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ON THE CONVERGENCE OF SERIES OF AUTOREGRESSIVE SEQUENCES IN BANACH SPACES

Necessary and sufficient conditions for the almost sure convergence of a series of autoregressive sequences in separable Banach spaces are studied. As an application of the obtained results, the condition for the admissible shift of a zero-mean Gaussian Markov measure is considered.

1. Introduction

Consider the recurrence relation

$$X_0 = 0, \quad X_k = \alpha_k X_{k-1} + Z_k, \quad k \geq 1, \tag{1}$$

where (α_k) is a nonrandom real sequence such that $\alpha_1 = 1$, and (Z_k) is a sequence of independent symmetric random elements of some separable Banach space $\mathfrak{X}$ such that

$$\mathsf{P}\{Z_k = 0\} < 1, \qquad k \geq 1.$$

We recall that a random element Z is called *symmetric* if Z and $(-Z)$ are identically distributed.

In particular, if $Z_k = \beta_k \theta_k$, $k \geq 1$, where (β_k) is a nonrandom real sequence and (θ_k) is *a standard Gaussian sequence*, i.e. (θ_k) is a sequence of independent N(0,1)-distributed Gaussian random variables, then (X_k) is a zero-mean Gaussian Markov sequence [5].

For a given sequence (X_k), we now consider the random series

$$\sum_{k=1}^{\infty} X_k. \tag{2}$$

This paper deals with necessary and sufficient conditions for the convergence *almost surely* (*a.s.*) of series (2).

It is worth noting that the necessary and some sufficient conditions for the convergence a.s. of series (2) in the case of autoregressive sequences of random variables were obtained in [2]. The method introduced in [2] is based on the theory of random series with independent symmetric terms in Banach spaces. This allows one to consider the more general case of sequences (X_k).

In the present paper, the special attention is devoted to the summation of autoregressive sequences with weighted coefficients. The sufficient conditions obtained for such a series were applied to the problem of absolute continuity and singularity of two Gaussian Markov measures.

In order to find the necessary conditions for the convergence a.s. of series (2), we consider the sequence of its partial sums

$$S_n = \sum_{k=1}^{n} X_k, \quad n \geq 1, \tag{3}$$

as a series with independent symmetric terms in the separable Banach space of convergent sequences. Such an approach allows one to use the theory of random series in separable

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Banach sequence spaces [3, 4]. By finding the sufficient conditions, we use the theory of infinite-summability matrices [3, 4].

In particular, the general results immediately yield the necessary and sufficient conditions for convergence a.s. of series

$$\sum_{k=1}^{\infty} \xi_k \tag{4}$$

for zero-mean Gaussian Markov sequences (ξ_k) described by the recurrence relation

$$\xi_0 = 0, \quad \xi_k = \alpha_k \xi_{k-1} + \beta_k \theta_k, \quad k \geq 1. \tag{5}$$

Some of them were already considered in [2].

2. Preliminaries

Let $\mathfrak{X}$ be the separable Banach space, let $\mathfrak{X}^\infty$ be the space of all sequences of elements of $\mathfrak{X}$, and let $c(\mathfrak{X})$ be the space of all convergent sequences from $\mathfrak{X}^\infty$. The space $c(\mathfrak{X}^\infty)$ is a separable Banach space if it is endowed with the norm $\|x\|_\infty = \sup\limits_{k\geq 1}|x_k|$, $x = (x_k) \in c(\mathfrak{X}^\infty)$ [4].

The recurrence equation (1) implies that sequence (3) can be represented in the form

$$\begin{pmatrix} S_1 \\ S_2 \\ \vdots \\ S_n \\ \vdots \end{pmatrix} = \begin{pmatrix} A(1,1) \\ A(2,1) \\ \vdots \\ A(n,1) \\ \vdots \end{pmatrix} Z_1 + \begin{pmatrix} 0 \\ A(2,2) \\ \vdots \\ A(n,2) \\ \vdots \end{pmatrix} Z_2 + ... + \begin{pmatrix} 0 \\ \vdots \\ 0 \\ A(n,n) \\ \vdots \end{pmatrix} Z_n + \dots, \tag{6}$$

where

$$A(n,k) = \begin{cases} 1 + \sum_{l=1}^{n-k} \left(\prod_{j=k+1}^{k+l} \alpha_j \right), & 1 \leq k \leq n-1, \\ 1, & k = n, \\ 0, & k > n, \end{cases}$$

and

$$S_n = \sum_{k=1}^{n} A(n,k) Z_k, \ \ n \geq 1. \tag{7}$$

It is convenient to rewrite series (6) as

$$\overrightarrow{S} = \sum_{k=1}^{\infty} Z_k \overrightarrow{A}_k, \tag{8}$$

where

$$\overrightarrow{S} = \begin{pmatrix} S_1 \\ S_2 \\ \vdots \\ S_n \\ \vdots \end{pmatrix}, \quad \overrightarrow{A}_k = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ A(k,k) \\ A(k+1,k) \\ \vdots \end{pmatrix}, \quad k \geq 1.$$

It is worth noting that series (8) converges in the coordinate-wise sense. Thus, the sequence of partial sums (S_n) is represented in the form of series (8) with independent symmetric random terms in the space $\mathfrak{X}^\infty$. Therefore, the sequence (S_n) satisfies the conditions of Theorem 2.1.1 [3]. This theorem states that *if* (Z_k) *is a sequence of independent symmetric random elements of the separable Banach space* $\mathfrak{X}$, $\overrightarrow{S} = \sum_{k=1}^{\infty} Z_k \overrightarrow{A}_k$, *and if this series converges in the coordinate-wise sense, then, given that* $\overrightarrow{S} \in c(\mathfrak{X}^\infty)$

a.s., one has $Z_k\overrightarrow{A}_k \in c(\mathfrak{X}^\infty)$, $k \geq 1$, *and the series* $\sum_{k=1}^\infty Z_k\overrightarrow{A}_k$ *converges a.s. in the norm of the space* $c(\mathfrak{X}^\infty)$ *a.s.*

The theorem above gives a criterion for the convergence a.s. of series (2).

Lemma 2.1. *The random series* (2) *converges a.s. if and only if* $\overrightarrow{A}_k \in c(\mathfrak{X}^\infty)$, $k \geq 1$, *and the random series* (8) *converges a.s. in the norm of the space* $c(\mathfrak{X}^\infty)$.

3. Necessary conditions

For $k \geq 1$, let us consider the nonrandom series

$$A(\infty, k) = 1 + \sum_{l=1}^{\infty} \prod_{j=k+1}^{k+l} \alpha_j. \tag{9}$$

We note that $A(\infty, k) = \lim_{n\to\infty} A(n, k)$, if $\lim_{n\to\infty} A(n, k)$ exists, i.e. series (9) converges.

The next result follows from Lemma 2.1 and Theorem 2.8.1 [4].

Theorem 3.1. *If the random series* (2) *converges a.s., then the nonrandom series* (9) *converges for any* $k \geq 1$, *and the random series*

$$\sum_{k=1}^{\infty} A(\infty, k) Z_k \tag{10}$$

converges a.s. Moreover, the equality

$$\sum_{k=1}^{\infty} X_k = \sum_{k=1}^{\infty} A(\infty, k) Z_k \quad a.s. \tag{11}$$

holds true.

Consider the case where (Z_k) is a sequence of independent zero-mean Gaussian random elements in a separable Hilbert space H. For given (Z_k), the above result is specialized as follows.

Corollary 3.1. *Let* (Z_k) *be a sequence of independent zero-mean Gaussian random elements in a separable Hilbert space* H. *If the random series* (2) *converges a.s., then the nonrandom series* (9) *converges for any* $k \geq 1$, *and*

$$\sum_{k=1}^{\infty} (A(\infty, k))^2 \, \mathsf{E} \, \|Z_k\|^2 < \infty. \tag{12}$$

Moreover, equality (11) *holds true.*

Proof. Corollary 3.1 follows from Corollary 1.4.4 [3], since the random series (10) with independent zero-mean Gaussian random elements in a separable Hilbert space converges a.s. if and only if condition (12) holds true. □

Corollary 3.1 yields the necessary conditions for convergence a.s. of series (4) for a zero-mean Gaussian Markov sequence (ξ_k).

Corollary 3.2. *Suppose that* (θ_k) *is a standard Gaussian sequence, i.e.* (ξ_k) *is a zero-mean Gaussian Markov sequence. If the random series* (4) *converges a.s., then the non-random series* (9) *converges for any* $k \geq 1$, *and*

$$\sum_{k=1}^{\infty} (\beta_k A(\infty, k))^2 < \infty. \tag{13}$$

Moreover, the equality

$$\sum_{k=1}^{\infty} \xi_k = \sum_{k=1}^{\infty} \beta_k A(\infty, k) \theta_k \quad a.s. \tag{14}$$

holds true.

Introducing some more notation, we obtain the necessary conditions for the convergence a.s. of the random series (4) for zero-mean Gaussian Markov sequences (ξ_k) in "correlation" terms.

For a zero-mean Gaussian Markov sequence (ξ_k), we consider two sequences: *the sequence of variance*, (σ_k^2), and *the sequence of correlation coefficients*, $(r_{k,k+1})$, where $\sigma_k^2 = \mathsf{E}\,{\xi_k}^2$, $k \ge 1$, and $r_{k,k+1} = (\mathsf{E}\,\xi_k\xi_{k+1})/\sigma_k\sigma_{k+1}$ if $\sigma_k\sigma_{k+1} > 0$, and $r_{k,k+1} = 0$ if $\sigma_k\sigma_{k+1} = 0$, $k \ge 1$. It is well known [5] that

$$\mathsf{E}\,\xi_j\xi_m = \sigma_j\sigma_m \prod_{i=j}^{m-1} r_{i,i+1}$$

for any $m \ge 1$ and $1 \le j < m$. For $k \ge 1$, we consider the nonrandom series

$$B(k) = (1 - r_{k-1,k}^2)^{1/2}(\sigma_k + \sum_{l=k+1}^{\infty} \sigma_l \prod_{i=k}^{l-1} r_{i,i+1}), \quad k \ge 1, \tag{15}$$

where $r_{0,1} = 0$.

Corollary 3.3. *Suppose that (ξ_k) is a zero-mean Gaussian Markov sequence such that $\sigma_k^2 > 0$, $k \ge 2$. If the random series* (4) *converges a.s., then the nonrandom series* (15) *converges for any $k \ge 1$, and*

$$\sum_{k=1}^{\infty} (B(k))^2 < \infty. \tag{16}$$

Moreover, if (ξ_k) is generated by the standard Gaussian sequence (θ_k) (recall (2)*), then the random series*

$$\sum_{k=1}^{\infty} B(k)\theta_k \tag{17}$$

converges a.s., and the equality

$$\sum_{k=1}^{\infty} \xi_k = \sum_{k=1}^{\infty} B(k)\theta_k \quad a.s. \tag{18}$$

holds true.

Proof. Corollary 3.3 follows from Corollary 3.2, since

$$\alpha_k = \frac{\sigma_k}{\sigma_{k-1}} r_{k-1,k}, \quad k \ge 2; \quad \beta_1^2 = \sigma_1^2, \quad \beta_k^2 = \sigma_k^2(1 - r_{k-1,k}^2), \quad k \ge 2,$$

and

$$A(\infty, k) = B(k), \quad k \ge 1. \quad \square$$

4. Sufficient conditions

This section deals with sufficient conditions for the convergence a.s. of series (2). The method used in this section is based on the theory of infinite-summability matrices [3, 4].

Consider an *infinite-summability* real matrix $\Lambda = (\lambda_{n,k})_{n,k=1}^{\infty}$. This means that $\lim_{n\to\infty} \lambda_{n,k} = 1$ for all $k \ge 1$. Consider also a real series $\sum_{k=1}^{\infty} X_k$ in a separable Banach space $\mathfrak{X}$. To this series and to the matrix Λ, we relate the sequence of series $\sum_{k=1}^{\infty} \lambda_{n,k} X_k, n \ge 1$. Assume that all these series converge. We denote their sums by $\Xi_n, n \ge 1$. Then, if the sequence (Ξ_n) converges in the space $\mathfrak{X}$, the series $\sum_{k=1}^{\infty} X_k$ is called Λ-*summable*, and the limit $\lim_{n\to\infty} \Xi_n$ is called the Λ-*sum* of the series $\sum_{k=1}^{\infty} X_k$.

Let Λ be a summability matrix. If

$$Varn(\Lambda) = \sup_{n\ge1} \sup_{m\ge2} \Big[\sum_{k=1}^{m-1} \Big(|\lambda_{n,k} - \lambda_{n,k+1}| \Big) + |\lambda_{n,m}| \Big] < \infty,$$

then the matrix Λ is called *the matrix of bounded variation.*

In order to obtain the sufficient conditions, we use one result which asserts the equivalence of the summation by matrices of bounded variation (see [4], Theorem 2.8.2). Theorem 2.8.2 [4] says that *if the sequence* (X_k) *is a sequence of independent symmetric random variables in a separable Banach space and the series* $\sum_{k=1}^{\infty} X_k$ *is* Λ'*-summable a.s. by some matrix of bounded variation* Λ'*, then it is* Λ*-summable a.s. by all the matrices of bounded variation, and all* Λ*-sums are equal a.s. to each other.* In particular, this theorem says that *if the series* $\sum_{k=1}^{\infty} X_k$ *is convergent a.s., then it is* Λ*-summable a.s. by all the matrices of bounded variation, and all* Λ*-sums are equal a.s. to* $\sum_{k=1}^{\infty} X_k$.

This theorem provides the sufficient conditions for the convergence a.s. of series (2).

Theorem 4.1. *Suppose that* $\alpha_k \geq 0$, $k \geq 2$. *If the nonrandom series* (9) *converges for any* $k \geq 1$, *and the random series* (10) *converges a.s., then the random series* (2) *converges a.s., and the equality* (11) *holds true.*

Proof. Assume that the series $\sum_{k=1}^{\infty} A(\infty, k)\theta_k$ converges a.s. Consider the matrix $\Lambda = (\lambda_{n,k})_{n,k=1}^{\infty}$, where

$$\lambda_{n,k} = \begin{cases} \frac{A(n,k)}{A(\infty,k)}, & 1 \leq k \leq n, \\ 0, & k > n. \end{cases} \tag{19}$$

Observe that all $\lambda_{n,k}$ are well-defined, since the series $A(\infty, k)$ converges, and

$$A(\infty, k) \neq 0$$

for any $k \geq 1$. Since $\lim_{n\to\infty} A(n, k) = A(\infty, k)$ for any $k \geq 1$, we have

$$\lim_{n\to\infty} \lambda_{n,k} = \lim_{n\to\infty} \frac{A(n,k)}{A(\infty,k)} = 1, \quad k \geq 1.$$

Hence, the matrix Λ is a summability matrix and, for $Y_k = A(n, k)Z_k$, $k \geq 1$,

$$\sum_{k=1}^{\infty} \lambda_{n,k} Y_k = \sum_{k=1}^{n} \frac{A(n,k)}{A(\infty,k)} A(\infty,k) Z_k = \sum_{k=1}^{n} A(n,k) Z_k, \quad n \geq 1.$$

Thus, by (7) and (2),

$$\sum_{k=1}^{\infty} \lambda_{n,k} Y_k = \sum_{k=1}^{n} X_k, \quad n \geq 1. \tag{20}$$

Since $\alpha_k \geq 0$, $k \geq 2$, $\lambda_{n,k} \geq 0$ for any $k, n \geq 1$, and one can obtain

$$\lambda_{n,k} - \lambda_{n,k+1} = \frac{A(n,k)}{A(\infty,k)} - \frac{A(n,k+1)}{A(\infty,k+1)} = \frac{(\alpha_{k+2}\alpha_{k+3}\dots\alpha_{n+1})A(\infty,n+1)}{(1+\alpha_{k+1}A(\infty,k+1))A(\infty,k+1)} \geq 0$$

for any $1 \leq k \leq n-1$. Therefore,

$$Varn(\Lambda) = \sup_{n\geq 2} \Big[\sum_{k=1}^{n-1} \Big(\lambda_{n,k} - \lambda_{n,k+1} \Big) + \lambda_{n,n} \Big] = \sup_{n\geq 2} (\lambda_{n,1}) = \sup_{n\geq 2} \frac{A(n,1)}{A(\infty,1)} \leq 1.$$

Thus, the matrix Λ is a summability matrix of bounded variation.

Since the sequence $(A(\infty, k)Z_k)$ is a sequence of independent symmetric random elements and the series $\sum_{k=1}^{\infty} A(\infty, k)Z_k$ converges a.s., this sequence is, by Theorem 2.8.2 [4], Λ-summable a.s., and its Λ-sum is equal a.s. to $\sum_{k=1}^{\infty} A(\infty, k)Z_k$.

Therefore, by (20), the limit

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} \lambda_{n,k} Y_k = \lim_{n\to\infty} \sum_{k=1}^{n} X_k = \sum_{k=1}^{\infty} X_k$$

exists a.s., and equality (11) holds. □

Corollary 4.1. *Suppose that* $\alpha_k \geq 0$, $k \geq 2$, *and* (Z_k) *is a sequence of independent zero-mean Gaussian random elements in a separable Hilbert space* H. *The random series* (2) *converges a.s., if and only if the nonrandom series* (9) *converges for any* $k \geq 1$, *and condition* (12) *holds. Moreover, equality* (11) *holds true.*

For the zero-mean Gaussian Markov (ξ_k) sequence, the above results are specialized as follows.

Corollary 4.2. *Suppose that $\alpha_k \geq 0$, $k \geq 2$, and (θ_k) is a standard Gaussian sequence, i.e. (ξ_k) is a zero-mean Gaussian Markov sequence. The random series* (4) *converges a.s., if and only if the nonrandom series* (9) *converges for any $k \geq 1$, and condition* (13) *holds. Moreover, equality* (14) *holds true.*

Corollary 4.3. *Suppose that (ξ_k) is a zero-mean Gaussian Markov sequence such that $\sigma_k^2 > 0$, $k \geq 2$, and $r_{k-1,k} \geq 0, k \geq 2$. The random series* (4) *converges a.s., if and only if the nonrandom series* (15) *converges for any $k \geq 1$, and the condition* (16) *holds. Moreover, if (ξ_k) is generated by the standard Gaussian sequence (θ_k) (see* (5)*), then the random series* (17) *converges a.s., and the equality* (18) *holds true.*

In the next theorem, we consider the sequences (α_k) with elements of alternating signs.

Theorem 4.2. *Assume that the nonrandom series* (9) *converges for any $k \geq 1$. Let ε and M be two positive numbers such that*

$$0 < \varepsilon \leq |A(\infty, k)| \leq M < \infty \tag{21}$$

for any $k \geq 1$, and

$$H = \sup_{n\geq 1} \sum_{k=1}^{n} \prod_{j=k+2}^{n+1} |\alpha_j| < \infty. \tag{22}$$

If the random series (10) *converges a.s., then the random series* (2) *converges a.s., and equality* (11) *holds true.*

Proof. Consider the matrix $\Lambda = (\lambda_{n,k})_{n,k=1}^{\infty}$, which is defined at (19). By the proof of Theorem 4.1 above, we have

$$\lambda_{n,k} - \lambda_{n,k+1} = \frac{(\alpha_{k+2}\alpha_{k+3}\ldots\alpha_{n+1})A(\infty, n+1)}{A(\infty, k)A(\infty, k+1)},$$

for any $n \geq 2$ and $1 \leq k \leq n-1$. Hence, by (21) and (22),

$$Varn(\Lambda) = \sup_{n\geq 2} \Big[\Big(\sum_{k=1}^{n-1} |\lambda_{n,k} - \lambda_{n,k+1}|\Big) + |\lambda_{n,n}|\Big] =$$

$$= \sup_{n\geq 2} \Big[\Big(\sum_{k=1}^{n-1} \frac{|\alpha_{k+2}\alpha_{k+3}\ldots\alpha_{n+1}||A(\infty, n+1)|}{|A(\infty, k)A(\infty, k+1)|}\Big) + \frac{1}{|A(\infty, n)|}\Big] \leq \frac{MH}{\varepsilon^2} + \frac{1}{\varepsilon} < \infty.$$

Much of the following repeats the proof of Theorem 4.1. □

Example 4.1. *Suppose that $0 < q < 1$ and $\alpha_k = (-1)^k q^k$, $k \geq 2$. Then, for the sequence (α_k), all conditions of Theorem 4.2 hold.*

5. Series of autoregressive sequences with weighted coefficients

For the autoregressive sequences (X_k) (recall (1)) and a real sequence (c_k) such that $c_k \neq 0$, $k \geq 1$, consider the random series

$$\sum_{k=1}^{\infty} c_k X_k. \tag{23}$$

Denote $Y_k = c_k X_k$, $k \geq 1$. It is clear that (Y_k) is an autoregressive sequence, and

$$Y_0 = 0, \quad Y_k = \widetilde{\alpha}_k Y_{k-1} + c_k Z_k, \quad k \geq 1,$$

where

$$\widetilde{\alpha}_1 = 1, \quad \widetilde{\alpha}_k = \frac{c_k}{c_{k-1}}\,\alpha_k, \ k \geq 2,$$

and (Z_k) is a sequence of independent symmetric random elements of the separable Banach space $\mathfrak{X}$ (recall (1)). Put

$$\widetilde{A}_c(n,k) = \begin{cases} c_k + \sum_{l=1}^{n-k} \left(c_{k+l} \prod_{j=k+1}^{k+l} \alpha_j \right), & 1 \le k \le n-1, \\ c_k, & k = n, \\ 0, & k > n, \end{cases}$$

and denote, for $k \ge 1$,

$$\widetilde{A}_c(\infty,k) = \lim_{n\to\infty} \widetilde{A}_c(n,k) = c_k + \sum_{l=1}^{\infty} \left(c_{k+l} \prod_{j=k+1}^{k+l} \alpha_j \right) \tag{24}$$

if this limit exists. The results of Sections 3 and 4 yield the necessary and sufficient conditions for the convergence a.s. of series (23).

Theorem 5.1. *If the random series* (23) *converges a.s., then the nonrandom series* (24) *converges for any* $k \ge 1$, *and the random series*

$$\sum_{k=1}^{\infty} \widetilde{A}_c(\infty,k) Z_k \tag{25}$$

converges a.s. Moreover, the equality

$$\sum_{k=1}^{\infty} c_k X_k = \sum_{k=1}^{\infty} \widetilde{A}_c(\infty,k) Z_k \quad \text{a.s.} \tag{26}$$

holds true.

Theorem 5.2. *Suppose that* $\widetilde{\alpha}_k \ge 0$, $k \ge 2$. *The random series* (23) *converges a.s., if and only if the nonrandom series* (24) *converges for any* $k \ge 1$, *and the random series* (25) *converges a.s. Moreover, equality* (26) *holds true.*

Corollary 5.1. *Let* (α_k) *be a real sequence such that the sequence* $(|\alpha_k|)$ *satisfies the conditions of Theorem* 4.1. *Then there exists some real sequence* (c_k), *where* $c_k = \pm 1$, $k \ge 1$, *such that the random series* (23) *converges a.s.*

Corollary 5.2. *Let* (Z_k) *be a sequence of independent zero-mean Gaussian random elements in a separable Hilbert space* H. *If the random series* (23) *converges a.s., then the nonrandom series* (24) *converges for any* $k \ge 1$, *and*

$$\sum_{k=1}^{\infty} (\widetilde{A}_c(\infty,k))^2 \, \mathsf{E} \, \|Z_k\|^2 < \infty. \tag{27}$$

Moreover, equality (26) *holds true.*

Corollary 5.3. *Suppose that* $\widetilde{\alpha}_k \ge 0$, $k \ge 2$, *and* (Z_k) *is a sequence of independent zero-mean Gaussian random elements in a separable Hilbert space* H. *The random series* (23) *converges a.s., if and only if the nonrandom series* (24) *converges for any* $k \ge 1$, *and condition* (27) *holds. Moreover, equality* (26) *holds true.*

Theorems 5.1 and 5.2 provide the necessary and sufficient conditions of convergence a.s. of series

$$\sum_{k=1}^{\infty} c_k \xi_k \tag{28}$$

for a zero-mean Gaussian Markov sequence (ξ_k).

Corollary 5.4. *Suppose that* (θ_k) *is a standard Gaussian sequence, i.e.,* (ξ_k) *is a zero-mean Gaussian Markov sequence. If the random series* (28) *converges a.s., then the* *nonrandom series* (24) *converges for any* $k \ge 1$, *and*

$$\sum_{k=1}^{\infty} (\beta_k \widetilde{A}_c(\infty, k))^2 < \infty. \tag{29}$$

Moreover, the equality

$$\sum_{k=1}^{\infty} c_k \xi_k = \sum_{k=1}^{\infty} \beta_k \widetilde{A}_c(\infty, k)\theta_k \quad a.s. \tag{30}$$

holds true.

Corollary 5.5. *Let* (ξ_k) *be a zero-mean Gaussian Markov sequence such that* $\sigma_k^2 > 0$, $k \ge 2$. *If the random series* (28) *converges a.s., then the nonrandom series*

$$\widetilde{B}(k) = (1 - r_{k-1,k}^2)^{1/2} (c_k \sigma_k + \sum_{l=k+1}^{\infty} c_l \sigma_l \prod_{i=k}^{l-1} r_{i,i+1}), \tag{31}$$

converges for any $k \ge 1$, *and*

$$\sum_{k=1}^{\infty} (\widetilde{B}(k))^2 < \infty. \tag{32}$$

Moreover, if (ξ_k) *is generated by the standard Gaussian sequence* (θ_k) (*recall* (5)), *then the random series*

$$\sum_{k=1}^{\infty} \widetilde{B}(k)\theta_k \tag{33}$$

converges a.s., and the equality

$$\sum_{k=1}^{\infty} c_k \xi_k = \sum_{k=1}^{\infty} \widetilde{B}(k)\theta_k \quad a.s. \tag{34}$$

holds true.

Corollary 5.6. *Suppose that* (θ_k) *is a standard Gaussian sequence, i.e.,* (ξ_k) *is a zero-mean Gaussian Markov sequence. Suppose also that* $\widetilde{\alpha}_k \ge 0$, $k \ge 2$. *The random series* (28) *converges a.s., if and only if the nonrandom series* (24) *converges for any* $k \ge 1$, *and* (29) *holds. Moreover, equality* (30) *holds true.*

Corollary 5.7. *Let* (ξ_k) *be a zero-mean Gaussian Markov sequence such that* $\sigma_k^2 > 0$, $k \ge 2$, *and* $c_k c_{k-1} r_{k-1,k} \ge 0$, $k \ge 2$. *The random series* (28) *converges a.s., if and only if the nonrandom series* (31) *converges for any* $k \ge 1$, *and* (32) *holds. Moreover, if* (ξ_k) *is generated by the standard Gaussian sequence* (θ_k) (*recall* (5)), *then the random series* (33) *converges a.s., and equality* (34) *holds true.*

6. On the admissible shift of Gaussian Markov measures

As an application of the general results consider the problem of finding the conditions providing the equivalence and the singularity of Gaussian Markov measures.

Let (ξ_k) be a zero-mean Gaussian Markov sequence (recall (5)), such that $\beta_k^2 > 0, k \ge 1$. Consider another Gaussian Markov sequence $(\widehat{\xi}_k)$, $\widehat{\xi}_k = \xi_k + s_k$, $k \ge 1$, where (s_k) is nonrandom real sequence. Suppose also that

$$\overline{\lim_{n\to\infty}} \frac{\prod_{j=k}^{k+n} |\alpha_j|}{\beta_{k+n}} < \infty, \quad k \ge 2, \quad \lim_{k\to\infty} \frac{s_k^2}{\beta_k^2} = 0. \tag{35}$$

Note that conditions (35) in "correlation" terms are of the following form:

$$\overline{\lim_{n\to\infty}} \frac{|r_{k,k+n+1}|}{\sqrt{1 - r_{k+n,k+n+1}^2}} < \infty, \quad k \ge 2, \quad \lim_{k\to\infty} \frac{s_k^2}{\sigma_k^2(1 - r_{k-1,k}^2)} = 0.$$

By P and $\widehat{P}$, we denote the distributions of (ξ_k) and $(\widehat{\xi}_k)$ in $(\mathbb{R}^\infty, B(\mathbb{R}^\infty))$, respectively.

Recall [5] that the probabilistic measure $\widehat{P}$ is called *absolutely continuous* with respect to another probabilistic measure P ($\widehat{P} \ll P$), if $P(G) = 0$ implies $\widehat{P}(G) = 0$ for $G \in B(\mathbb{R}^\infty)$. If both $\widehat{P} \ll P$ and $P \ll \widehat{P}$, then P and $\widehat{P}$ are called *equivalent* ($\widehat{P} \sim P$). The measures P and $\widehat{P}$ are called *singular* ($\widehat{P} \perp P$), if there exists $G \in B(\mathbb{R}^\infty)$, such that both $P(G) = 1$ and $\widehat{P}(G) = 1$.

The sequence (s_k) is called the *admissible shift* of a zero-mean Gaussian measure P if the distributions P and $\widehat{P}$ are equivalent.

The well-known Gaec–Feldmann alternative says that two Gaussian measures in a locally convex space are equivalent or singular. Using the results of Section 5, the conditions of equivalence for the given Gaussian Markov measures P and $\widehat{P}$ are found.

Proposition 6.1. *Let (ξ_k) and $(\widehat{\xi}_k)$ be two Gaussian Markov sequences given above. Let P and $\widehat{P}$ be the distributions of (ξ_k) and $(\widehat{\xi}_k)$ in $(\mathbb{R}^\infty, B(\mathbb{R}^\infty))$. If $\widetilde{\alpha}_k = \frac{c_k}{c_{k-1}}\,\alpha_k \geq 0, \quad k \geq 2$, where*

$$c_k = \frac{s_k - \alpha_k s_{k-1}}{\beta_k^2} - \frac{\alpha_{k+1}(s_{k+1} - \alpha_{k+1}s_k)}{\beta_{k+1}^2},$$

then $\widehat{P} \sim P$ if and only if the condition

$$\sum_{n=1}^{\infty} \frac{(s_n - \alpha_n s_{n-1})^2}{\beta_n^2} < \infty, \tag{36}$$

holds true. Otherwise, $\widehat{P} \perp P$.

Proof. Note, that there exists the Radon–Nikodym density $\frac{d\widehat{P}}{dP}(x_{(n)})$ of a shifted Gaussian Markov measure $\widehat{P}$ with respect to the zero-mean Gaussian Markov measure P if and only if $\widehat{P} \sim P$.

Let P_n and $\widehat{P}_n$ be the distributions of finite Gaussian Markov sequences (ξ_k^n) and $(\widehat{\xi}_k^n)$ in $(\mathbb{R}^n, B(\mathbb{R}^n))$. Since $\beta_k^2 > 0, k \geq 1$, consider the finite Radon–Nikodym density of the shifted Gaussian Markov measure $\widehat{P}_n$ with respect to a zero-mean Gaussian Markov measure P_n which is of the following form [1]:

$$\frac{d\widehat{P_n}}{dP_n}((x^n)) = \exp\Big(\sum_{k=1}^{n}\Big[\frac{s_k - \alpha_k s_{k-1}}{\beta_k^2} - \frac{\alpha_{k+1}(s_{k+1} - \alpha_{k+1}s_k)}{\beta_{k+1}^2}\Big]x_k +$$

$$+\sum_{k=1}^{n}\Big[-\frac{(s_{k+1} - \alpha_{k+1}s_k)^2}{2\beta_{k+1}^2} + \frac{1}{2}\Big(\frac{s_{k+1}^2}{\beta_{k+1}^2} - \frac{s_k^2}{\beta_k^2}\Big)\Big]\Big), \quad mod \;\; P((x^n) \in \mathbb{R}^n). \tag{37}$$

Since the sequence $\left(\frac{d\widehat{P_n}}{dP_n}\right)$ is a martingale, we can find the Radon–Nikodym density $\frac{d\widehat{P}}{dP}(x_{(n)})$ making the passage to the limit in (37). Therefore, the Radon–Nikodym density $\frac{d\widehat{P}}{dP}(x_{(n)})$ exists if and only if as $n \to \infty$, all the series under the exponent sign in (37) are convergent.

Consider the nonrandom series

$$\sum_{k=1}^{\infty}\Big[-\frac{(s_{k+1} - \alpha_{k+1}s_k)^2}{2\beta_{k+1}^2} + \frac{1}{2}\Big(\frac{s_{k+1}^2}{\beta_{k+1}^2} - \frac{s_k^2}{\beta_k^2}\Big)\Big].$$

It is convergent by (35) and (36). Let us prove the convergence a.s. of the series

$$\sum_{k=1}^{\infty}\Big[\frac{s_k - \alpha_k s_{k-1}}{\beta_k^2} - \frac{\alpha_{k+1}(s_{k+1} - \alpha_{k+1}s_k)}{\beta_{k+1}^2}\Big]\xi_k, \tag{38}$$

using the results obtained in Section 5. Since $\widetilde{\alpha}_k = \frac{c_k}{c_{k-1}}\,\alpha_k \geq 0, \quad k \geq 2$, then, by (24),

$$\widetilde{A}_c(\infty, k) = \frac{s_k - \alpha_k s_{k-1}}{\beta_k^2} - \frac{\alpha_{k+1}\big(s_{k+1} - \alpha_{k+1} s_k\big)}{\beta_{k+1}^2} + $$

$$+ \sum_{l=1}^{\infty} \Big(\Big[\frac{s_{k+l} - \alpha_{k+l} s_{k+l-1}}{\beta_{k+l}^2} - \frac{\alpha_{k+l+1}\big(s_{k+l+1} - \alpha_{k+l+1} s_{k+l}\big)}{\beta_{k+l+1}^2} \Big] \prod_{j=k+1}^{k+l} \alpha_j \Big).$$

The partial sum of this series is of the following form:

$$\widetilde{S}_n = \frac{s_k - \alpha_k s_{k-1}}{\beta_k^2} - \frac{\big(s_{k+n+1} - \alpha_{k+n+1} s_{k+n}\big)}{\beta_{k+n+1}^2} \prod_{j=k+1}^{k+n+1} \alpha_j .$$

Therefore, by (35) and (36), one has

$$\lim_{n\to\infty} \widetilde{S}_n = \frac{s_k - \alpha_k s_{k-1}}{\beta_k^2} - \lim_{n\to\infty} \Big(\frac{\big(s_{k+n+1} - \alpha_{k+n+1} s_{k+n}\big)}{\beta_{k+n+1}} \cdot \frac{\prod_{j=k+1}^{k+n+1} \alpha_j}{\beta_{k+n+1}} \Big) = \frac{s_k - \alpha_k s_{k-1}}{\beta_k^2}$$

for any $k \geq 2$. Hence, the nonrandom series $\widetilde{A}_c(\infty, k)$ converges for any $k \geq 2$. Moreover, condition (29) holds true by (36). Therefore, according to Corollary 5.7, the random series (38) converges a.s.

Thus, for the given Gaussian Markov measures P and $\widehat{P}$, the corresponding Radon–Nikodym density $\frac{d\widehat{P}}{dP}(x_{(n)})$ exists, i.e., $\widehat{P} \sim P$, if and only if condition (36) holds true. Otherwise, $\widehat{P} \perp P$. □

Note that (36) provides the condition of admissible shift for the given zero-mean Gaussian Markov measure P.

Remark also that the necessary and sufficient conditions of admissible shift for a zero-mean Gaussian Markov measure were obtained in [1] with the use of the theory of absolute continuity and singularity of probabilistic measures [6]. In the considered case, the condition of admissible shift (36) coincides with the corresponding result obtained in [1].

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