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THE FOKKER–PLANCK–KOLMOGOROV EQUATIONS FOR SOME DEGENERATE DIFFUSION PROCESSES

We clarify the connection between diffusion processes and partial differential equations of the parabolic type. The emphasis is on degenerate parabolic equations. These equations are a generalization of the classical Kolmogorov equation of diffusion with inertia which may be treated as the Fokker-Planck-Kolmogorov equations for the respectively degenerate diffusion processes. The basic results relating to the fundamental solution and the correct solvability of the Cauchy problem are presented.

1. Diffusion processes (DPs) and partial differential equations (PDE)

DPs occupy, probably, a central place in the theory of Markov processes. This can be explained by a few reasons.

Firstly, DPs are exact enough models of important physical processes. In particular, we mention a model of diffusible particle in liquids — the phenomenon opened by the English botanist R. Brown in 1828.

Secondly, just DP is a binding link between the theory of stochastic processes and the theory of PDE. Thus, it follows to mark the reciprocity of influences of these theories: while studying the properties of DP, the analytical results from the theory of PDE, and vice versa, while studying the Cauchy problem, the boundary-value problems for PDE probabilistic methods are used.

One of the important problems of DP theory is the development of methods of DP construction at the known diffusion coefficients: the diffusion matrix b, drift vector a, and coefficient of absorption c. If the process isn't torn off, then $c = 0$.

From the point on the phenomenon of diffusion, the drift vector a is a macroscopic velocity of the liquid, and the diffusion matrix b characterizes the random movement of a particle as a result of collisions with molecules of liquid which are in a thermal motion. Which conditions should be satisfied by a vector function a and a matrix function b characteristic of DP? What does the fact of existence of local characteristics of motion for some DP mean?

A.N. Kolmogorov was the first who gave a solution to this problem in 1931. In his work [1], he selected the class of continuous Markov processes which later got the name of DP. By definition, a process which has the value in $\mathbb{R}^m$ and the transition density $P(s,\, x,\, t,\, \Gamma)$, $0 \le s < t$, $x \in \mathbb{R}^m$, where Γ is a Borel subset of $\mathbb{R}^m$, is called DP if the following conditions are satisfied:

1) for any $\varepsilon > 0$,

$$(1) \qquad \lim_{\Delta t \to 0} \frac{1}{\Delta t} \int_{\{|y-x|>\varepsilon\}} P(t,\, x,\, t+\Delta t,\, dy) = 0,$$

2) there exist the functions a and b such that, for some $\varepsilon > 0$,

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$$\begin{aligned}&\lim_{\Delta t\to 0}\frac{1}{\Delta t}\int_{\{|y-x|<\varepsilon\}}(y-x)\,P\,(t,\,x,\;t+\Delta t,\;dy)=a\,(t,\,x)\,,\\ &\lim_{\Delta t\to 0}\frac{1}{\Delta t}\int_{\{|y-x|<\varepsilon\}}(y-x,\,\theta)^2\,P\,(t,\,x,\;t+\Delta t,\;dy)=(b\,(t,\,x)\,\theta,\,\theta)\;,\end{aligned} \tag{2}$$

where θ is an arbitrary vector in $\mathbb{R}^m$.

The functions a and b defined by formulas (2) are local characteristics of DP. Obviously, the matrix b is nonnegative definite. In the mentioned work, A.N. Kolmogorov showed that DP are closely connected with the second-order PDE of the parabolic type. Namely, if a function $P\,(s,\,x,\,t,\,\Gamma)$ determines the transition density to DP with the drift vector a and the diffusion matrix b, then, under such conditions, the function

$$u\,(s,\,x):=\int_{\mathbb{R}^m}\varphi\,(y)\,P\,(s,\,x,\,t,\,dy), \tag{3}$$

is the solution in a layer $\{\,(s,\,x)|\;s\in[0,\,t)\,,\;\;x\in\mathbb{R}^m\}$ for the equation

$$\partial_s u+\frac{1}{2}\sum_{j,\,k=1}^{m}b_{jk}\,(s,\,x)\,\partial_{x_j}\partial_{x_k}u+\sum_{j=1}^{m}a_j\,(s,\,x)\partial_{x_j}u=0 \tag{4}$$

with the initial condition

$$\lim_{s\to t}u\,(s,\,x)=\varphi\,(x)\,. \tag{5}$$

Here, $b_{jk}\,(s)$ are the elements of the matrix b, a_j are coordinates of a vector a, and x_j are coordinates of a vector x.

Except the inverse equation (4), the so-called direct equation is obtained in that work as well. Both equations are PDE of the parabolic type.

Let us pay attention to the fact that direct equations for some special cases have been got by the physicists Fokker and Planck a bit earlier, who studied the phenomenon of diffusion. That's why it is also called the Fokker–Planck equation.

Kolmogorov's result showed the way, following which one could hope to find a solution to the problem of DP construction with given diffusion coefficients. In this way, a lot of important analytical results connected with the study of Cauchy problem (4), (5) was obtained. First of all, it is:

– existence of a fundamental solution of the Cauchy problem (FSCP);
– properties of the FSCP:
  • estimates of the FSCP and its derivatives,
  • positivity,
  • normality ,
  • the convolution formula;
– properties of the Poisson and volume potentials;
– conditions for the existence of integral representations of solutions;
– correct solvability for the Cauchy problem in the corresponding functional spaces.

It is a natural to expect that such results can be obtained at minimal assumptions about the diffusion coefficients.

This definition of DP is based only on the transition density function. In a more advanced theory of Markov processes [2], DP in $\mathbb{R}^m$ is a continuous strictly Markov process, whose characteristic operator A has the following property: $(Af)\,(x)$ is defined for all functions f which are given and twice continuously differentiable in any neighborhood of $x\in\mathbb{R}^m$. The characteristic operator A of DP on the set of twice continuously differentiable functions coincides with the elliptic operator L which is defined by the formula

$$(Lf)(x) := \frac{1}{2}\sum_{j,\,k=1}^{m} b_{jk}(x)\partial_{x_j}\partial_{x_k} f + \sum_{j=1}^{m} a_j(x)\partial_{x_j} f + c(x) f.$$

The operator L is called a generating differential operator for DP.

Together with the elaboration of analytical methods, the direct stochastic methods of DP construction were developed as well. While using such an approach, the path of a process satisfies the stochastic differential equation

$$dx(t) = a(t,\ x(t))\ dt + B(t,\ x(t))\ dW(t),$$

where W is the m-dimensional Wiener process, and B is such m-order square matrix that $b = BB^*$.

Many scientists studied the development and the usage of analytical methods of DP construction according to their local characteristics under different assumptions. Among them, we mention such scientists as N. Wiener, A.N. Kolmogorov, V. Feller, M.V. Krylov, A. Tanaka, I.I. Gikhman, A.V. Skorokhod, and M.I. Portenko. Because of their efforts, not only different theorems on the DP existence where proved under general assumptions about the diffusion coefficients, but also different generalizations of the very notion for DP were obtained. Thus, in monograph [3], the processes with a local unbounded drift vector and a sufficiently smooth diffusion matrix were constructed. Such processes the author call the generalized DP.

Note that Eq. (4), which is a starting point in the analytical approach, is the uniformly parabolic equation (in the sense of the theory of PDE), so it is not degenerate as well.

2. Degenerate parabolic Kolmogorov-type equations

These equations are a natural generalization of the classical Kolmogorov equation of diffusion with inertia. This equation appears in the study of models for the Brownian motion. In the classical theory of the Brownian motion developed by Einstein and Smoluchowski [4], [5], the inertia of a Brownian particle is neglected, i.e., the mass of a particle is actually assumed equal to 0. Therefore, a Brownian particle in the Einstein–Smoluchowski theory cannot have a finite velocity. The Brownian motion of a physical system in the Einstein–Smoluchowski approximation is a continuous Markov process in the coordinate space (the Wiener process, for the case of a free particle).

The fact of the non-differentiability of Brownian paths in the Einstein–Smoluchowski theory is closely related to the idealization made in this theory (the neglect of inertia) making it invalid on small time intervals. For the simplest case of the Brownian motion of a free particle, a theory taking inertia into account was developed as early as 1930 by Uhlenbeck and Ornstein [6]. In this more precise theory, paths are already differentiable (but do not have the second derivative, so that now the acceleration becomes infinite).

In fact, the same generalization is contained in paper [7] by A.N. Kolmogorov. He considered the general case of the Brownian motion for an arbitrary physical system with n degrees of freedom. According to Kolmogorov, inertia is taken into account if a state of the system is described by values of n coordinates $q_1, \dots, q_n$ and n their time derivatives (velocities) $\dot{q}_1, \dots, \dot{q}_n$. Here, the model of the Brownian motion is a continuous Markov process in the $2n$- dimensional phase space of coordinates and velocities.

In paper [7], it was assumed that, whenever we know the values of $q := (q_1, \dots, q_n)$ and $\dot{q} := (\dot{q}_1, \dots, \dot{q}_n)$ at an instant of time t, we can find the probability density

$$G(t, q, \dot{q}; t', q', \dot{q}')$$

of possible values q and $\dot{q}$ of the coordinates and their time derivatives at an arbitrary instant $t' > t$. It is assumed that G does not depend on the behavior of the system before the instant t (there is no aftereffect, the process is of the Markov type). It is proved that the function G is a FSCP for the Fokker–Planck differential equation

$$\partial_{t'} g = -\sum_{j=1}^{n} \dot{q}'_j \partial_{q'_j} g - \sum_{j=1}^{n} \partial_{\dot{q}'_j} \left(f_j(t', q', \dot{q}') g \right) + \sum_{j,l=1}^{n} \partial_{\dot{q}'_j} \partial_{\dot{q}'_l} \left(k_{jl}(t', q', \dot{q}') g \right). \tag{6}$$

If $n = 1$, then this equation has the form

$$\partial_{t'} g = -\dot{q} \partial_{q'} g - \partial_{\dot{q}'} \left(f\left(t',\, q',\, \dot{q}'\right) g \right) + \partial^2_{\dot{q}'} \left(k\left(t',\, q',\, \dot{q}'\right) g \right). \tag{7}$$

If f and k are constants, then, as A.N. Kolmogorov showed, a FSCP for Eq. (7) is given by the formula

$$\begin{gathered} G\left(t,\, q,\, \dot{q};\, t',\, q',\, \dot{q}'\right) = \\ = 2\sqrt{3}\pi^{-1} k^{-2} \left(t' - t\right)^{-2} \exp\Big\{ -\left(4k\left(t' - t\right)\right)^{-1} \left(\dot{q}' - \dot{q} - f\left(t' - t\right)\right)^2 - \\ -3k^{-1}\left(t' - t\right)^{-3} \left(q' - q - 2^{-1}\left(\dot{q}' + \dot{q}\right)\left(t' - t\right)\right)^2 \Big\}, \\ t < t', \;\; \{q,\, \dot{q},\, q',\, \dot{q}'\} \subset \mathbb{R}. \end{gathered}$$

This is a classical formula for a FSCP for Kolmogorov's equation of diffusion with inertia.

Equations (6) and (7) are a prototype for the family of evolution equations arising in the kinetic theory of gases that take the following general form:

$$Su = I\left(u\right). \tag{8}$$

Here, the function $\mathbb{R}^{2n} \ni x \mapsto u\left(x,\, t\right) \in \mathbb{R}$ is the density of particles which have velocity $x_1, \ldots, x_n$ and position $x_{n+1}\,, \ldots,\, x_{2n}$ at the time t;

$$Su := \partial_t u + \sum_{j=1}^{n} x_j \partial_{x_{n+j}} u$$

is the so-called total derivative of u, and $I\left(u\right)$ describes some kind of collision. This last term $I\left(u\right)$ can take different forms, either linear or nonlinear. For instance, in the usual Fokker–Planck equation, we have

$$I\left(u\right) = -\sum_{j,\,l=1}^{n} a_{jj} \partial^2_{x_j x_l} u + \sum_{j=1}^{n} a_j \partial_{x_j} u + au$$

.

The term $I\left(u\right)$ may also occur in the divergence form

$$I\left(u\right) = -\sum_{j,l=1}^{n} \partial_{x_j} \left(a_{jl}(t,\, x) \partial_{x_j} u + b_j(t,\, x) u \right) + \sum_{j=1}^{n} a_j(t,\, x) \partial_{x_j} u + c(t,\, x) u.$$

The nonlinear collision operator in the Fokker–Planck–Landau equation has the form

$$I\left(u\right) = \sum_{j.l=1}^{n} \partial_{x_j} \left(a_{jl}\left(z,\, u\right) \partial_{x_l} u + b_j\left(z,\, u\right) \right).$$

While studying mathematical models of options, a model Markov type appears in the case where the dynamics is defined by a stochastic differential equation in the N-dimensional space of states

$$dx(t) = \left(Bx(t) + b\left(t,\, x(t)\right)\right) dt + \sigma\left(t,\, x(t)\right) dW(t), \tag{9}$$

where W is a d-dimensional Wiener process, $d \leq N$, $B := (b_{jl})$ is an $N \times N$ matrix which has constant real entries, σ is an $N \times d$ matrix, and the vector $b := (b_1, \ldots, b_N)$ is such that $0 = b_{d+1} = \cdots = b_N$.

Under some assumptions about the matrices σ, B, and b, the transition density for Eq. (9) is a FSCP for the equation

$$\begin{aligned} &\frac{1}{2}\sum_{j,l=1}^{d} a_{jl}\,(t,\,x)\,\partial_{x_j}\partial_{x_l}u + \sum_{j,l=1}^{N} b_{jl}x_l\partial_{x_l}u + \\ &\qquad + \sum_{j=1}^{d} b_j\,(t,\,x)\,\partial_{x_j}u + \partial_t u = 0\,. \end{aligned} \tag{10}$$

Many works have been recently devoted to the study of mathematical models of options (see a review article [8], where a wide survey of the related literature is presented). In these works, different problems connected with such models are investigated by analytical, probabilistic, and numerical methods.

Differential equations with a nonlinear total derivative term of the form

$$\Delta_x u + \partial_y g\,(u) - \partial_t u = f \tag{11}$$

have been considered for convection-diffusion models and for pricing models of options with memory feedback. The linearized equation (11) can be reduced, if the derivative of the function g is different from zero and smooth enough, to the Kolmogorov equation.

Equations (6)–(8), (10), and their generalization are degenerate equations of the parabolic type. Such equations belong to a class of ultraparabolic or elliptic-parabolic equations. They are called the Fokker–Planck–Kolmogorov equations for the respectively degenerate diffusion processes.

The development of the theory of ultraparabolic equations of the Kolmogorov type in subsequent investigations is aimed at finding as weaker conditions as possible for the existence of a FSCP, obtaining its precise estimates, and considering equations with a more complicated structure. The investigation of the FSCP and the correct solvability of the Cauchy problem for degenerate parabolic equations of the Kolmogorov type under different assumptions for coefficients of equations was executed by M. Weber, A.M. Il'in, I.M. Sonin, Ya.I. Shatyro, L.P. Kuptsov, S.D. Eidelman, A.P. Malitska, L.M. Tychinska. S.D. Ivasyshen, L.N. Androsova, and V.S. Dron' (see monograph [9]).

Equations of the Kolmogorov type which have the form

$$\begin{aligned} Lu :\;=\; &\sum_{j,\,l=1}^{p_0} a_{jl}\,(t,\,x)\;\partial_{x_j}\partial_{x_l}u + \sum_{j=1}^{p_0} a_j\,(t,\,x)\;\partial_{x_j}u + c\,(t,\,x)\;u + \\ &+ \sum_{j,\,l=1}^{N} b_{jl}x_j\,\partial_{x_l}u - \partial_t u = 0\,, \end{aligned} \tag{12}$$

were investigated in the series of papers of Italian mathematicians (see the review article [10]). Here, $1 \leq p_0 < N$, the matrix $A_0 := (a_{jl})_{j,\,l=1}^{p_0}$ is symmetric and positive definite, and $B := (b_{jl})_{j,\,l=1}^{N}$ is constant real matrix taking the block form

$$\begin{pmatrix} * & B_1 & 0 & \ldots & 0 \\ * & * & B_2 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ * & * & * & \ldots & B_r \\ * & * & * & \ldots & * \end{pmatrix}.$$

Here B_j-blocks are $p_{j-1} \times p_j$ matrices of rank p_j, where $p_0,\ p_1, \dots,\ p_r$ are positive integers such that $p_0 \geq p_1 \geq \dots \geq p_r \geq 1$, $p_0 + p_1 + \dots + p_r = N$, and the blocks denoted by $*$ are arbitrary.

Note that, under the conditions imposed upon B, the well-known Hörmander's hypoellipticity condition is satisfied by the operator L with the coefficients in (12) which are fixed at any point $(t,\ x)$.

It is well known [2], [11] that a nondegenerate DP with sufficiently regular characteristics possesses a smooth transition density. In general, a degeneration of the diffusion matrix leads to the nonexistence of the transition probability density. However, there exist some classes of degenerate processes with smooth transition densities. As was mentioned above, the processes of the Brownian motion with inertia considered by A.N. Kolmogorov, A.M. Il'in, R.Z. Khasminsky, and others are among such processes.In paper [12], I.M. Sonin studied a natural generalization of the process of diffusion with inertia. For the class of processes considered, the transition densities are constructed as a FSCP for the equations

$$\begin{aligned} \partial_t u = \Big(\Big(\tfrac{1}{2} \textstyle\sum_{j,\, l=1}^{n} a_{jl}\,(t,\ x,\ y,\ z)\,\partial_{y_j}\partial_{y_l} + \sum_{j=1}^{n} a_j\,(t,\ x,\ y,\ z)\,\partial_{y_j} + \\ + \textstyle\sum_{j=1}^{n} b_j\,(t,\ x,\ y,\ z)\,\partial_{x_j} + \sum_{j=1}^{n} c_j\,(t,\ x,\ y,\ z)\,\partial_{z_j}\Big)\, u, \quad \{x,\ y,\ z\} \subset \mathbb{R}^n\,, \end{aligned} \tag{13}$$

where $b_j\,(t,\ x,\ y,\ z)$ behave "approximately" like y_j, and $c_j\,(t,\ x,\ y,\ z)$ as x_j. The existence of a FSCP for Eq. (13) under condition that the coefficients of the equation are sufficiently smooth functions is proved.

Note that the results of construction and investigation of properties of a FSCP for equations which have form (13) under weaker conditions for their coefficients are contained in monograph [9]. Some novel results about the FSCP and the correct solvability of the Cauchy problem for equations like (13) are presented in the following sections.

3. FSCP for the Fokker–Planck–Kolmogorov equations of degenerate diffusion processes

The main object of our study in this section, the FSCP for some degenerate parabolic equations, can be interpreted in a natural way as the transition density of a respective diffusion process with a value from the n-dimensional phase space $\mathbb{R}^n$ of points x with three different groups of phase coordinates $x_s := (x_{s1}, \dots, x_{sn_s}) \in \mathbb{R}^{n_s}$, $s \in \{1, 2, 3\}$, $n = n_1 + n_2 + n_3$, and $1 \leq n_3 \leq n_2 \leq n_1$.

We consider the equation with real-valued coefficients

$$\begin{aligned} (S - \textstyle\sum_{j,\, l=1}^{n_1} a_{jl}\,(t,\ x)\,\partial_{x_{1j}}\partial_{x_{1l}} - \sum_{j=1}^{n_1} a_j\,(t,\ x)\,\partial_{x_{1j}} - \\ - a_0\,(t,\ x))u(t,\ x) = f(t,\ x), \qquad (t,\ x) \in \Pi_{(0,\, T]}\ \ , \end{aligned} \tag{14}$$

and the corresponding adjoint equation

$$\begin{aligned} S^* v\,(\tau,\ \xi)\ - \textstyle\sum_{j,l=1}^{n_1} \partial_{\xi_{1j}}\partial_{\xi_{1l}}\,(a_{jl}\,(\tau,\ \xi)\ v\,(\tau,\ \xi)) + \sum_{j=1}^{n_1} \partial_{\xi_{1j}}\,(a_j\,(\tau,\ \xi)\ v\,(\tau,\ \xi)) - \\ - a_0\,(\tau,\ \xi)\ v\,(\tau,\ \xi) = g\,(\tau,\ \xi)\,, \qquad (\tau,\ \xi) \in \Pi_{[0,\, T)}\,, \end{aligned} \tag{15}$$

where $\Pi_H := \{(t,\ x)|\, t \in H,\ x \in \mathbb{R}^n\}$, if $H \subset \mathbb{R}$; T is a given positive number; S is a differential expression determined as

$$S := \partial_t - \sum_{j=1}^{n_2} x_{1j}\partial_{x_{2j}} - \sum_{j=1}^{n_3} x_{2j}\partial_{x_{3j}}$$

or the Lie derivative of a vector field, and S^* is the expression adjoint to S.

Before making assumption on the coefficients of Eq. (14), we introduce the following notation:

$$d\,(x;\,\xi) := \sum_{s=1}^{3} |x_s - \xi_s|^{1/(2s-1)}\,, \quad d\,(t,\,x;\,\tau,\,\xi) := |t-\tau|^{1/2} + d\,(x;\,\xi)$$

are the distances between points x and ξ, $(t,\,x)$ and $(\tau,\,\xi)$, respectively;

$\Delta_x^{\xi} f\,(\cdot,\,x) := f\,(\cdot,\,x) - f\,(\cdot,\,\xi)\,,\ \Delta_{t,\,x}^{\tau,\,\xi} f(t,\,x,\,\cdot) := f(t,\,x,\,\cdot) - f(\tau,\,\xi,\,\cdot)$;

$X\,(t) := (X_1\,(t)\,, X_2\,(t)\,, X_3\,(t))\,,\ X_1\,(t) := x_1,\ X_s\,(t) := (X_{s1}\,(t)\,,\ldots,\,X_{sn_s}\,(t)),\ s \in \{2,3\},\ X_{2j}\,(t) := x_{2j} + tx_{1j},\ \ X_{3j}\,(t) := x_{3j} + tx_{2j} + 2^{-1}t^2x_{1j}$.

A function $f(t,\,x)$, $(t,\,x) \in \Pi_{[0,\,T]}$, is said to be Hölder continuous in the exponent $\alpha \in (0,1)$ in $\Pi_{[0,\,T]}$ if there exist a constant $C > 0$ such that

$$|\Delta_{t,\,x}^{\tau,\,\xi} f(t,\,x)| \le C(d(t, X(t-\tau); \tau, \xi))^{\alpha},$$

for all $\{(t,\,x),\,(\tau,\,\xi)\} \subset \Pi_{[0,\,T]}$.

For real-valued coefficients of Eq. (14), we use the following conditions:

A_1) there exist a constant $\delta > 0$ such that, for any $(t,\,x) \in \Pi_{[0,\,T]}$ and $\sigma_1 \in \mathbb{R}^{n_1}$, the following inequality is valid:

$$\sum_{j,\,l=1}^{n_1} a_{jl}\,(t,\,x)\ \sigma_{1j}\sigma_{1l} \ge \delta\,|\sigma_1|^2\,;$$

A_2) the coefficients a_{jl}, a_j, and a_0 are bounded and Hölder continuous in the exponent $\alpha \in (0,1)$in $\Pi_{[0,\,T]}$;

A_3) there exist the derivatives $\partial x_{1j}\,\partial x_{1l}\,a_{jl}$, and $\partial x_{1j}\,a_j$ bounded and Hölder continuous in the exponent $\alpha \in (0,1)$in $\Pi_{[0,\,T]}$

The following theorem is valid.

Theorem 1. *If conditions A_1, A_2 are satisfied, then there exists a FSCP Z for Eq. (14) with the estimates*

$$\left|\partial_{x_1}^{k_1} Z\,(t,\,x;\,\tau,\,\xi)\right| \le C\,(t-\tau)^{-M-|k_1|/2}\,E_c\,(t,\,x;\,\tau,\,\xi)\,,\ \ |k_1| \le 2,$$

$$|SZ\,(t,\,x;\,\tau,\,\xi)| \le C\,(t-\tau)^{-M-1}\,E_c\,(t,\,x;\,\tau,\,\xi)\,,$$

$$|\Delta_x^{x'}\,\partial_{x_1}^{k_1} Z(t,\,x; \tau,\,\xi)| \le C(d(x;x')^{\alpha}(t-\tau)^{-M-(|k_1|+\alpha)/2} \times$$
$$\times(E_c(t,\,x; \tau,\,\xi) + E_c(t,\,x'; \tau,\,\xi)), \quad |k_1| \le 2,$$

$$|\Delta_x^{x'}\,SZ(t,\,x; \tau,\,\xi)| \le C(d(x;x')^{\alpha}(t-\tau)^{-M-1-\alpha/2} \times$$
$$\times(E_c(t,\,x; \tau,\,\xi) + E_c(t,\,x'; \tau,\,\xi)),$$

$$|\int_{\mathbb{R}^n} \partial_{x_1}^{k_1} Z(t,\,x; \tau,\,\xi)\,d\xi| \le C(t-\tau)^{-(|k_1|-\alpha)/2},\ 0 < |k_1| \le 2,$$

$$|\int_{\mathbb{R}^n} \partial_{x_1}^{k_1} Z(t,\,x; \tau,\,\xi)\,d\xi| \le C(t-\tau)^{-1+\alpha/2}, \tag{16}$$

where $0 \le \tau < t \le T$, $\{x,\,x',\,\xi\} \subset \mathbb{R}^n, C$, *and* c *are positive constants,* $M := (n_1 + 3n_2 + 5n_3)/2$, $E_c\,(t,\,x;\,\tau,\,\xi) := \exp\{-c\sum_{s=1}^{3}\,(t-\tau)^{1-2s}|X_s(t-\tau) - \xi|^2\}$.

If condition A_3 is additionally satisfied, then a FSCP Z has such properties:

1) the FSCP is normal, i.e., a function

$$Z^*\,(\tau,\,\xi;\,t,\,x) := Z\,(t,\,x;\,\tau,\,\xi)\,, \quad 0 \le \tau < t \le T, \quad \{\xi,\,x\} \subset \mathbb{R}^n.$$

is the FSCP for the adjoint equation (15);

2) the following convolution formula holds:

$$Z\,(t,\,x;\,\tau,\,\xi) = \int_{\mathbb{R}^n} Z\,(t,\,x;\,\lambda,\,y)\ \ Z\,(\lambda,\,y;\,\tau,\,\xi)\,dy, 0 \le \tau < \lambda < t \le T,\ \ \{x,\,\xi\} \subset \mathbb{R}^n\,;$$

3) there exists only one normal FSCP for Eq. (14);

4) the coefficients of Eq. (14) have a representation via the function Z:

$$a_{jl}(t,x) = 2^{-1}\lim_{\tau\to t}\Big((t-\tau)^{-1}\int_{\mathbb{R}^n}(y_{1j}-x_{1j})(y_{1l}-x_{1l})\,Z(t,x;\tau,y)\,dy\Big),$$
$$a_j(t,x) = \lim_{\tau\to t}\Big((t-\tau)^{-1}\int_{\mathbb{R}^n}(y_{1j}-x_{1j})\,Z(t,x;\tau,y)\,dy\Big),$$
$$a_0(t,x) = \lim_{\tau\to t}\Big((t-\tau)^{-1}\Big(\int_\tau^t d\theta\int_{\mathbb{R}^n} Z(t,x;\theta,y)\,dy - 1\Big)\Big),\quad (t,x)\in\Pi_{(0,T]};$$

5) the function Z is positive;

6) there exists such a number $\Delta\in(0,T)$, that, for any $t_0\in[0,T-\Delta]$, $(t,x)\in\Pi_{(t_0,\,t_0+\Delta)}$, and $\delta\in(0,\,t-t_0)$, there exist such numbers $\omega>0$ and $\gamma>0$ that the following lower estimate is valid:

$$Z(t,x;\tau,\xi)\ge\omega\exp\left\{-\gamma|\xi|^2\right\},\quad (\tau,\xi)\in\Pi_{[t_0,\,t-\delta]}.$$

4. Correct solvability of the Cauchy problem

The results of Section 3 allow us to investigate properties of the potentials generating by FSCP Z and then, by basing on these properties, to prove various theorems about the correct solvability of the Cauchy problem for Eq. (14). Below, we present some of them.

We use the necessary norms and spaces and define the functions

$$\vec{k}(t,\vec{a}) := (k_1(t,a_1),\ k_2(t,a_2),\ k_3(t,a_3));\ \vec{l}(t) := (l_1(t),\ l_2(t),\ l_3(t));$$

$$k_s(t,a_s) := c_0 a_s\left(c_0 - a_s t^{2s-1}\right)^{-1},\quad s\in\{1,2,3\};$$

$$l_1(t) := k_1(t,a_1) + 2t^2k_2(t,a_2) + t^4k_3(t,a_3),$$

$$l_2(t) := 2k_2(t,a_2) + 4t^2k_3(t,a_3),\quad l_3(t) := 4k_3(t,a_3),\quad t\in[0,T],$$

where $c_0\in(0,c)$, c is a constant from the estimates of FSCP (16), and $\vec{a}:=(a_1,a_2,a_3)$ are a set of nonnegative numbers such that $T\le\min_{s\in\{1,2,3\}}(c_0/a_s)^{1/(2s-1)}$.

Let $p\in[1,\infty]$, and let $u(t,x)$, $(t,x)\in\Pi_{[0,T]}$, be a given function measurable in x for any $t\in[0,T]$. For every $t\in[0,T]$, we define the norms

$$\|u(t,\cdot)\|_p^{\vec{k}(t,\vec{a})} := \left\|u(t,x)\,\exp\left\{-\sum_{s=1}^{3}k_s(t,a_s)|X_s(t)|^2\right\}\right\|_{L_p(\mathbb{R}^n)},$$

$$\|u(t,\cdot)\|_p^{\vec{l}(t)} := \left\|u(t,x)\,\exp\left\{-\sum_{s=1}^{3}l_s(t)|x_s|^2\right\}\right\|_{L_p(\mathbb{R}^n)}.$$

We use the following spaces: $L_p^{\vec{k}(t,\vec{a})}$, $t\in[0,T]$, $p\in[1,\infty]$, the spaces of measurable functions $\varphi:\mathbb{R}^n\to R$ with finite norms $\|\varphi\|_p^{\vec{k}(t,\vec{a})}$; $L_p^{\vec{a}} := L_p^{\vec{k}(0,\vec{a})}$; $M^{\vec{a}}$ is the space of generalized Borel measures μ on $\mathbb{R}^n$ satisfying the condition

$$\|\mu\|^{\vec{a}} := \int_{\mathbb{R}^n}\exp\left\{-\sum_{s=1}^{3}a_s|x_s|^2\right\}d|\mu|(x)<\infty,$$

where $|\mu|$ is the total variation of μ; $L_1^{-\vec{l}(T)}$ is the space of measurable functions $\psi:\mathbb{R}^n\to\mathbb{R}$ with a finite norm

$$\left\|\psi(x)\exp\left\{\sum_{s=1}^{3}l_s(T)|x_s|^2\right\}\right\|_{L_1(\mathbb{R}^n)};$$

$C_0^{-\vec{l}(T)}$ is the space of continuous functions $\psi : \mathbb{R}^n \to \mathbb{R}$ such that

$$|\psi\,(x)|\exp\left\{\sum_{s=1}^{3} l_s(T)|x_s|^2\right\} \to 0,$$

as $|x| \to \infty$.

For the term on the right-hand side of Eq. (14), we use the following conditions:

B_p) the function f is continuous, satisfies the local Hölder condition on x respect to the distance d uniformly in t in $\Pi_{[0,\,T]}$, and, for any $t \in (0,\,T]$, the expressions $\|f\,(t,\,\cdot\,)\|_p^{\vec{k}(t,\,\vec{a})}$ and $F_p\,(t) := \int_0^t \|f\,(\tau,\,\cdot\,)\|_p^{\vec{k}(\tau,\,\vec{a})} d\tau$ are finite, where $p \in [1,\,\infty]$.

Theorem 2. *Suppose that conditions A_1-A_3 are satisfied. Then:*

1) for any function $\varphi \in L_p^{\vec{a}}$ and any function f that satisfy condition B_p, $p \in [1,\,\infty]$, the formula

$$u\,(t,\,x) = \int_{\mathbb{R}^n} Z\,(t,\,x;\,0,\,\xi)\,\varphi\,(\xi)\;d\xi\,+$$

$$+\int_0^t d\tau \int_{\mathbb{R}^n} Z\,(t,\,x;\,\tau,\,\xi) f\,(\tau,\,\xi)\,d\xi,,\quad (t,\,x) \in \Pi_{(0,\,T]}\,, \tag{17}$$

defines the unique solution of Eq. (14) satisfying the estimates

$$\|u\,(t,\,\cdot\,)\|_p^{\vec{k}(t,\,\vec{a})} \le C\left(\|\varphi\|_p^{\vec{a}} + F_p\,(t)\right),\quad t \in (0,\,T]\,,$$

and the relations

$$\lim_{t\to 0} \|u\,(t,\,\cdot\,) - \varphi\,(\,\cdot\,)\|_p^{\vec{l}(t)} = 0, \tag{18}$$

and, for any function $\psi \in L_1^{-\vec{l}(T)}$,

$$\lim_{t\to 0}\int_{\mathbb{R}^n} \psi\,(x)\,u\,(t,\,x)\;dx = \int_{\mathbb{R}^n} \psi\,(x)\,\varphi\,(x)\;dx \tag{19}$$

for $p \in [1,\,\infty)$ and $p = \infty$, respectively;

2) for an arbitrary generalizing measure $\mu \in M^{\vec{a}}$ and for any function f which satisfy condition B_1, the formula

$$u\,(t,\,x) = \int_{\mathbb{R}^n} Z\,(t,\,x;\,0,\,\xi)\,d\mu\,(\xi)\;+$$

$$+\int_0^t d\tau \int_{\mathbb{R}^n} Z\,(t,\,x;\,\tau,\,\xi) f\,(\tau,\,\xi)\,d\xi,\quad (t,\,x) \in \Pi_{(0,\,T]}\,, \tag{20}$$

defines the unique solution of Eq. (14) satisfying the estimates

$$\|u\,(t,\,\cdot\,)\|_1^{\vec{k}(t,\,\vec{a})} \le C\left(\|\mu\|^{\vec{a}} + F_1\,(t)\right),\quad t \in (0,\,T]\,,$$

and, for any function $\psi \in C_0^{-\vec{l}(T)}$, the relations

$$\lim_{t\to 0}\int_{\mathbb{R}^n} \psi\,(x)\,u\,(t,\,x)\;dx = \int_{\mathbb{R}^n} \psi\,(x)\,d\mu\,(x)\;. \tag{21}$$

The following theorem is inverse (in some sense) to Theorem 2.

Theorem 3. *Suppose that the coefficients of Eq. (14) satisfy conditions A_1-A_3, function f satisfies condition B_p, and the solution u is defined in the layer $\Pi_{(0,\,T]}$ and satisfies the conditions*

$$\|u\,(t,\,\cdot\,)\|_p^{\vec{k}(t,\,\vec{a})} \le C, \quad t \in (0,\,T]\,, \tag{22}$$

with some constants $C > 0$ and $p \in [1,\,\infty]$. Then, if $p \in (1,\,\infty]$, then exists a unique function $\varphi \in L_p^{\vec{a}}$, and if $p = 1$, then there exists a unique generalized measure $\mu \in M^{\vec{a}}$ such that the solution u is represented by formula (17) or (20).

Corollary 1. *Theorems 2 and 3, under conditions on the coefficients and the right-hand side f of Eq. (14), yield*

1) the spaces $L_p^{\vec{a}}$ and $M^{\vec{a}}$ are the sets of initial values of the solutions of Eq. (14) if and only if these solutions satisfy condition (22) with $p \in (1,\infty]$ and $p = 1$, respectively;

2) solution (14) has representations in form (17) or (20) with $\phi \in L_p^{\vec{a}}$ and $\mu \in M^{\vec{a}}$ if and only if condition (22) is valid;

3) under condition (22), solutions of Eq. (14) satisfy the initial condition with t=0 in the sense of (18), (19), and (21).

By concluding, we formulate a theorem, whose proof is based on the maximum principle for solutions of Eq. (14).

Theorem 4. *Suppose that conditions A_1-A_3 are satisfied. Then the Cauchy problem for Eq. (14) cannot have more than one nonnegative solution.*

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