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ON SOME GENERALIZATIONS OF THE POLLACHEK–KHINCHINE FORMULA

For the skip-free Poisson process $\xi(t)$ $(t \geq 0, \xi(0) = 0)$,

$$\xi(t) = at + S(t),\ a < 0,\ S(t) = \sum_{k \leq \nu(t)} \xi_k,\ \xi_k > 0, \xi(0) = 0,$$

where $\nu(t)$ is a simple Poisson process with intensity $\lambda > 0$, the moment generating function (m.g.f.) of $\xi^+ = \sup_{0 \leq t < \infty} \xi(t)$ is defined by the well-known Pollachek–Khinchine formula under the condition $m = E\xi(1) < 0$ (see [1-3]).

For a homogeneous process $\xi(t)$ with bounded variation, we establish prelimit and limit generalizations of this formula, which define the m.g.f. of

$$\xi^+(\theta_s) = \sup_{0 \leq t \leq \theta_s} \xi(t),\ \xi^+ = \lim_{s \to 0} \xi^+(\theta_s) \left(P\{\theta_s > t\} = e^{-st},\ s > 0 \right).$$

These generalizations are essentially based on the condition $P\{\tau^+(0) = \gamma^+(0) = 0\} = 0$, where $(\tau^+(0), \gamma^+(0))$ is the initial ladder point of $\xi(t)$ $(t \geq 0, \xi(0) = 0)$.

Some another relations for the m.g.f. of $\xi^+(\theta_s)$ and ξ^+ are established for the general lower semicontinuous process $\xi(t)$ on the base of results in [3-5].

1. Introduction

Let $\xi(t)$ be a lower continuous compound Poisson process with cumulant function $\psi(\alpha)$:

$$Ee^{i\alpha\xi(t)} = e^{t\psi(\alpha)},\ t \geq 0;\ \psi(\alpha) = i\alpha a + \lambda(\varphi(\alpha) - 1),$$

$$a < 0, \lambda > 0, \varphi(\alpha) = Ee^{i\alpha\xi_k},\ F(x) = P\{\xi_k < x\},\ x \geq 0, \overline{F}(x) = 1 - F(x).$$

Under the condition

$$m = E\xi(1) = a + \lambda\mu_1 < 0,\ \mu_1 = E\xi_k,\ F(0) = 0,$$

the moment generating function (m.g.f) of $\xi^+ = \sup_{0 \leq t < \infty} \xi(t)$ is defined by the classic Pollachek–Khinchine formula

$$Ee^{-z\xi^+} = \frac{p_+}{1 - q_+ \widetilde{F}(z)/\mu_1},\ p_+ = P\{\xi^+ = 0\} = 1 - q_+,\ \widetilde{F}(z) = \int_0^\infty \overline{F}(x) e^{-zx} dx. \quad (1)$$

At first, we consider the process $\xi(t)$ with a bounded variation

$$\psi(\alpha) = i\alpha a + \int_{-\infty}^{\infty} (e^{i\alpha x} - 1)\Pi(dx),\ \int_{|x| \leq 1} |x|\Pi(dx) < \infty,\ a \leq 0, \quad (2)$$

and denote its extremal values and ladder points as

$$\xi^\pm(t) = \sup_{0 \leq t' \leq t}(\inf)\, \xi(t'),\ \xi^\pm = \sup_{0 \leq t < \infty}(\inf)\, \xi(t),\ \tau^-(-x) = \inf\{t > 0 : \xi(t) < -x\},$$

$$\tau^+(x) = \inf\{t > 0 : \xi(t) > x\},\ \gamma^+(x) = \xi(\tau^+(x)) - x,\ x \geq 0.$$

$$P\left\{\tau^+(x) < \infty\right\} = 1,\ \text{if } m \geq 0, P\left\{\tau^+(x) < \infty\right\} < 1,\ \text{if } m < 0.$$

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If $\int_{-\infty}^{0} \Pi(dx) = 0$, then $a < 0$. We suppose that, for the processes $\xi(t)$ with cumulant (2), the following condition is realized:

$$P\{\tau^+(0) = 0\} = 0. \tag{3}$$

It should be pointed out that $P\{\tau^+(0) = 0\} = 0 \Leftrightarrow P\{\tau^+(0) > 0\} = 1$ and then, by definition, $\gamma^+(0) = \xi(\tau^+(0)) > 0$ with probability 1. This means

$$P\left\{\tau^+(0) = 0, \gamma^+(0) = 0\right\} = 0.$$

For such processes, we establish generalizations of the Pollachek–Khinchine formula (1).

2. The m.g.f. of $\xi^+(\theta_s)$ and ξ^+ for $\xi(t)$ with a bounded variation

Let θ_s be an exponentially distributed random variable with parameter $s > 0$ which is independent of $\xi(t)$. Then Laplace–Carson transforms of the characteristic function (ch.f.) of $\xi(t)$ and $\xi^\pm(t)$ can be written as

$$\varphi(s,\alpha) = s\int_0^\infty e^{-st} E e^{i\alpha\xi(t)} dt = \frac{s}{s-\psi(\alpha)},$$
$$\varphi_\pm(s,\alpha) = s\int_0^\infty E e^{i\alpha\xi^\pm(t)} e^{-st} dt = E e^{i\alpha\xi^\pm(\theta_s)}. \tag{4}$$

Under condition (3), the joint m.g.f. of $\{\tau^+(0), \gamma^+(0)\}$ is represented by the expression

$$f_+(s,z) = E e^{-z\gamma^+(0) - s\tau^+(0)} \mathrm{I}_{\tau^+(0)<\infty} = E e^{-z\gamma^+(0)} \mathrm{I}_{\xi^+(\theta_s)>0} < q_+(s), \tag{5}$$
$$f_+(s,z) \underset{z\to 0}{\to} P\{\xi^+(\theta_s) > 0\} = q_+(s),\ p_+(s) = 1 - q_+(s) > 0.$$

Theorem 2.1. *Let $\xi(t)$ be a process with stationary independent increments and bounded variation (see (2)). Then, under condition(3), the following prelimit generalization of Pollachek–Khinchine formula is true ($s > 0$) which defines the m.g.f. of $\xi^+(\theta_s)$:*

$$E e^{-z\xi^+(\theta_s)} =: \varphi_+(s, iz) = \frac{p_+(s)}{1 - q_+(s) E[e^{-z\gamma^+(0)} | \xi^+(\theta_s) > 0]}. \tag{6}$$

From Spitzer's formula [4, Theorem 2.2] *after the limit passage (as $z \to \infty$), it follows that*

$$p_+(s) = \lim_{z\to\infty} \varphi_+(s, iz) = \exp\left\{-\int_0^\infty e^{-st} t^{-1} P\left\{\xi(t) > 0\right\} dt\right\}.$$

If $m = E\xi(1) < 0$, then the m.g.f. of ξ^+ is defined by the limit generalization of the Pollachek–Khinchine formula ($s \to 0$)

$$E e^{-z\xi^+} = \lim_{s\to 0} E\varphi_+(s, iz) = \frac{p_+}{1 - q_+ E[e^{-z\gamma^+(0)} | \xi^+ > 0]}, \tag{7}$$
$$p_+ = 1 - q_+ = P\{\xi^+ = 0\} = \exp\left\{-\int_0^\infty t^{-1} P\left\{\xi(t) > 0\right\} dt\right\} > 0.$$

Proof. Let θ'_μ be the exponentially distributed random variable which is independent of θ_s and $\xi(t)$. By virtue of the second factorization formula (see (2.26) in [4]),

$$E[e^{-s\tau^+(\theta'_\mu) - z\gamma^+(\theta'_\mu)}, \tau^+(\theta'_\mu) < \infty] =$$
$$= \frac{\mu}{\mu - z}\left\{1 - \frac{\varphi_+(s, i\mu)}{\varphi_+(s, iz)}\right\},\ s > 0, z > 0, \mu > 0. \tag{8}$$

Accordingly to (2.29) in [4], the joint m.g.f. of $\{\tau^+(0), \gamma^+(0)\}$ is defined from (8) ($\mu \to \infty$) by the relation

$$f_+(s,z) = E e^{-z\gamma^+(0) - s\tau^+(0)} \mathrm{I}_{\tau^+(0)<\infty} =$$

$$= E[e^{-z\gamma^+(0)}, \xi^+(\theta_s) > 0] = 1 - \frac{p_+(s)}{\varphi_+(s, iz)}. \tag{9}$$

Since $f_+(s,z) = q_+(s)\tilde{g}_s(z)$, where $\tilde{g}_s(z) = E[e^{-z\gamma^+(0)}|\xi^+(\theta_s) > 0]$, relation (9) yields

$$\varphi_+(s, iz) = \frac{p_+(s)}{1 - f_+(s,z)} = \frac{p_+(s)}{1 - q_+(s)\tilde{g}_s(z)}, \tag{10}$$

and formula (6) is proved.

If $m < 0$, then, after the limit passage as $s \to 0$, relation (10) yields (7) with $\tilde{g}_0(z) = E[e^{-z\gamma^+(0)}|\xi^+ > 0]$. □

3. The m.g.f of $\xi^+(\theta_s)$ and ξ^+ for the lower continuous processes

Under condition (3), the lower continuous process $\xi(t) = at + \xi_1(t)$ is defined by the cumulant

$$\psi(\alpha) = i\alpha a + \int_0^\infty (e^{i\alpha x} - 1)\Pi(dx), \ \int_0^1 x\Pi(dx) < \infty, \ a < 0. \tag{11}$$

If $\int_0^\infty \Pi(dx) = \lambda < \infty$, then $\Pi(dx) = \lambda dF(x), x > 0$,

$$\psi(\alpha) = i\alpha a + \lambda \int_0^\infty (e^{i\alpha x} - 1)dF(x) = i\alpha a + \psi_1(\alpha), \tag{12}$$

$$\Pi(x) = \int_x^\infty \Pi(dy) = \lambda\overline{F}(x), \ x > 0, \psi_1(\alpha) = \ln Ee^{i\alpha\xi_1(1)} = i\alpha\lambda\int_0^\infty \overline{F}(x)e^{i\alpha x}dx.$$

By Theorem 3.2 in [4], the Lunberg's equation

$$k(r) = \psi(-ir) = s \tag{13}$$

has negative root $r_s = -\rho_-(s) < 0$ which defines the ch.f. of $\xi^-(\theta_s)$:

$$\varphi_-(s, \alpha) = Ee^{i\alpha\xi^-(\theta_s)} = \frac{\rho_-(s)}{\rho_-(s) + i\alpha}\left(p_+(s) = \frac{s}{|a|\rho_-(s)}\right). \tag{14}$$

Accordingly to Corollary 5.2 ([4], § 5.1) for the lower continuous processes, $\tilde{g}_s(z)$ is defined by the relation

$$\tilde{g}_s(z) = \frac{z\widetilde{\Pi}(z) - \rho_-(s)\widetilde{\Pi}(\rho_-(s))}{(z - \rho_-(s))\widetilde{\Pi}(\rho_-(s))}, \ \widetilde{\Pi}(z) = \int_0^\infty e^{-zx}\Pi(x)dx. \tag{15}$$

The conditional m.g.f. of $\gamma^+(0)$ is defined for $m = E\xi(1) = k'(0) < 0$, $\rho_-(s) \underset{s\to 0}{\to} 0$ by the limit passage ($s \to 0$)

$$\tilde{g}_0(z) = \lim_{s\to 0}\tilde{g}_s(z) = E[e^{-z\gamma^+(0)}|\zeta^+ > 0] = \frac{\widetilde{\Pi}(z)}{\widetilde{\Pi}(0)}. \tag{16}$$

If $\lambda = \Pi(0) < \infty$, then $\widetilde{\Pi}(z) = \lambda\widetilde{F}(z) = \lambda\int_0^\infty e^{-zx}\overline{F}(x)dx$,

$$\tilde{g}_0(z) = \frac{\widetilde{F}(z)}{\widetilde{F}(0)}, \ \widetilde{F}(0) = \int_0^\infty \overline{F}(x)dx = \mu_1. \tag{17}$$

Taking these relations into account, Theorem 1 yields the following corollary

Corollary 3.1. *If $\xi(t)$ is the lower semicontinuous process with cumulant (11), then the prelimit generalization of the Pollachek–Khinchine formula*

$$\varphi_+(s, iz) = \frac{p_+(s)}{1 - q_+(s)\dfrac{z\widetilde{\Pi}(z) - \rho_-(s)\widetilde{\Pi}(\rho_-(s))}{(z - \rho_-(s))\widetilde{\Pi}(\rho_-(s))}}, \ q_+(s) = \frac{1}{|a|}\widetilde{\Pi}(\rho_-(s)) \tag{18}$$

holds true. If $\lambda = \Pi(0) < \infty$ *(with cumulant (12)), then* $(p_+(s) = 1 - q_+(s))$

$$\varphi_+(s, iz) = \frac{p_+(s)}{1 - q_+(s)\dfrac{z\widetilde{F}(z) - \rho_-(s)\widetilde{F}(\rho_-(s))}{(z - \rho_-(s))\widetilde{F}(\rho_-(s))}}, \; q_+(s) = \frac{\lambda\widetilde{F}(\rho_-(s))}{|a|}. \tag{19}$$

If $m = E\xi(1) < 0$, *then we obtain the following limit relations from (18) and (19) as* $s \to 0$*:*

$$Ee^{-z\xi^+} = \frac{p_+}{1 - q_+\widetilde{\Pi}(z)/\widetilde{\Pi}(0)}, \; q_+ = \frac{1}{|a|}\widetilde{\Pi}(0), \tag{20}$$

$$Ee^{-z\xi^+} = \frac{p_+}{1 - q_+\widetilde{F}(z)/\widetilde{F}(0)}, \; q_+ = \frac{\lambda}{|a|}\widetilde{F}(0) = \frac{\lambda\mu_1}{|a|}. \tag{21}$$

Proof. After the corresponding substitution $\widetilde{g}_s(z)$ (15) and $\widetilde{g}_0(z)$ (16)–(17) into (6) and (7), we deduce relations (19)–(21). It should be mentioned that formula (21) coincides with the Pollachek–Khinchine formula (1). □

4. The m.g.f. of $\xi^+(\theta_s)$ and ξ^+ for almost lower semicontinuous stepwise processes

An almost lower semicontinuous stepwise process is defined as

$$\xi(t) = \xi_1(t) + \xi_2(t), \; \xi_{1,2}(t) = \sum_{k \le \nu_{1,2}(t)} \xi'_k(\xi''_k), \tag{22}$$

$$F_1(x) = P\{\xi'_k < x\}, x > 0; \; F_2(x) = P\{\xi''_k < x\} = e^{bx} \; (b > 0, x < 0),$$

where processes $\xi_{1,2}(t)$ are independent, and $\nu_{1,2}(t)$ are simple independent Poisson processes with rates $\lambda_{1,2} > 0$.

$$\begin{aligned}
&\psi(\alpha) = \ln Ee^{i\alpha\xi(t)} = \psi_1(\alpha) + \psi_2(\alpha),\\
&\psi_1(\alpha) = \lambda_1\int_0^\infty (e^{i\alpha x} - 1)dF(x) = i\alpha\lambda_1\int_0^\infty e^{i\alpha x}\overline{F}(x)dx,\\
&\psi_2(\alpha) = \lambda_2\int_{-\infty}^0 (e^{i\alpha x} - 1)be^{bx}dx = -\frac{i\alpha\lambda_2}{b + i\alpha}, \; \lambda_2 > 0, b > 0.
\end{aligned} \tag{23}$$

Following Lemma 3.4 (see §5.2 in [4]) for the process $\xi(t)$ with culumant (22), the root $r_s = -\rho_-(s) = -bp_-(s) < 0$ of Eq. (13) defines the ch.f. of $\xi^-(\theta_s)$:

$$\varphi_-(s, \alpha) = \frac{b(p_-(s) + i\alpha)}{\rho_-(s) + i\alpha} \; \left(p_+(s)p_-(s) = \frac{s}{s + \lambda}, \lambda = \lambda_1 + \lambda_2\right).$$

By Theorem 5.9 (see §5.2 in [4]),

$$\int_0^\infty e^{-zy}P\{\gamma^+(0) > y, \xi^+(\theta_s) > 0\}dy =$$

$$= \frac{1}{s + \lambda}[\widetilde{\Pi}(z) + bq_-(s)\frac{\widetilde{\Pi}(z) - \widetilde{\Pi}(\rho_-(s))}{\rho_-(s) - z}], \; \widetilde{\Pi}(z) = \lambda\widetilde{F}(z). \tag{24}$$

Note that $1 - z\widetilde{F}_1(z) = \varphi_1(iz)$. Then, after the integration by parts, relation (24) yields

$$f_+(s, z) = E[e^{-z\gamma^+(0)}, \xi^+(\theta_s) > 0] =$$

$$= \frac{\lambda_1}{s + \lambda}[\varphi_1(iz) + bq_-(s)\frac{z\widetilde{F}_1(z) - \rho_-(s)\widetilde{F}_1(\rho_-(s))}{z - \rho_-(s)}], \tag{25}$$

$$q_+(s) = \frac{\lambda_1}{s + \lambda}[1 + bq_-(s)\widetilde{F}_1(\rho_-(s))], \; \varphi_1(iz) = \int_0^\infty e^{-zx}dF_1(x);$$

From (25) under the condition $m < 0$ by the limit passage $s \to 0$ ($\rho_-(s) \to 0$), we deduce

$$f_+(0,z) = E[e^{-z\gamma^+(0)}, \xi^+ > 0] = \frac{\lambda_1}{\lambda}[\varphi_1(iz) + b\widetilde{F}_1(z)], \tag{26}$$

$$q_+ = \frac{\lambda_1}{\lambda}[1 + b\widetilde{F}_1(0)],\ \widetilde{F}_1(0) = E\xi_k' = \mu_1',\ \widetilde{F}_1(z) = \int_0^\infty e^{-zx}\overline{F}_1(x)dx.$$

Corollary 4.1. *If $\xi(t)$ is the almost semicontinuous process with cumulant (23), then*

$$\widetilde{g}_s(z) = E[e^{-z\gamma^+(0)}|\xi^+(\theta_s) > 0] = \frac{1}{1 + bq_-(s)\widetilde{F}_1(\rho_-(s))}\times \tag{27}$$

$$\times\left[\varphi_1(iz) + bq_-(s)\frac{z\widetilde{F}_1(z) - \rho_-(s)\widetilde{F}_1(\rho_-(s))}{z - \rho_-(s)}\right] \ \text{as}\ \ s > 0;$$

and, for $m < 0$, $s \to 0$,

$$\widetilde{g}_0(z) = E[e^{-z\gamma^+(0)}|\xi^+ > 0] = \frac{\varphi_1(iz) + b\widetilde{F}_1(z)}{1 + b\widetilde{F}_1(0)}. \tag{28}$$

For the m.g.f. of ξ^+, we have the following generalization of the Pollachek–Khinchine formula:

$$Ee^{-z\xi^+} = \frac{p_+}{1 - q_+\dfrac{\varphi_1(iz) + b\widetilde{F}_1(z)}{1 + b\widetilde{F}_1(0)}},\ q_+ = \frac{\lambda_1}{\lambda}(1 + b\mu_1'). \tag{29}$$

Proof. From (25) and (26), we get the representations of the conditional m.g.f. of $\gamma^+(0)$ (27) and (28) which correspond to prelimit formula (6) and limit formula (7). □

For general processes with stationary independent increments, we denote

$$K(s,x) = \int_{-\infty}^0 \Pi(x - y)dP_-(s,y), x > 0, \Pi(x) = \int_x^\infty \Pi(dy), x \geq 0,$$

$$k(s,\alpha) = \int_0^\infty e^{i\alpha x}K(s,x)dx,\ k'(0,\alpha) = (k(s,\alpha))'_{s=0}.$$

Then, according to Theorem 2.5. in [4] for $\varphi_+(s,\alpha)$, we have

Theorem 4.1. *If the process $\xi(t)$ has the cumulant*

$$\psi(\alpha) = i\alpha a - \alpha^2\sigma^2/2 + \int_{-\infty}^\infty \left(e^{i\alpha x} - 1 - i\alpha x I_{|x|\leq 1}\right)\Pi(dx), \tag{30}$$

then

$$\varphi_+(s,\alpha) = \frac{1}{1 - i\alpha\left(C_*(s) + s^{-1}k(s,\alpha)\right)};$$

$$\varphi_+(iz) = \frac{1}{1 + z\left(C_*(0) + k'(0,iz)\right)}\text{as } m < 0; \tag{31}$$

$$C_*(s) = \begin{cases} (2s)^{-1}\sigma^2 P_-'(s,0), & \sigma > 0; \\ s^{-1}p_-(s)\max\{0,a\}, & \sigma = 0. \end{cases}$$

$$zk'(0,iz) = \int_0^\infty (1 - e^{-zx})dM(x),\ M(x) = \int_{-\infty}^0 \Pi(x - y)dE\tau^-(y).$$

If $\int_{|x|\leq 1}|x|\Pi(dx) = \infty$, then $p_-(s) = 0$ and $C_*(s) = 0$.

If $\xi(t)$ is the lower semicontinuous process with the cumulant

$$\psi(\alpha)=i\alpha a-\sigma^2\alpha^2/2+\int_0^\infty\left(e^{i\alpha x}-1-i\alpha xI_{|x|\le 1}\right)\Pi(dx) \qquad (32),$$

then $k(s,iz)$ is defined by the relation

$$k(s,iz)=\frac{\rho_-(s)}{\rho_-(s)-z}\left[\tilde\Pi(z)-\tilde\Pi(\rho_-(s))\right].$$

In addition, $P'_-(s,0)=\rho_-(s)$. Then, according to (31), the following assertion is proved:

Corollary 4.2. *If the process $\xi(t)$ has cumulant (32), then*

$$\varphi_+(s,iz)=Ee^{-z\xi^+(\theta_s)}=\left[1+s^{-1}z\rho_-(s)\left(\frac{\sigma^2}{2}+\frac{\tilde\Pi(z)-\tilde\Pi(\rho_-(s))}{\rho_-(s)-z}\right)\right]^{-1}. \qquad (33)$$

If $m<0$, then

$$\varphi_+(iz)=\lim_{s\to 0}\varphi_+(s,iz)=Ee^{-z\xi^+}=\left[1+|m|^{-1}\left(\frac{\sigma^2}{2}z+\tilde\Pi(0)-\tilde\Pi(z)\right)\right]^{-1}. \qquad (34)$$

If $\xi(t)$ is the almost lower semicontinuous process with the cumulant

$$\psi(\alpha)=i\alpha a+b\lambda_1\int_{-\infty}^0\left(e^{iax}-1\right)e^{bx}dx+\int_0^\infty\left(e^{i\alpha x}-1\right)\Pi(dx),\ a\ge 0, \qquad (35)$$

then, with regard for Theorem 5.8 (see (5.53) in [4]), relations (31) are true with values

$$C_*(s)=ap_-(s)/s;$$

$$k(s,iz)=\left[(b-z)\tilde\Pi(z)-bq_-(s)\tilde\Pi(\rho_-(s))\right]p_-(s)/(\rho_-(s)-z),$$

$$C_*(0)=a(b|m|)^{-1},\ k'(0,iz)=(b|m|)^{-1}\left[b\tilde\Pi(0)-(b-z)\tilde\Pi(z)\right],\ m<0. \qquad (36)$$

Particularly, if $m<0$, then $k'(0,iz)=-\int_0^\infty\left(e^{-zx}-1\right)(\Pi(dx)+b\Pi(x)dx)$. Using notations $a_*=a(b|m|)^{-1}$, $\Pi_*(dx)=[\Pi(dx)+b\Pi(x)dx](b|m|)^{-1}$ $(x>0)$ the m.g.f. of ξ^+ can be rewritten in a compact form

$$\varphi_+(iz)=Ee^{-z\xi^+}=\frac{1}{1-k_*(z)},\ k_*(z)=-a_*z+\int_0^\infty\left(e^{-zx}-1\right)\Pi_*(dx). \qquad (37)$$

From results of §11[1], §3.1-§3.2[4] and §23[6], the following propositions hold:

Theorem 4.2. *Let $\xi(t)$ be an upper continuous process with the cumulant*

$$\psi(\alpha)=i\alpha a-\frac{\sigma^2\alpha^2}{2}+\int_{-\infty}^0\left(e^{i\alpha x}-1-\mathbb{I}_{|x|\le 1}i\alpha x\right)\Pi(dx), \qquad (38)$$

if $\sigma=0$ and $\int_{-1}^0|x|\Pi(dx)<\infty$. Then we suppose that

$$\psi(\alpha)=i\alpha a'+\int_{-\infty}^0\left(e^{i\alpha x}-1\right)\Pi(dx),\ where\ a'=a+\int_{-1}^0|x|\Pi(dx)>0.$$

Then $T(x)=\tau^+(x)\,(T(0)=0)$ is a nondecreasing process with respect to $x\ge 0$ with stationary independent increments which can be considered as a "inverted" function of $x=\xi^+(t)$ $(t\ge 0)$.

If $m\ge 0$, then $-\rho_+(s)=k_T(-s)$ is the cumulant of $T(x)$, and, by the formula (23.2) (§23,[6]), it is represented as

$$-\rho_+(s)=-\gamma_+s+\int_0^\infty\left(e^{-sx}-1\right)dN(x),\gamma_+\ge 0;\ \gamma_+=\frac{1}{a'}\mathbb{I}_{a'\ge 0},\ if\ \sigma=0. \qquad (39)$$

If $m > 0$, *then* $\rho'_+(0) = 1/m = k'_T(0) = \gamma_+ + \int_0^\infty x dN(x) < \infty$, *and the mean value* $ET(x)$ *exists:* $ET(x) = x/m$, $x \geq 0$.

If $m = 0$, *then* $\rho_+(s) \approx \sqrt{2s/k''(0)} = \sqrt{2s/D\xi(1)}$ *as* $s \to 0$, $ET(x) = \infty$.

If $m < 0$, *then* $\rho_+(s) \to \rho_+ > 0$, *and* $T(x) = \tau^+(x)$ *has a degenerate distribution, because of* $P\{T(x) < \infty\} = e^{-\rho_+(s)x} < 1$ *for* $x > 0$. *Hence,*

$$-\rho_+(s) = -\rho_+ - \gamma_+ s + \int_0^\infty \left(e^{-sx} - 1\right) dN(x), \; \gamma_+ = 0, \text{ if } \sigma^2 > 0 \text{ or } a' = \infty. \tag{40}$$

The conditional m.g.f. of $T(x)$ *is defined by the relation*

$$E\left[e^{-sT(x)} | T(x) < \infty\right] = e^{xk^*_T(-s)}, \; k^*_T(-s) = \rho_+ - \rho_+(s), x \geq 0. \tag{41}$$

Theorem 4.3. *Let* $\xi(t)$ *be an almost upper semicontinuous process with the cumulant*

$$\psi(\alpha) = i\alpha a + \int_{-\infty}^0 \left(e^{i\alpha x} - 1\right) \Pi(dx) + \frac{\lambda_1 i\alpha}{c - i\alpha}, \; a < 0; c, \lambda_1 > 0. \tag{42}$$

Then $T(x) = \tau^+(0) + T_0(x)$ ($T_0(x) = T(x) - \tau^+(0)$, $T(0) = \tau^+(0) = 0$) *is a nondecreasing random function with stationary independent increments.*

If $m \geq 0$, *then* $-\rho_+(s) = k_{T_0}(-s)$ *is the cumulant for* $T_0(x)$

$$-\rho_+(s) = \int_0^\infty \left(e^{-sx} - 1\right) dN(x). \tag{43}$$

If $m > 0$, *then* $\rho'_+(0) = 1/m = k'_{T_0}(0) = \int_0^\infty x dN(x) < \infty$ *and* $ET(x) = x/m$, $x \geq 0$.

If $m = 0$, *then* $\rho_+(s) \to 0$, $\rho_+(s) \approx \sqrt{2s/D\xi(1)}$ *as* $s \to 0$, $ET_0(x) = \infty$.

If $m < 0$, *then* $\rho_+(s) \to \rho_+ > 0$, *and* $T_0(x)$ *has the degenerate distribution, because of*

$$P\{T_0(x) < \infty\} = P\left\{\xi^+ > x\right\} = q_+ e^{-\rho_+ x} < 1, x \geq 0.$$

Hence, the conditional m.g.f. of $T_0(x)$ *also has the exponential form (see* (41)*):*

$$\begin{aligned} &E[e^{-sT_0(x)} | T_0(x) < \infty] = e^{x(\rho_+ - \rho_+(s))}, x \geq 0, \\ &E[T_0(x) | T_0(x) < \infty] = x \int_0^\infty y dN(y). \end{aligned} \tag{44}$$

The similar assertions hold for the lower continuous processes $\xi(t)$ with cumulant (32) and for the almost lower semicontinuous processes $\xi(t)$ with cumulant (35), for which

$$T(x) = \tau^-(-x) = \inf\{t > 0 : \xi(t) < -x\}, x \geq 0.$$

For example (see Theorem 2.10 in [4,§ 2.5]), if $W(t)$ is the Wiener process ($EW(t) = 0$, $DW(t) = 2$), then $T(x) = \tau^\pm(\pm x)$ ($x \geq 0$) are the stable processes with the parameter $\alpha_* = 1/2$. The cumulant functions of $T(x)$ have representations

$$k_T(-s) = -\rho_\pm(s) = -\sqrt{s} = \frac{1}{\sqrt{2\pi}} \int_0^\infty \left(e^{-sy} - 1\right) y^{-\frac{3}{2}} dy, \tag{45}$$

which means

$$\gamma_\pm = 0, \; N(y) = -\sqrt{\frac{2}{\pi}} y^{-\frac{1}{2}}, \; \Pi(dy) = \frac{1}{\sqrt{2\pi}} y^{-\frac{3}{2}} dy, y > 0.$$

The next example for the illustration of the homogeneity of $\tau^+(x)$ with respect to x is connected with the process

$$\xi(t) = \xi_*(t) + S(t), \; S(t) = \sum_{k \leq \nu(t)} \xi_k, \; \varphi(\alpha) = Ee^{i\alpha\xi_k} = \frac{1}{1 - i\alpha},$$

$\xi_*(t)$ is a stable process with negative jumps and with the parameter of stability $\alpha_* = 1/2$.
$\xi(t)$ is the almost upper semicontinuous process with the cumulant

$$\psi(\alpha) = \ln e^{i\alpha\xi(1)} = \frac{i\alpha}{1-i\alpha} - 2|\alpha|^{\frac{1}{2}},\ k(r) = \frac{r}{1-r} - 2\sqrt{|r|}.$$

Accordingly to (44), the conditional m.g.f. of $T_0(x)$ has the representation

$$E[e^{-sT_0(x)}|T_0(x) < \infty] = e^{x\hat{k}_{T_0}(-s)}, E\,[T_0(x)|T_0(x) < \infty] = xE\tau^+(0)\mathbb{I}_{\tau^+(0)<\infty},$$

$$\hat{k}_{T_0}(-s) = \rho_+ - \rho_+(s) = \pi(s) - \pi(0),\ \pi(s) = Ee^{-s\tau^+(0)}\mathbb{I}_{\tau^+(0)<\infty},$$

where $0 < \rho_+(s) = p_+(s) < 1$ is the root of Lundberg's equation (13), which is defined by the intersection of the curves $(0 \le r < 1)$

$$y = \frac{r}{1-r} - s,\ y = 2\sqrt{r},\ m = E\xi(1) = -\infty < 0,$$

$$\rho_+(s) \underset{s\to 0}{\longrightarrow} \rho_+ = p_+ = \frac{9-\sqrt{17}}{8} < 1,$$

ρ_+ is the root of the limit Lundberg's equation $\frac{r}{1-r} - 2\sqrt{r} = 0$ $(s = 0, 0 \le r < 1)$.

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