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ON THE BEHAVIOUR OF METRICS H_s ON LOOP GROUPS

The heat measures with respect to metric H_s on loop groups were introduced by P. Malliavin; its behaviour as $s \downarrow 1/2$ of finite dimensional distributions was studied by Y. Inahama [J. Funct. Anal., 198 (2003), p. 311-340]. In this note, we shall transfer this problem to the long time behaviour of diffusion processes. We conclude the result by using a metric equivalent form for the lower bound of Ricci tensors.

Let G be a semi-simple compact Lie group and $\mathcal{G}$ its Lie algebra endowed with an Ad_G invariant metric $\langle\ ,\ \rangle_{\mathcal{G}}$. Let S^1 be the unit circle and denote by $C^\infty(S^1, \mathcal{G})$ the space of smooth functions defined on S^1, taking values in $\mathcal{G}$. Each $h \in C^\infty(S^1, \mathcal{G})$ has the Fourier series expansion

$$h(\theta) = \sum_{n\in\mathbb{Z}} \hat{h}(n)\, e^{in\theta}, \quad \text{with } \hat{h}(n) = \int_{S^1} h(\theta) e^{-in\theta}\, \frac{d\theta}{2\pi}.$$

The metric H_s on $C^\infty(S^1, \mathcal{G})$ is defined by

$$|h|_s^2 = \sum_{n\in\mathbb{Z}} (1+n^2)^s |\hat{h}(n)|_{\mathcal{G}}^2.$$

Let

$$H_s(\mathcal{G}) = \big\{ h \in L^2(S^1, \mathcal{G});\ |h|_s < +\infty \big\}.$$

By Sobolev embedding theorem, when $s > 1/2$, $H_s(\mathcal{G}) \subset C(S^1, \mathcal{G})$. In this case, there is a Gaussian measure μ_s on $C(S^1, \mathcal{G})$ such that $(C(S^1, \mathcal{G}), H_s(\mathcal{G}), \mu_s)$ becomes an abstract Wiener space in the sense of L. Gross [7].

1. Green functions associated to H_s

Let $\{u_n;\ n \geq 1\}$ be an orthonormal basis of $H_s(\mathbb{R})$ and $\{\varepsilon_1, \cdots, \varepsilon_d\}$ an orthonormal basis of $\mathcal{G}$. Define

$$e_{n,\alpha}(\theta) = u_n(\theta)\varepsilon_\alpha, \quad n \geq 1,\, \alpha = 1, \cdots, d. \tag{1}$$

Then $\{e_{n,\alpha};\ n \geq 1, \alpha = 1, \cdots, d\}$ is an orthonormal basis of $H_s(\mathcal{G})$. Let $G^{(s)}(\theta_1, \theta_2)$ be the Green function on S^1 associated to the operator $(1 - \frac{d^2}{d\theta^2})^s$; that is the solution in the distribution sense of

$$(1 - \frac{d^2}{d\theta^2})^s G^{(s)}(\theta_1, \cdot) = \delta_{\theta_1},$$

where δ_{θ_1} is the Dirac mass at θ_1. We have the relation

$$G^{(s)}(\theta_1, \theta_2) = \sum_{n\geq 1} u_n(\theta_1) u_n(\theta_2). \tag{2}$$

If we consider

$$u_0 = 1, u_{2n-1}(\theta) = \frac{\sqrt{2}\cos n\theta}{(1+n^2)^{s/2}},\ u_{2n}(\theta) = \frac{\sqrt{2}\sin n\theta}{(1+n^2)^{s/2}},\ n \geq 1,$$

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we obtain the following expression (see [9])

$$G^{(s)}(\theta_1,\theta_2)=\sum_{n\in\mathbb{Z}}\frac{e^{in(\theta_1-\theta_2)}}{(1+n^2)^s}. \tag{3}$$

We have

$$(G^{(s)}(\theta_1,\cdot),u)_s=u(\theta_1)\quad \text{for } u\in H_s. \tag{4}$$

2. Brownian motion on the loop group

Let $\{x_{n,\alpha};\ n\geq 1,\alpha=1,\ldots,d\}$ be a sequence of independent standard real valued Brownian motion, defined on a probability space $(\Omega,\mathcal{F},P)$. Consider the Random series

$$x(t,\theta)=\sum_{n,\alpha}x_{n,\alpha}(t)e_{n,\alpha}(\theta). \tag{5}$$

It is known that for $s>1/2$, almost surely the series (5) converges uniformly with respect to $(t,\theta)\in[0,1]\times S^1$. We have the relation, using (2)

$$\mathbb{E}\Big(\langle x(t,\theta_1),a\rangle_{\mathcal{G}}\langle x(s,\theta_2),b\rangle_{\mathcal{G}}\Big)=(t\wedge s)G(\theta_1,\theta_2)\,\langle a,b\rangle_{\mathcal{G}},\quad a,b\in\mathcal{G}. \tag{6}$$

For $\theta\in S^1$ given, following P. Malliavin [10], we consider the SDE on G

$$d_tg_x(t,\theta)=g_x(t,\theta)\circ d_tx(t,\theta),\quad g_x(0,\theta)=e. \tag{7}$$

It has been proved that in [10] and in [2] that $(t,\theta)\to g_x(t,\theta)$ has a continuous version. Then $t\to g_x(t,\cdot)$ is a continuous process taking values in the loop group

$$\mathbb{L}(G)=C(S^1,G).$$

The geometry of $\mathbb{L}(G)$ endowed with the metric H_s was investigated in [6]. For $s>1/2$, geometric stochastic analysis was developed in [3, 2, 1, 4, 5, 8, 9], to mention but a few.

3. A geometric result

Let M be a compact smooth Riemannian manifold. Denote by ρ the Riemannian distance and m the normalized Riemannian volume of M. Let Ric be the Ricci tensor on M. There are several equivalent descriptions for the lower bound of Ric. Let $P_t(x,dy)$ be the heat measure on M starting from $x\in M$, then $P_t(x,dy)=p_t(x,y)m(dy)$. For a probability measure μ on M, we define $P_t^*\mu$ by

$$\int_M f\,dP_t^*\mu=\int_M f(y)P_t(x,dy)\,d\mu(x).$$

μ is said to invariant under P_t if $P_t^*\mu=\mu$. It is known that the Riemannian volume m is the unique invariant measure of P_t. The following result is taken from [11]:

Theorem 3.1. *The following properties are equivalent:*

(i) Ric $\geq K$*, which means that* $\langle \mathrm{Ric}_x v,v\rangle_x\geq K|v|_x^2$*, where* $\langle\ ,\ \rangle_x$ *is the inner product in the tangent space* T_xM*.*

(ii) For any probability measures μ,ν *on* M*,*

$$W_2(P_t^*\mu,P_t^*\nu)\leq e^{-Kt}\,W_2(\mu,\nu),\quad t\geq 0 \tag{8}$$

where W_2 *denotes the Wasserstein distance between two probability measures, which is defined by*

$$W_2^2(\mu,\nu)=\inf\Big\{\int_{M\times M}\rho(x,y)^2\,\pi(dx,dy);\ \pi\in\mathcal{C}(\mu,\nu)\Big\} \tag{9}$$

where $\mathcal{C}(\mu,\nu)$ denotes the totality of probability measures on the product space having μ and ν as marginal laws.

In particular, taking $\mu = m$ and $\nu = \delta_x$ in (8), we get

$$W_2(m, P_t(x,dy)) \le e^{-Kt}\, W_2(m,\delta_x), \quad t \ge 0 \tag{10}$$

4. Main result

Fix a partition $\mathcal{P} = \{\theta_1 < \cdots < \theta_N\}$ of S^1. Consider the stochastic process

$$t \to (g_x(t,\theta_1), \cdots, g_x(t,\theta_N))$$

on $G^{\mathcal{P}}$. In this note, we will give a new proof to the following result, due to [9].

Theorem 4.1. *Let $\mu_t^{(s)}$ be the law of $(g_x(t,\theta_1),\cdots,g_x(t,\theta_N))$ on $G^{\mathcal{P}}$. Then for any $t > 0$, when $s \downarrow 1/2$, $\mu_t^{(s)}$ converges weakly to the normalized Haar measure $dg_1 \cdots dg_N$ on $G^{\mathcal{P}}$.*

Proof. Denote by $\mathbb{G}^{(s)} = (G^{(s)}(\theta_i,\theta_j))_{1\le i,j\le N}$, which is a positive definite matrix. Let $\mathbb{Q}^{(s)} = (Q_{ij}^{(s)})_{1\le i,j\le N}$ be the inverse matrix of $\mathbb{G}^{(s)}$. Equipe $\mathcal{G}^{\mathcal{P}}$ with the metric

$$|a|_{\mathcal{P}}^2 = \sum_{i,j=1}^N Q_{ij}^{(s)}\, \langle a_i, a_j\rangle_{\mathcal{G}}, \quad a = (a_1,\cdots,a_N) \in \mathcal{G}^{\mathcal{P}}. \tag{11}$$

According to (6), under this metric, $t \to (x(t,\theta_1),\cdots,x(t,\theta_N))$ is a standard Brownian motion on $\mathcal{G}^{\mathcal{P}}$:

$$\mathbb{E}\Big(|(x(t,\theta_1),\cdots,x(t,\theta_N))|_{\mathcal{G}}^2\Big) = tN.$$

In order to explain our idea of proof, consider first $B_\theta(t) = \frac{x(t,\theta)}{\sqrt{\alpha_s}}$, where according to (3)

$$\alpha_s^2 = G(\theta,\theta) = \sum_{n\in\mathbb{Z}} \frac{1}{(1+n^2)^s}.$$

We see that

$$\lim_{s\to 1/2} \alpha_s = +\infty. \tag{12}$$

Consider the SDE on G:

$$d_t g_x(t,\theta) = \sqrt{\alpha_s}\, g_x(t,\theta) \circ dB_\theta(t), \quad g_x(0,\theta) = e.$$

Then the law of $x \to g_x(t,\theta)$ has the density $p_{\alpha_s t}(g)$, where p_t is the heat kernel on G. Therefore for any $t > 0$, as $s \downarrow 1/2$, $p_{\alpha_s t}(g)$ converges weakly to the normalized Haar measure dg on G. For the general case, let

$$U^{(s)} = \frac{1}{\alpha_s}\mathbb{G}^{(s)}.$$

Then as $s \downarrow 1/2$, the matrix $U^{(s)}$ converges to Id. Define

$$B_{\mathcal{P}}(t) = \frac{1}{\sqrt{\alpha_s}} \sqrt{U^{(s)}}^{-1} \begin{pmatrix} x(t,\theta_1) \\ \cdot \\ \cdot \\ \cdot \\ x(t,\theta_N)) \end{pmatrix}, \tag{13}$$

where $\sqrt{U^{(s)}}$ denotes the square root of $U^{(s)}$. Then by (11), $t \to B_{\mathcal{P}}(t)$ is a standard Brownian motion on $\mathcal{G}^{\mathcal{P}}$ endowed with the direct product metric

$$\mathbb{E}\Big(\langle B_{\mathcal{P}}^i(t), B_{\mathcal{P}}^j(s)\rangle_{\mathcal{G}}\Big) = d\,(t \wedge s)\delta_{ij}, \tag{14}$$

where d is the dimension of $\mathcal{G}$. Set $g_{\mathcal{P}}(t) = (g_x(t,\theta_1), \cdots, g_x(t,\theta_N))$. Then $g_{\mathcal{P}}$ solves the SDE on $G^{\mathcal{P}}$

$$dg_{\mathcal{P}}(t) = \sqrt{\alpha_s}\, g_{\mathcal{P}}(t) \circ \sqrt{U^{(s)}} dB_{\mathcal{P}}(t), \quad g_{\mathcal{P}}(0) = (e, \cdots, e).$$

Now consider the SDE on $G^{\mathcal{P}}$

$$d\hat{g}_{\mathcal{P}}(t) = \hat{g}_{\mathcal{P}}(t) \circ \sqrt{U^{(s)}} dB_{\mathcal{P}}(t), \quad \hat{g}_{\mathcal{P}}(0) = (e, \cdots, e).$$

Then by the above two SDE, the law $\hat{\mu}_t^{(s)}$ of $\hat{g}_{\mathcal{P}}(t)$ satisfies the relation

$$\mu_t^{(s)} = \hat{\mu}_{\alpha_s t}^{(s)}. \tag{15}$$

We know that $U^{(s)}$ induces a left invariant metric on $G^{\mathcal{P}}$. Let $\mu_\infty^{(s)}$ be the normalized Riemannian measure on $G^{\mathcal{P}}$. Then by uniqueness of the Haar measure

$$\mu_\infty^{(s)} = dg_1 \cdots dg_N. \tag{16}$$

Lemma 4.2. *There exists a $\delta > 0$ such that for each $f \in C(G^{\mathcal{P}})$*

$$\lim_{t\to+\infty} \sup_{s\in[\frac{1}{2},\frac{1}{2}+\delta]} \Big| \int_{G^{\mathcal{P}}} f d\hat{\mu}_t^{(s)} - \int_{G^{\mathcal{P}}} f dg_1 \cdots dg_N \Big| = 0. \tag{17}$$

Proof. We give first a general look. Let G be a general compact Lie group with the Lie algebra $\mathcal{G}$. Let $\{e_1, \cdots, e_d\}$ be a basis of $\mathcal{G}$. Consider the structure equation

$$[e_i, e_j] = \sum_{k=1}^d c_{ij}^k e_k.$$

Let $g_{ij} = \langle e_i, e_j \rangle$. Then the Christoffel coefficients are given by

$$\Gamma_{ij}^q = \frac{1}{2} \sum_{k,\ell=1}^d g^{kq} \Big(c_{ij}^\ell g_{\ell,k} - c_{ik}^\ell g_{\ell j} - c_{jk}^\ell g_{\ell i} \Big)$$

where (g^{kq}) is the inverse matrix of (g_{ij}). It follows that the Ricci tensor Ric on G depend continuously of the metric (g_{ij}).

Now returning to our situation, denote by $\mathrm{Ric}_s^{\mathcal{P}}$ the Ricci tensor on $\mathcal{G}^{\mathcal{P}}$ associated to the metric $U^{(s)}$ and $\mathrm{Ric}^{\mathcal{P}}$ associated to the direct product metric. Then as $s \downarrow 1/2$,

$$\mathrm{Ric}_s^{\mathcal{P}} \to \mathrm{Ric}^{\mathcal{P}} = \frac{1}{4} \mathrm{Id} \text{ on } \mathcal{G}^{\mathcal{P}}. \tag{18}$$

Therefore it exists $\delta > 0$ such that $\mathrm{Ric}_s^{\mathcal{P}} \geq \frac{1}{8}\mathrm{Id}$ for $s \in [\frac{1}{2}, \frac{1}{2} + \delta]$. For the simplicity, denote by h the Haar measure on $G^{\mathcal{P}}$. Let $F \in C^1(G^{\mathcal{P}})$ and any $\pi \in \mathcal{C}(\hat{\mu}_t^{(s)}, h)$ (see section 3), we have

$$\int_{G^{\mathcal{P}}} F d\hat{\mu}_t^{(s)} - \int_{G^{\mathcal{P}}} F dh = \int_{G^{\mathcal{P}} \times G^{\mathcal{P}}} \big(F(x) - F(y)\big) \pi(dx, dy).$$

It follows that for constant $C_F > 0$

$$\Big| \int_{G^{\mathcal{P}}} F d\hat{\mu}_t^{(s)} - \int_{G^{\mathcal{P}}} F dh \Big| \leq C_F\, W_2(\hat{\mu}_t^{(s)}, h)$$

which is dominated by

$$C_F e^{-t/8} W_2(\delta_{\mathbf{e}}, h)$$

due to Theorem 3.1, where $\mathbf{e}$ is the unit element of $G^{\mathcal{P}}$. It follows that for any $f \in C(G^{\mathcal{P}})$,

$$\lim_{t\to+\infty} sup_{s\in[\frac{1}{2},\frac{1}{2}+\delta]} \Big| \int_{G^{\mathcal{P}}} f d\hat{\mu}_t^{(s)} - \int_{G^{\mathcal{P}}} dh \Big| = 0. \tag{19}$$

The proof is complete. ∎

We continue the proof of Theorem 4.1. Let $f \in C(G^{\mathcal{P}})$. Then for $s_0 \in]\frac{1}{2}, \frac{1}{2}+\delta]$ and according to (15)

$$\Big|\int_{G^{\mathcal{P}}} f d\mu_t^{(s_0)} - \int_{G^{\mathcal{P}}} f\, dg_1 \cdots dg_N\Big| = \Big|\int_{G^{\mathcal{P}}} f d\hat{\mu}^{(s_0)}_{\alpha_{s_0} t} - \int_{G^{\mathcal{P}}} f\, dg_1 \cdots dg_N\Big|$$
$$\leq sup_{s\in[\frac{1}{2},\frac{1}{2}+\delta]}\Big|\int_{G^{\mathcal{P}}} f d\hat{\mu}^{(s)}_{\alpha_{s_0} t} - \int_{G^{\mathcal{P}}} dg_1 \cdots dg_N\Big|,$$

which converges to 0 as $s_0 \to 1/2$ due to Lemma 4.2. □

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