\lambda) = (c+d) - \frac{dP_2^2(\alpha_2+\alpha_3)}{A^2(\alpha_2+\beta)} \frac{\alpha_3 + i\lambda}{\beta + i\lambda}.$$

Analogously

$$h_2(\lambda) = \frac{dP_2^2(\alpha_2+\alpha_3)}{A^2(\alpha_2+\beta)} \frac{\alpha_3 + i\lambda}{\beta + i\lambda}.$$

The value of the mean square error of the optimal estimate is calculated by the formula

$$\Delta(F, G) = \langle c_F, a \rangle - \|C_F b^*\|^2 = \langle \vec{c}_F(0), \vec{a}(0) \rangle - \|C_F b^*\|^2.$$

Since

$$\vec{c}_F(0) = \vec{a}(0) \int_0^\infty \varphi(u)\varphi^*(u)du,$$

we will have

$$\langle \vec{c}_F(0), \vec{a}(0)\rangle = \frac{(c+d)^2P_1^2}{2\alpha_1} + \frac{d^2P_2^2}{2\alpha_2}.$$

The second summand

$$\|C_F b^*\|^2 = \|(C_F b^*)(t)\|^2,$$

where

$$(C_F b^*)(t) = \left(\left[\frac{(c+d)P_1}{\alpha_1 + i\lambda}\right]_t, \left[\frac{dP_2^2(\alpha_3 - i\lambda)}{A(\alpha_2 + i\lambda)(\beta - i\lambda)}\right]_t \right) =$$

$$= \left((c+d)P_1 e^{-\alpha_1 t}, \frac{dP_2^2}{A}\left(e^{-\alpha_1 t} + \frac{\alpha_3 - \beta}{\alpha_2 + \beta} e^{-\alpha_2 t}\right)\right).$$

Finally

$$\Delta(F,G) = \frac{d^2P_2^2}{2\alpha_2} - \frac{d^2P_2^4}{2\alpha_2 A^2}\left(\frac{\alpha_2 + \alpha_3}{\alpha_2 + \beta}\right)^2.$$

Example 2. Consider the problem of estimation of the value $\vec{a}(t)\vec{\xi}(-t) = c\xi_1(-t) + d\xi_2(-t)$ under conditions of Example 1. In this case $c_F(u) = (c_1(u), c_2(u))$, where

$$c_1(u) = \left[\frac{(c+d)P_1^2 e^{-i\lambda t}}{|\alpha_1 + i\lambda|^2}\right]_u, \quad c_2(u) = c_1(u) + \left[\frac{dP_2^2 e^{-i\lambda t}}{|\alpha_2 + i\lambda|^2}\right]_u.$$

We will have

$$h(\lambda) = ((c+d)e^{-i\lambda t} - q(\lambda), q(\lambda)),$$

$$q(\lambda) = \frac{dP_2^2}{A^2}\frac{(\alpha_2 + i\lambda)(\alpha_3 + i\lambda)}{\beta + i\lambda}\left[\frac{\alpha_3 - i\lambda}{(\alpha_2 + i\lambda)(\beta - i\lambda)} e^{-i\lambda t}\right]_- =$$

$$= \frac{dP_2^2}{A^2}\frac{\alpha_3 + i\lambda}{\beta + i\lambda}\left(e^{-i\lambda t} + e^{\alpha_2 t}\frac{\alpha_3 - \beta}{\alpha_2 + \beta}\right).$$

The value of the mean square error of the optimal estimate is calculated by the formula

$$\Delta(F,G) = \langle \vec{c}_F(t), \vec{a}(t)\rangle - \|(C_F b^*)(u)\|^2 = \frac{d^2P_2^2}{2\alpha_2} - \|(C_F b^*)_2(u)\|^2,$$

where

$$(C_F b^*)_2(u) = \frac{dP_2^2}{A}\left[\frac{1}{\alpha_2 + i\lambda}\left(e^{-i\lambda t} + e^{\alpha_2 t}\frac{\alpha_3 - \beta}{\alpha_2 + \beta}\right)\right]_u =$$

$$= \begin{cases} \frac{dP_2^2(\alpha_3-\beta)}{A(\alpha_2+\beta)} e^{\alpha_2(t-u)}, \ 0 \le u < t\,; \\ \frac{dP_2^2(\alpha_2+\alpha_3)}{A(\alpha_2+\beta)} e^{\alpha_2(t-u)}, \ u \ge t\,. \end{cases}$$

The mean-square error of the optimal estimate can be calculated by the formula

$$\Delta(F,G) = \frac{d^2P_2^2}{2\alpha_2} - \frac{d^2P_2^4}{2\alpha_2 A^2(\alpha_2+\beta)^2}((\alpha_3-\beta)^2(e^{2\alpha_2 t}-1)+(\alpha_2+\alpha_3)^2).$$

Example 3. Consider the problem of estimation of the value $A_1\vec{\xi} = \int_0^1 \vec{a}(t)\vec{\xi}(-t)\,dt$. By using results of the previous examples we will get that the spectral characteristic

$$h(\lambda) = \left(\int\limits_0^1 (a_1(t) + a_2(t)e^{-i\lambda t}dt - q(\lambda)\ ,\ q(\lambda) \right),$$

where

$$q(\lambda) = \frac{P_2^2(\alpha_3+i\lambda)}{A^2(\beta+i\lambda)} \left(\int\limits_0^1 a_2(t)e^{-i\lambda t}dt + \frac{\alpha_3-\beta}{\alpha_2+\beta}\int\limits_0^1 a_2(t)e^{\alpha_2 t}dt \right).$$

The mean-square error of the optimal estimate can be calculated by the formula

$$\Delta(F,G) = \frac{P_2^2}{2\alpha_2}\int\limits_0^1 a_2^2(t)\,dt - \frac{P_2^4}{2\alpha_2 A^2(\alpha_2+\beta)^2}\times$$

$$\times\left((\alpha_3-\beta)^2\int\limits_0^1 a_2^2(t)(e^{2\alpha_2 t}-1)\,dt + (\alpha_2+\alpha_3)^2\int\limits_0^1 a_2^2(t)\,dt \right).$$

Let $\vec{a}(t) = (1-t\ ,\ e^{-at})$. In this case the spectral characteristic is of the form

$$h(\lambda) = \left(\frac{1}{i\lambda} + \frac{1-e^{-i\lambda}}{\lambda^2} + \frac{1-e^{-(a+i\lambda)}}{a+i\lambda} - q(\lambda)\ ,\ q(\lambda) \right),$$

where

$$q(\lambda) = \frac{P_2^2(\alpha_3+i\lambda)}{A^2(\beta+i\lambda)} \left(\frac{1-e^{-(a+i\lambda)}}{a+i\lambda} + \frac{\alpha_3-\beta}{\alpha_2+\beta}\frac{e^{\alpha_2-a}-1}{\alpha_2-a} \right).$$

The mean-square error of the optimal estimate can be calculated by the formula

$$\Delta(F,G) = \frac{P_2^2}{2\alpha_2}\frac{1-e^{-2a}}{2a} - \frac{P_2^4}{2\alpha_2 A^2(\alpha_2+\beta)^2}\left(\frac{1-e^{-2a}}{2a}(\alpha_2+\beta)(2\alpha_3+\alpha_2-\beta)+ \right.$$

$$+ (\alpha_3-\beta)^2\frac{e^{2(\alpha_2-a)}-1}{2(\alpha_2-a)}\Bigg).$$

Example 4. Consider the problem of estimation of the value $A\vec{\xi} = \int_0^\infty \vec{a}(t)\vec{\xi}(-t)\,dt$, where $\vec{a}(t) = (e^{-at}, e^{-bt})$, $b > \alpha_2$. In this case the spectral characteristic is of the form

$$h(\lambda) = \left(\frac{1}{i\lambda+a+b} - q(\lambda)\ ,\ q(\lambda)\right),$$

$$q(\lambda) = \frac{P_2^2(\alpha_3+i\lambda)}{A^2(\beta+i\lambda)}\left(\frac{1}{i\lambda+b} + \frac{\alpha_3-\beta}{(\alpha_2+\beta)(b-\alpha_2)}\right).$$

The mean-square error of the optimal estimate can be calculated by the formula

$$\Delta(F,G) = \frac{P_2^2}{4b\alpha_2} - \frac{P_2^4}{2\alpha_2A^2(\alpha_2+\beta)^2}\left(\frac{(\alpha_2+\alpha_3)^2}{2b} + \right.$$
$$\left. +(\alpha_3-\beta)^2\left(\frac{1}{2(b-\alpha_2)} - \frac{1}{2b}\right)\right).$$

3. Minimax-robust method of filtering

Formulas (l)-(10) may be used to determine the mean-square error and the spectral characteristic of the optimal linear estimate of the functional $A\vec{\xi}$ when the spectral density matrices $F(\lambda)$ and $G(\lambda)$ of multidimensional stationary stochastic processes $\vec{\xi}(t)$, $\vec{\eta}(t)$ are known. In the case where the spectral density matrices are unknown, but a set $D = D_F \times D_G$ of admissible spectral density matrices is given, the minimax-robust method of estimation of the unknown values of the functional $A\vec{\xi}$ is reasonable (see, for example, the survey article by S. A. Kassam and H. V. Poor (1985)). By means of this method it is possible to determine an estimate that minimizes the mean-square error for all spectral density matrices $F(\lambda)$, $G(\lambda)$ from the class $D = D_F \times D_G$ simultaneously.

Definition 3.1. Spectral density matrices $F^0(\lambda)$, $G^0(\lambda)$ are called the least favorable in the class $D = D_F \times D_G$ for the optimal linear filtering of the functional $A\vec{\xi}$ if the following relation holds true $\Delta(h(F^0,G^0);F^0,G^0) =$

$$= \max_{(F,G)\in D}\Delta(h(F,G);F,G) = \max_{F\in D}\min_{h\in L_2^-(F+G)}\Delta(h,F,G).$$

Definition 3.2. A spectral characteristic $h^0(\lambda)$ of the optimal linear estimate of the functional $A\vec{\xi}$ is called the minimax-robust in the class $D = D_F \times D_G$ if the conditions

$$h^0(\lambda) \in H_D = \bigcap_{(F,G)\in D} L_2^-(F+G),$$

$$\min_{h\in H_D}\max_{(F,G)\in D}\Delta(h;F,G)=\max_{(F,G)\in D}\Delta(h^0;F,G).$$

are satisfied.

Taking into account relations (1)–(10), it is possible to verify the following propositions.

Proposition 3.1 *The spectral density matrices $F^0(\lambda)\in D_F$ and $G^0(\lambda)\in D_G$ are the least favorable in the class $D=D_F\times D_G$ for the optimal linear filtering of the functional $A\vec{\xi}$ if the density matrices $F^0(\lambda)$, $G^0(\lambda)$ admit the canonical factorization (3)-(5) with functions $d(u),0\le u\le\infty$, $\psi(u),0\le u\le\infty$, $\varphi(u),0\le u\le\infty$, which are solutions to the conditional extremum problem*

$$\Delta(F,G)=\langle c_G,a\rangle-\|C_Gb^*\|^2\to\sup,\tag{11}$$

$$G(\lambda)=\left(\int_0^\infty\psi(u)e^{-iu\lambda}du\right)\cdot\left(\int_0^\infty\psi(u)e^{-iu\lambda}du\right)^*\in D_G,\tag{12}$$

$$F(\lambda)=\left(\left(\int_0^\infty d(u)e^{-iu\lambda}du\right)\cdot\left(\int_0^\infty d(u)e^{-iu\lambda}du\right)^*-G(\lambda)\right)\in D_F.\tag{13}$$

or the conditional extremum problem

$$\Delta(F,G)=\langle c_F,a\rangle-\|C_Fb^*\|^2\to\sup,\tag{14}$$

$$F(\lambda)=\left(\int_0^\infty\varphi(u)e^{-iu\lambda}du\right)\cdot\left(\int_0^\infty\varphi(u)e^{-iu\lambda}du\right)^*\in D_F,\tag{15}$$

$$G(\lambda)=\left(\left(\int_0^\infty d(u)e^{-iu\lambda}du\right)\cdot\left(\int_0^\infty d(u)e^{-iu\lambda}du\right)^*-F(\lambda)\right)\in D_G.\tag{16}$$

In the case where one of the spectral density matrices is fixed we have conditional extremum problems with respect to the function $b(u),0\le u\le\infty$. In this case the following lemmas hold true.

Proposition 3.2 *Let the spectral density matrix $G(\lambda)\in D_G$ be given. The spectral density matrix $F^0(\lambda)\in D_F$ is the least favorable in the class D_F for the optimal linear filtering of the functional $A\vec{\xi}$ if the density matrix $F^0(\lambda)+G(\lambda)$ admits the canonical factorization*

$$F^0(\lambda)+G(\lambda)=\left(\int_0^\infty d^0(u)e^{-iu\lambda}du\right)\cdot\left(\int_0^\infty d^0(u)e^{-iu\lambda}du\right)^*.$$

Here $d^0(u),0\le u\le\infty$, is determined by the function $b^0(u),0\le u\le\infty$, with the help of equation $b^0(\lambda)d^0(\lambda)=I_m$, $b^0(\lambda)=\int_0^\infty b^0(u)e^{-iu\lambda}du$, where

$b^0(u), 0 \leq u \leq \infty$, and $d^0(u), 0 \leq u \leq \infty$, gives a solution to the conditional extremum problem

$$\|C_G b^*\|^2 \to \inf, \tag{17}$$

$$F(\lambda) = \left(\int_0^\infty d(u)e^{-iu\lambda}du\right) \cdot \left(\int_0^\infty d(u)e^{-iu\lambda}du\right)^* - G(\lambda) \in D_F. \tag{18}$$

Proposition 3.3 *Let the spectral density matrix $F(\lambda) \in D_F$ be given. The spectral density matrix $G^0(\lambda) \in D_G$ is the least favorable in the class D_G for the optimal linear filtering of the functional $A\vec{\xi}$ if the density matrix $F(\lambda) + G^0(\lambda)$ admits the canonical factorization*

$$F(\lambda) + G^0(\lambda) = \left(\int_0^\infty d^0(u)e^{-iu\lambda}du\right) \cdot \left(\int_0^\infty d^0(u)e^{-iu\lambda}du\right)^*.$$

Here $d^0(u), 0 \leq u \leq \infty$, is determined by the function $b^0(u), 0 \leq u \leq \infty$, with the help of equation $b^0(\lambda)d^0(\lambda) = I_m$, $b^0(\lambda) = \int_0^\infty b^0(u)e^{-iu\lambda}du$, where $b^0(u), 0 \leq u \leq \infty$, and $d^0(u), 0 \leq u \leq \infty$, gives a solution to the conditional extremum problem

$$\|C_F b^*\|^2 \to \inf, \tag{19}$$

$$G(\lambda) = \left(\int_0^\infty d(u)e^{-iu\lambda}du\right) \cdot \left(\int_0^\infty d(u)e^{-iu\lambda}du\right)^* - F(\lambda) \in D_G. \tag{20}$$

The least favorable spectral density matrices $F^0(\lambda) \in D$ and $G^0(\lambda) \in D_G$ and the minimax-robust spectral characteristic $h^0(\lambda) \in H_D$ form a saddle point of the function $\Delta(h; F, G)$ on the set $H_D \times D$. The saddle point inequalities

$$\Delta(h; F^0, G^0) \geq \Delta(h^0; F^0, G^0) \geq \Delta(h^0; F, G),\ \forall (F, G) \in D,\ \forall h \in H_D$$

hold true if $h^0 = h(F^0, G^0) \in H_D$ and (F^0, G^0) give a solution to the conditional extremum problem

$$\Delta(h(F^0, G^0); F^0, G^0) = \sup_{(F,G)\in D} \Delta(h(F^0, G^0); F, G), \tag{21}$$

where

$$\Delta(h(F^0, G^0); F, G) = \frac{1}{2\pi}\int_{-\infty}^{\infty} r_G(\lambda)b^0(\lambda)F(\lambda)(b^0(\lambda))^*(r_G(\lambda))^*d\lambda +$$

$$+\frac{1}{2\pi}\int_{-\infty}^{\infty} r_F(\lambda)b^0(\lambda)G(\lambda)(b^0(\lambda))^*(r_F(\lambda))^*d\lambda.$$

The functions $r_F(\lambda)$, $r_G(\lambda)$ are calculated by formulas (8), (10) with $F(\lambda) = F^0(\lambda)$, $G(\lambda) = G^0(\lambda)$.

This conditional extremum problem is equivalent to the unconditional extremum problem

$$\Delta_D(F, G) = -\Delta(h(F^0, G^0); F, G) + \delta((F, G)|D) \to \inf, \qquad (22)$$

where $\delta((F, G)|D)$ is the indicator function of the set $D = D_F \times D_G$. A solution to this problem is determined by the condition $0 \in \partial\Delta_D(F^0, G^0)$, where $\partial\Delta_D(F^0, G^0)$ is the subdifferential of the convex functional $\partial\Delta_D(F, G)$ at the point (F^0, G^0).

4. Least favorable spectral densities in the class $D_{0,0}$

Consider the problem of minimax filtering for the set of spectral density matrices

$$D_{0,0} = \left\{ (F(\lambda), G(\lambda)) | \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\lambda) d\lambda = P_1, \ \frac{1}{2\pi} \int_{-\infty}^{\infty} G(\lambda) d\lambda = P_2 \right\}.$$

With the help of the Lagrange multipliers method we can find the following relations that determine the least favorable spectral density matrices $(F^0(\lambda), G^0(\lambda)) \in D_{0,0}$:

$$r_G(\lambda) b^0(\lambda) (b^0(\lambda))^* (r_G(\lambda))^* = \vec{\alpha} \cdot \vec{\alpha}^*,$$

$$r_F(\lambda) b^0(\lambda) (b^0(\lambda))^* (r_F(\lambda))^* = \vec{\beta} \cdot \vec{\beta}^*.$$

Here $\vec{\alpha} = (\alpha_1, \ldots, \alpha_T)^T$, $\vec{\beta} = (\beta_1, \ldots, \beta_T)^T$ are the Lagrange multipliers. It follows from these relations that the least favorable density matrices are such that

$$F^0(\lambda) + G^0(\lambda) = \vec{\gamma} \left(\int_0^{\infty} (C_G b^*)(t) e^{-it\lambda} dt \right) \cdot \left(\int_0^{\infty} (C_G b^*)(t) e^{-it\lambda} dt \right)^* \vec{\gamma}^*, \quad (23)$$

$$F^0(\lambda) + G^0(\lambda) = \vec{\delta} \left(\int_0^{\infty} (C_F b^*)(t) e^{-it\lambda} dt \right) \cdot \left(\int_0^{\infty} (C_F b^*)(t) e^{-it\lambda} dt \right)^* \vec{\delta}^*. \quad (24)$$

The unknown $\vec{\beta} = (\beta_1, \ldots, \beta_T)^\top$, $\vec{\delta} = (\delta_1, \ldots, \delta_T)^\top$, $b = \{b(u) : u \geq 0\}$ are calculated with the help of the canonical factorization equations (3)-(5) of the spectral density matrices $F^0(\lambda)$, $G^0(\lambda)$, $F^0(\lambda) + G^0(\lambda)$ and conditions

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} F(\lambda) d\lambda = P_1, \quad \frac{1}{2\pi} \int_{-\infty}^{\infty} G(\lambda) d\lambda = P_2. \qquad (25)$$

In the particular case where one of the spectral density matrices is fixed we may use only one of these relations. If the spectral density matrix $G(\lambda)$ is given, then the least favorable density matrix $F^0(\lambda) \in D_0$ is of the form $F^0(\lambda) =$

$$\max\left\{\vec{\gamma}\left(\int_0^{\infty}(C_G b^*)(t)e^{-it\lambda}dt\right)\left(\int_0^{\infty}(C_G b^*)(t)e^{-it\lambda}dt\right)^* \vec{\gamma}^* - G(\lambda), 0\right\}. \quad (26)$$

If the spectral density matrix $F(\lambda)$ is given, then the least favorable density matrix $G^0(\lambda) \in D_0$ is of the form $G^0(\lambda) =$

$$\max\left\{\vec{\delta}\left(\int_0^{\infty}(C_F b^*)(t)e^{-it\lambda}dt\right)\left(\int_0^{\infty}(C_F b^*)(t)e^{-it\lambda}dt\right)^* \vec{\delta}^* - F(\lambda), 0\right\}. \quad (27)$$

The unknown $\vec{\gamma}, \vec{\delta}, b(u)$, are calculated with the help of the canonical factorization equations (3)-(5) of the spectral density matrices $F^0(\lambda)$, $G(\lambda)$, $F^0(\lambda) + G(\lambda)$ (or $F(\lambda)$, $G^0(\lambda)$, $F(\lambda) + G^0(\lambda)$) and conditions (25).

The preceding reasonings show us that the following theorem holds true.

Theorem 4.1. *The least favorable density matrices $F^0(\lambda)$, $G^0(\lambda)$ in the class $D_{0,0}$ for the optimal linear estimation of the functional $A\vec{\xi}$ are determined by relations (23), (24), (3)-(5), (11)–(16), (25). If the spectral density matrix $F(\lambda)$ ($G(\lambda)$) is given, then the least favorable density matrix $G^0(\lambda) \in D_0$ ($F^0(\lambda) \in D_0$) is determined by relations (26), (3)-(5), (17), (18), (25) (or (27), (3)-(5), (19), (20), (25)). The minimax spectral characteristic $h(F)$ of the optimal linear estimate of the functional $A\vec{\xi}$ is calculated by formulas (8), (10).*

5. Least favorable spectral densities in the class $D_V^U \times D_\varepsilon$

Consider the problem of minimax estimation of the functional $A\vec{\xi}$ under the condition that spectral density matrices $(F(\lambda), G(\lambda))$ of the multidimensional stationary processes $\vec{\xi}(t)$, $\vec{\eta}(t)$ are from the set of spectral density matrices $D_V^U \times D_\varepsilon$, where

$$D_V^U = \left\{F(\lambda)\middle| V(\lambda) \leq F(\lambda) \leq U(\lambda), \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\lambda)d\lambda = P_1\right\},$$

$$D_\varepsilon = \left\{G(\lambda)\middle| G(\lambda) = (1-\varepsilon)G_1(\lambda) + \varepsilon W(\lambda),\ \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\lambda)d\lambda = P_2\right\},$$

where $V(\lambda),\ U(\lambda),\ G_1(\lambda)$ are given fixed spectral density matrices, $W(\lambda)$ ia an unknown spectral density matrix, and expression $B(\lambda) \geq D(\lambda)$ means that $B(\lambda) - D(\lambda) \geq 0$ (positive definite matrix function). The set D_V^U describes the 'band' model of stochastic processes while the set D_ε describes the ε-contamination model of stochastic processes. For the set $D_V^U \times D_\varepsilon$ from the condition $0 \in \partial\Delta_D(F^0, G^0)$ we can get the following relations which determine the least favorable spectral density matrices

$$r_G(\lambda)b^0(\lambda)(b^0(\lambda))^*(r_G(\lambda))^* = \vec{\alpha}\cdot\vec{\alpha}^* + \Gamma_1(\lambda) + \Gamma_2(\lambda); \tag{28}$$

$$r_F(\lambda)b^0(\lambda)(b^0(\lambda))^*(r_F(\lambda))^* = \vec{\beta}\cdot\vec{\beta}^* + \Gamma_3(\lambda). \tag{29}$$

The coefficients $\vec{\alpha} = (\alpha_1, \ldots, \alpha_T)^T$, $\vec{\beta} = (\beta_1, \ldots \beta_T)^T$, the matrix function $b^0(\lambda)$, the unknown functions $r_F(\lambda), r_G(\lambda)$ are calculated with the help of the canonical factorization equations (3)-(5) of the spectral density matrices $F^0(\lambda)$, $G^0(\lambda)$, $F^0(\lambda) + G^0(\lambda)$ and conditions

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} F(\lambda)d\lambda = P_1, \quad \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\lambda)d\lambda = P_2. \tag{30}$$

The matrix functions $\Gamma_1(\lambda) \geq 0,\ \Gamma_2(\lambda) \geq 0,\ \Gamma_3(\lambda) \geq 0$ are determined by the conditions

$$V(\lambda) \leq F^0(\lambda) \leq U(\lambda), \quad G^0(\lambda) = (1-\varepsilon)G_1(\lambda) + \varepsilon W(\lambda), \tag{31}$$

$$\Gamma_1(\lambda) = 0 \text{ if } F^0(\lambda) \geq V(\lambda), \quad \Gamma_2(\lambda) = 0 \text{ if } F^0(\lambda) \leq U(\lambda), \tag{32}$$

$$\Gamma_3(\lambda) = 0 \text{ if } G^0(\lambda) \geq (1-\varepsilon)G_1(\lambda). \tag{33}$$

In the case where the spectral density matrix $G(\lambda)$ that admits the canonical factorization is given, the least favorable in the class $D = D_V^U$ spectral density matrix $F^0(\lambda)$ is of the form

$$F^0(\lambda) = \min\left\{U(\lambda),\ \max\left\{V(\lambda), \vec{\gamma}\, r_G(\lambda)(r_G(\lambda))^*\vec{\gamma}^* - G(\lambda)\right\}\right\}. \tag{34}$$

In the case where the spectral density matrix $F(\lambda)$ that admits the canonical factorization is given, the least favorable in the class D_ε spectral density matrix $G^0(\lambda)$ is of the form

$$G^0(\lambda) = \max\left\{(1-\varepsilon)G_1(\lambda),\ \vec{\delta}\, r_F(\lambda)(r_F(\lambda))^*\vec{\delta}^* - F(\lambda)\right\}. \tag{35}$$

In both cases $\vec{\gamma} = (\gamma_1, \ldots, \gamma_T)^T$, $\vec{\delta} = (\delta_1, \ldots, \delta_T)^T$, $b(u),\ r_F(\lambda),\ r_G(\lambda)$ are determined by the factorization of the densities $F^0(\lambda),\ G(\lambda),\ F^0(\lambda) + G(\lambda)$ (or $F(\lambda),\ G^0(\lambda),\ F(\lambda) + G^0(\lambda)$).

The preceding reasonings show us that the following theorem holds true.

Theorem 5.1. *The least favorable density matrices $F^0(\lambda)$, $G^0(\lambda)$ in the class $D_V^U \times D_\varepsilon$ for the optimal linear estimation of the functional $A\vec{\xi}$ are determined by relations (3)-(5), (11)–(16), (28)–(33). If the spectral density matrix $F(\lambda)$ $(G(\lambda))$ is given, then the least favorable density matrix $G^0(\lambda) \in D_0$ $(F^0(\lambda) \in D_0)$ is determined by relations (34), (3)-(5), (17), (18), (30) (or (35), (3)-(5), (19), (20), (30)). The minimax spectral characteristic $h(F)$ of the optimal linear estimate of the functional $A\vec{\xi}$ is calculated by formulas (8), (10).*

6. Conclusions

We propose formulas for calculation the mean square errors and the spectral characteristic of the optimal linear estimate of the unknown value of the functional $A\vec{\xi} = \int_0^\infty \vec{a}(t)\vec{\xi}(-t)\,dt$ which depends on the unknown values of a multidimensional stationary stochastic process $\vec{\xi}(t)$ based on observations of the process $\vec{\xi}(t) + \vec{\eta}(t)$ for $t \leq 0$ under the condition that spectral density matrix $F(\lambda)$ and spectral density matrix $G(\lambda)$ of the noise process $\vec{\eta}(t)$ are known. Formulas are proposed that determine the least favorable spectral densities and the minimax-robust spectral characteristic of the optimal estimate of the functional for concrete classes $D = D_F \times D_G$ of spectral densities under the condition that spectral density matrices $F(\lambda)$ and $G(\lambda)$ are not known, but classes $D = D_F \times D_G$ of possible spectral densities are given.

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