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GEOMETRIC GAUSSIAN MARTINGALES WITH DISORDER

We propose the scheme of a geometric Gaussian martingale with "disorder" as a model of a stock price evolution and investigate the problem of finding a forecasting estimation optimal in mean square sense within this scheme.

1. Introduction and model

We investigate a stochastic process in discrete time described by geometric Gaussian martingales with "disorder" and consider it as a model of stock price evolution. At first, we analyze this scheme and show that the kurtosis coefficient of a logarithmic return is positive for any moment of time, as it is usually true for real financial time series. Then we find a forecasting estimation of stock prices which is optimal in mean square sense.

Such a kind of models with disorder was considered in [1,2,6], where we study problems of constructing the minimal relative entropy martingale measures.

On the probability space $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{0\le n\le N}, P)$, we consider the following real-valued stochastic process in discrete time as a model of stock price evolution:

$$S_n = S_{n-1}\exp\{I(\theta > n)\Delta M_n^{(1)} + I(\theta \le n)\Delta M_n^{(2)}\}, \quad S_0 > 0, \tag{1}$$

where $S_0 > 0$, $M^{(1)} = (M_n^{(1)})$ and $M^{(2)} = (M_n^{(2)})$, $n = 1, 2, \dots, N$, are the independent Gaussian martingales with quadratic characteristics $\langle M^{(1)}\rangle_n = E(M_n^{(1)})^2$ and $\langle M^{(2)}\rangle_n = E(M_n^{(2)})^2$, respectively, θ is the random variable which takes the values $1, 2, \dots, N$ with known probabilities $\pi_i = P(\theta = i)$, $1, 2, \dots, N$ and represents the random disorder moment, $M^{(1)}$ and $M^{(2)}$ are jointly independent of θ, i.e. the vector $(M^{(1)}, M^{(2)})$ is independent of θ, and $I(A)$ is an indicator of $A \in \mathcal{F}$.

2. Coefficient of kurtosis

From (1), the logarithmic return of stock for our model, $h_n = \ln \frac{S_n}{S_{n-1}}$, $n = 1, 2, \dots, N$, is as follows:

$$h_n = I(\theta > n)\Delta M_n^{(1)} + I(\theta \le n)\Delta M_n^{(2)}.$$

It is clear that

$$h_n^2 = I(\theta > n)[\Delta M_n^{(1)}]^2 + I(\theta \le n)[\Delta M_n^{(2)}]^2$$

and

$$h_n^4 = I(\theta > n)[\Delta M_n^{(1)}]^4 + I(\theta \le n)[\Delta M_n^{(2)}]^4.$$

We now find a representation of the coefficient of kurtosis (excess) of h_n, $n = 1, 2, \dots, N$.

Note that, in our model, $M^{(1)} = (M_n^{(1)}, \mathcal{F}_n)$ and $M^{(2)} = (M_n^{(2)}, \mathcal{F}_n)$ are the independent Gaussian martingales with square characteristics $\langle M^{(1)}\rangle_n = E(M_n^{(1)})^2$ and $\langle M^{(2)}\rangle_n = E(M_n^{(2)})^2$, respectively, and are independent of the disorder random moment θ.

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Therefore,

$$Eh_n = 0, \tag{2}$$

$$\begin{aligned} &Eh_n^2 = E[I(\theta > n)[\Delta M_n^{(1)}]^2 + I(\theta \le n)[\Delta M_n^{(2)}]^2] = \\ &E[E[I(\theta > n)[\Delta M_n^{(1)}]^2 + I(\theta \le n)[\Delta M_n^{(2)}]^2]/\theta] = \\ &E[I(\theta > n)E[\Delta M_n^{(1)}]^2/\theta] + E[I(\theta \le n)E[\Delta M_n^{(2)}]^2/\theta] = \\ &\Delta\langle M^{(1)}\rangle_n P(\theta > n) + \Delta\langle M^{(2)}\rangle_n P(\theta \le n), \end{aligned} \tag{3}$$

$$\begin{aligned} &Eh_n^4 = E[\Delta M_n^{(1)}]^4 P(\theta > n) + E[\Delta M_n^{(2)}]^4 P(\theta \le n) = \\ &3[\Delta\langle M^{(1)}\rangle_n]^2 P(\theta > n) + 3[\Delta\langle M^{(2)}\rangle_n]^2 P(\theta \le n). \end{aligned} \tag{4}$$

The kurtosis coefficient of h_n is

$$K_n = \frac{Eh_n^4}{(Eh_n^2)^2} - 3. \tag{5}$$

Substituting Eh_n^2 and Eh_n^4 from (3) and (4) in (5), we obtain

$$K_n = \frac{3[\Delta\langle M^{(1)}\rangle_n]^2 P(\theta > n) + 3[\Delta\langle M^{(2)}\rangle_n]^2 P(\theta \le n)}{[[\Delta\langle M^{(1)}\rangle_n]P(\theta > n) + [\Delta\langle M^{(2)}\rangle_n]P(\theta \le n)]^2} - 3. \tag{6}$$

Denote

$$\frac{\Delta\langle M^{(1)}\rangle_n}{\Delta\langle M^{(2)}\rangle_n} = a_n. \tag{7}$$

Then it follows from (6) that

$$\begin{aligned} &K_n = 3\{P(\theta > n) + a_n^2 P(\theta > n) - [P(\theta \le n)]^2 - \\ &2a_n P(\theta > n)P(\theta \le n)\} - a_n^2[P(\theta \le n)][P(\theta > n) + \\ &a_n P(\theta \le n)]^{-2} = 3\{P(\theta > n)P(\theta \le n) + \\ &a_n^2 P(\theta \le n)P(\theta > n) - 2aP(\theta \le n)P(\theta > n)\}[P(\theta > n) + \\ &P(\theta \le n)]^{-2} = \frac{3\{P(\theta > n)P(\theta \le n)(1 - a_n)^2}{[P(\theta > n) + a_n P(\theta \le n)]^2}. \end{aligned} \tag{8}$$

If, for any n, $a_n \ne 1$, then we have from (8) that K_n is positive for each n, $n = 1, 2, \dots, N$. For example, $a_n \ne 1$ if $\langle M^{(2)}\rangle_n = n$ and $\langle M^{(2)}\rangle_n = an$, where a is some positive constant. In the case without disorder, we have $\langle M^{(1)}\rangle_n = \langle M^{(2)}\rangle_n$ and the coefficient of kurtosis $K_n = 0$ for any n, because $a_n = 1$ in this case.

It is known (see [4],[5]) that, for real financial time series, the empirical kurtosis coefficient of a logarithmic return

$$\hat{K}_n = \frac{\frac{1}{n}\sum_{k=1}^n (h_k - \bar{h}_n)^4}{(\frac{1}{n}\sum_{k=1}^n (h_k - \bar{h}_n)^2)^2} - 3,$$

where $\bar{h}_n = \frac{1}{n}\sum_{k=1}^n h_k$ is usually positive.

3. Optimal forecasting

We consider the problem of finding the optimal in mean square sense forecasting estimation of a stochastic process S described by (1).

In sequel, we use the following trivial fact:

$$\begin{aligned} &1 = I(\theta \le (n-m)+1) + I(\theta = (n-m)+2) + \cdots + I(\theta = n-1) + \\ &I(\theta = n) + I(\theta > n), \end{aligned} \tag{9}$$

$n \le m,\ n = 1, 2, \ldots, N$.

From (1) and (9), we obtain

$$
\begin{aligned}
&S_n = S_{n-m} \exp[\textstyle\sum_{k=(n-m)+1}^{n} h_k] = S_{n-m} \exp\{[\sum_{k=(n-m)+1}^{n} h_k] \times \\
&[I(\theta \le (n-m)+1) + I(\theta = (n-m)+2) + \cdots + I(\theta = n-1) + \\
&I(\theta = n) + I(\theta \ge n)\} = \\
&S_{n-m} \exp\{I(\theta \le (n-m)+1) \textstyle\sum_{k=(n-m)+1}^{n} \Delta M_k^{(2)} + \\
&I(\theta = (n-m)+2)[\textstyle\sum_{k=(n-m)+1}^{n-1} \Delta M_k^{(2)} + \Delta M_n^{(1)}] + \cdots + \\
&I(\theta = n-1)[\Delta M_{n-1}^{(2)} + \Delta M_n^{(2)} + \textstyle\sum_{k=(n-m)+1}^{n-1} \Delta M_k^{(1)}] + \\
&I(\theta = n)[\Delta M_n^{(2)} + \textstyle\sum_{k=(n-m)+1}^{n-1} \Delta M_k^{(1)}] + \\
&I(\theta > n) \textstyle\sum_{k=(n-m)+1}^{n} \Delta M_k^{(1)}\}.
\end{aligned}
\tag{10}
$$

Relation (10) yields

$$
\begin{aligned}
&S_n = S_{n-m}\{I(\theta \le (n-m)+1) \exp[\textstyle\sum_{k=(n-m)+1}^{n} \Delta M_k^{(2)}] + \\
&I(\theta = (n-m)+2) \exp[\textstyle\sum_{k=(n-m)+1}^{n-1} \Delta M_k^{(2)} + \Delta M_k^{(1)}] + \cdots + \\
&I(\theta = n-1) \exp[\Delta M_{n-1}^{(2)} + \Delta M_n^{(2)} + \textstyle\sum_{k=(n-m)+1}^{n-2} \Delta M_k^{(1)}] + \\
&I(\theta = n) \exp\{\Delta M_n^{(2)} + \textstyle\sum_{k=(n-m)+1}^{n-1} \Delta M_k^{(1)}\} + \\
&I(\theta > n) \textstyle\sum_{k=(n-m)+1}^{n} \Delta M_k^{(1)}\}.
\end{aligned}
\tag{11}
$$

From (11), the m-optimal forecasting optimal in mean square sense has the form

$$
\begin{aligned}
&\hat{S}_n = E(S_n/\mathcal{F}_{n-m}^S) = S_{n-m}[P(\theta \le (n-m)+1)/\mathcal{F}_{n-m}^S] \times \\
&\exp\{\tfrac{1}{2} \textstyle\sum_{k=(n-m)+1}^{n} \Delta\langle M_k^{(2)}\rangle\} + P(\theta = (n-m)+2)/\mathcal{F}_{n-m}^S \times \\
&\exp\{[\tfrac{1}{2} \textstyle\sum_{k=(n-m)+1}^{n-1} \Delta\langle M_k^{(2)}\rangle + \Delta\langle M_n^{(1)}\rangle]\} + \cdots + \\
&P(\theta = (n-1)/\mathcal{F}_{n-m}^S) \exp\{\tfrac{1}{2}[\Delta\langle M_{n-1}^{(2)}\rangle + \Delta\langle M_n^{(2)}\rangle + \\
&\textstyle\sum_{k=(n-m)+1}^{n-2} \Delta\langle M_k^{(1)}\rangle]\} + P(\theta = n) \exp\{\tfrac{1}{2}[\sum_{k=(n-m)+1}^{n-1} \Delta\langle M_k^{(1)}\rangle + \\
&\Delta\langle M_n^{(2)}\rangle]\} + P(\theta > n/\mathcal{F}_{n-m}^S) \exp\{\tfrac{1}{2} \textstyle\sum_{k=(n-m)+1}^{n} \Delta\langle M_k^{(1)}\rangle\}.
\end{aligned}
\tag{12}
$$

Here, we use the fact that $M^{(1)}$ and $M^{(2)}$ are independent Gaussian martingales and are jointly independent of θ as well.

It is clear from (12) that, in order to solve the forecasting problem, it is necessary to find the conditional probabilities $P(\theta \le (n-m)+1/\mathcal{F}_{n-m}^S)$, $P(\theta = (n-m)+2/\mathcal{F}_{n-m}^S)$, $P(\theta = n-1/\mathcal{F}_{n-m}^S)$, $P(\theta = m/\mathcal{F}_{n-m}^S)$ and $P(\theta > n/\mathcal{F}_{n-m}^S)$.

We denote $P(\theta = n/\mathcal{F}_r^S) = P_n^r,\ n = 0, 1, 2, \ldots, N;\ r = 0, 1, 2, \ldots, N$.

It is clear that, for each n, $\mathcal{F}_n^S = \mathcal{F}_n^h$, where $\mathcal{F}_n^S = \sigma\{S_0, S_1, \ldots, S_n\}$ and $\mathcal{F}_n^h = \sigma\{h_0, h_1, \ldots, h_n\}$.

Using the Bayes formula

$$
P(\theta/h_1, h_2, \ldots, h_r) = \frac{P(h_1, h_2, \ldots, h_n/r)\pi_2}{\sum_{i=1}^{N} P(h_1, h_2, \ldots, h_n/i)\pi_i},
$$

where $P(x_1, x_2, \ldots, x_n/r) = P_{h_1, h_2, \ldots, h_n}(x_1, x_2, \ldots, x_n/\theta = r)$ is the conditional density probability function of a random vector $(h_1, h_2, \ldots, h_n)$, taking the condition $\{\theta = r\}$ into account, and performing direct calculations, we obtain the following result.

Lemma 1. If $r < n$, then

$$P_n^r = \frac{\pi_r u_n}{L_r}; \tag{13}$$

and if $r \geq n$,

$$P_n^r = \frac{\pi_n u_{n-1}}{L_r}, \tag{14}$$

where

$$u_r = \frac{P_{h_1^{(1)}}(h_1) P_{h_2^{(1)}}(h_2) \dots P_{h_k^{(1)}}(h_k)}{P_{h_1^{(2)}}(h_2) P_{h_2^{(2)}}(h_2) \dots P_{h_k^{(2)}}(h_k)}, \quad u_0 = 1, \; h_k = \ln \frac{\Delta S_k}{S_{k-1}}, \tag{15}$$

$$h_l^{(i)} = \Delta M_l^{(i)}, \; i = 1, 2; \; l = 1, 2, \dots, N; L_r = \sum_{i=1}^{r} \pi_i u_{i-1} + (1 - \Pi_r) u_r. \tag{16}$$

Here, $\Pi_r = P(\theta \leq n)$ and

$$P_{h_l^{(i)}} = \frac{1}{\sqrt{2\pi \Delta \langle M^i \rangle_l}} \exp\{-\frac{x^2}{2\Delta \langle M^i \rangle_l}\}, \; l = 1, 2, \dots, k; \; i = 1, 2. \tag{17}$$

We can obtain this result also using the method developed in [3], because the problem of finding P_n^r belongs to the general filtered probability–statistical experiment framework.

We now construct the optimal in mean square sense m-step forecasting estimation of $S : \hat{S}_n(m) = E(S_i / \mathcal{F}_{n-m}^S)$.

Theorem 1. The optimal forecasting estimation of S has the form

$$\begin{aligned} \hat{S}_n(m) = {} & S_{n-m}[\textstyle\sum_{k=n+1}^{N} P_k^{n-m} \exp[\frac{1}{2} \sum_{k=(n-m)+1}^{n} \Delta \langle M_k^{(1)} \rangle] + \\ & P_n^{n-m} \exp[\tfrac{1}{2}[\textstyle\sum_{k=(n-m)+1}^{n-1} \Delta \langle M^{(1)} \rangle_k + \Delta \langle M^{(2)} \rangle_n]] + \\ & P_{n-1}^{n-m} \exp[\tfrac{1}{2}[\textstyle\sum_{k=(n-m)+1}^{n-2} \Delta \langle M^{(1)} \rangle_k + \Delta \langle M^{(2)} \rangle_{n-1} + \\ & \Delta \langle M^{(2)} \rangle_n] + \cdots + P_{(n-m)+2}^{n-m} \exp[\tfrac{1}{2}[\Delta \langle M^{(1)} \rangle_{n-m+1} + \\ & \textstyle\sum_{k=1}^{n-1} \Delta \langle M^{(2)} \rangle_k]] + \sum_{k=1}^{n-m} P_k^{n-m} \exp[\frac{1}{2} \sum_{k=1}^{n} \Delta \langle M^{(2)} \rangle_k]], \end{aligned} \tag{18}$$

where P_k^{n-m} are determined from formulas (16),(17), and (18).

Proof. From (12), using notations (13) and results of Lemma 1, we obtain immediately (19).

Corollary. The one-step ($m = 1$) optimal estimation of S is

$$\hat{S}_n(1) = S_{n-1}[\sum_{k=n+1}^{N} P_k^{n-1} \exp\{\frac{1}{2} \Delta \langle M^{(1)} \rangle_n\} + \sum_{k=1}^{n} P_k^{n-1} \exp\{\frac{1}{2} \Delta \langle M^{(2)} \rangle_n\}]. \tag{19}$$

If $\Delta \langle M^{(1)} \rangle = a > 0$ and $\Delta \langle M^{(2)} \rangle = 1$, then relation (20) yields

$$\hat{S}_n(1) = S_{n-1}[\exp\{\frac{a}{2}\} \sum_{k=n+1}^{N} P_k^{n-1} + \exp\{\frac{1}{2}\} \sum_{k=1}^{n} P_k^{n-1}].$$

Remark. The problem of finding the minimal entropy martingale measure within our model (1) is investigated in [6].

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