Theory of Stochastic Processes
Vol. 16 (32), no. 1, 2010, pp. 17–28

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AN EXTENSION OF THE ITÔ INTEGRAL: TOWARD A GENERAL THEORY OF STOCHASTIC INTEGRATION

We introduce the class of instantly independent stochastic processes, which serves as the counterpart of the Itô theory of stochastic integration. This class provides a new approach to anticipating stochastic integration. The evaluation points for an adapted stochastic process and an instantly independent stochastic process are taken to be the left endpoint and the right endpoint, respectively. We present some new results on Itô's formula and stochastic differential equations.

1. Introduction

The purpose of this article is to explain the ideas introduced in [1] for an extension of the Itô integral. A crucial point is the discovery of the counterpart of the Itô theory, namely, the instantly independent stochastic processes versus the adapted stochastic processes. We will briefly review the background in Sections 2 and 3. In Section 4 we will give motivations for our viewpoint, define a new stochastic integral, and present some new results. In Section 5 we will give new results and examples for Itô's formula and stochastic differential equations involving the new stochastic integral. The ultimate goal is to develop a general theory of stochastic integration.

2. The Itô integral

Let $B(t)$ be a Brownian motion and let $\{\mathcal{F}_t\}$ be a filtration such that

- $B(t)$ is adapted to $\{\mathcal{F}_t\}$,
- $B(t) - B(s)$ and $\mathcal{F}_s$ are independent for $s \le t$.

Suppose $f(t)$ is a stochastic process satisfying the following conditions:

(1) $f(t)$ is adapted to $\{\mathcal{F}_t\}$,
(2) $E \int_a^b |f(t)|^2\, dt < \infty$.

Then the Itô integral $\mathcal{I} = \int_a^b f(t)\, dB(t)$ is defined (see, e.g., Chapter 4 of the book [13]) and we have the equalities:

$$E(\mathcal{I}) = 0, \quad E(|\mathcal{I}|^2) = E \int_a^b |f(t)|^2\, dt.$$

Moreover, we have the next theorem (see, e.g., Theorems 4.6.1 and 4.6.2 in the book [13].)

Theorem 2.1. *Let $f(t)$ be a stochastic process satisfying the above conditions* (1) *and* (2). *Then the stochastic process*

$$X_t = \int_a^t f(s)\, dB(s), \quad a \le t \le b,$$

is a continuous martingale.

2000 *Mathematics Subject Classification.* Primary 60H05, 60H20; Secondary 60H40.

Key words and phrases. Brownian motion, filtration, adapted stochastic process, Itô integral, Hitsuda-Skorokhod integral, anticipating, instantly independent stochastic processes, evaluation points, stochastic integral, Itô's formula, stochastic differential equations.

More generally, suppose $f(t)$ is a stochastic process satisfying the following conditions:

(a) $f(t)$ is adapted to $\{\mathcal{F}_t\}$,

(b) $\int_a^b |f(t)|^2\,dt < \infty$ almost surely.

Then the Itô integral $\int_a^b f(t)\,dB(t)$ is defined (see, e.g., Chapter 5 of the book [13]) and we have the next theorem (see, e.g., Theorems 5.5.2 and 5.5.5 in the book [13].)

Theorem 2.2. *Let $f(t)$ be a stochastic process satisfying the above conditions* (a) *and* (b)*. Then the stochastic process*

$$X_t = \int_a^t f(s)\,dB(s), \quad a \le t \le b,$$

is a continuous local martingale.

3. Anticipating stochastic integrals

From now on we will fix a Brownian motion $B(t)$ and a filtration $\{\mathcal{F}_t\}$ as specified in Section 1.

3.1. **K. Itô's ideas.** Suppose a stochastic process $f(t)$ is not adapted to this filtration. Then $\int_a^b f(t)\,dB(t)$ cannot be defined as an Itô integral. Below are some simple examples related to this yet to be defined stochastic integral:

1. Stochastic integral: $\int_0^1 B(1)\,dB(t) = ?$ (See Equations (1) and (5).)
2. Itô's formula: $\theta(B(t), B(1)) = ?$ for $0 \le t \le 1$. (See Theorem 5.1.)
3. SDE: $dX_t = X_t\,dB(t),\ 0 \le t \le 1,\ X_0 = \mathrm{sgn}(B(1))$. (See Example 3.8.)
4. SDE: $dX_t = X_t\,dB(t) + \frac{1}{B(1)} X_t\,dt,\ 0 \le t \le 1,\ X_0 = B(1)$. (See Example 5.1.)

We first describe K. Itô's ideas to define the stochastic integral $\int_0^1 B(1)\,dB(t)$ in his lecture at the 1976 Kyoto Symposium on SDE's [9]. Enlarge the filtration in order for the integrand $B(1)$ to be adapted, namely, let

$$\mathcal{G}_t = \sigma\{\mathcal{F}_t, B(1)\}.$$

Although $B(t)$ is not a Brownian motion with respect to the larger filtration $\{\mathcal{G}_t\}$, it can be decomposed as

$$B(t) = \left(B(t) - \int_0^t \frac{B(1) - B(u)}{1-u}\,du\right) + \int_0^t \frac{B(1) - B(u)}{1-u}\,du,$$

which shows that $B(t)$ is a quasimartingale with respect to the filtration $\{\mathcal{G}_t\}$. Then the stochastic integral $\int_0^1 B(1)\,dB(t)$ can be defined as a stochastic integral with respect to a quasimartingale and

$$\int_0^1 B(1)\,dB(t) = B(1)^2. \tag{1}$$

The above ideas of K. Itô inspire us to view anticipating stochastic integration from a different angle, which will be explained in Section 4.

3.2. **Literature.** We mention a few papers in the literature on anticipating stochastic integration. In the early stage there were papers by Hitsuda [7] in 1972, Skorokhod [29] in 1975, and Burger-Mizel [2] in 1980.

Then there was a period of silence for seven years. Since 1987 there have been many papers dealing with anticipating stochastic integration. We only mention a few of them with each one being the first paper published by the author(s) (to our best knowledge). The list below is obviously very incomplete:

- Buckdahn [3] (1987), Nualart–Pardoux [19] (1988), Kuo–Russek [15] (1988), Ocone–Pardoux [22] (1989), Dorogovtsev [5] (1990), Kuo–Potthoff [14] (1990), Ocone [21] (1990), Pardoux–Protter [24] (1990), Millet–Nualart–Sanz [17] (1991), Redfern [26] (1991).
- Léon–Protter [16] (1994), Russo–Vallois [27] (1994), Grorud–Nualart–Sanz-Solé [6] (1994), Øksendal–Zhang [23] (1996), Kohatsu-Higa–Léon [10] (1997), Privault [25] (1998).
- Shevchenko [28] (2005), Mishura–Shevchenko [18] (2008).

3.3. **White noise approach.** For the rest of this section we brief describe the white noise approach to anticipating stochastic integration. For detail, see the book [12].

Let $\mathcal{S}(\mathbb{R})$ denote the Schwartz space of rapidly decreasing functions on $\mathbb{R}$ and let $\mathcal{S}'(\mathbb{R})$ denote its dual space. Let μ be the standard Gaussian measure on $\mathcal{S}'(\mathbb{R})$. Then we have a Gel'fand triple

$$\mathcal{S}(\mathbb{R}) \subset L^2(\mathbb{R}) \subset \mathcal{S}'(\mathbb{R}).$$

The probability space $(\mathcal{S}'(\mathbb{R}), \mu)$ is often referred to as a *white noise space* because the stochastic process defined by

$$B(t,x) = \langle x, 1_{[0,t)} \rangle, \quad t \geq 0, \ x \in \mathcal{S}'(\mathbb{R}), \tag{2}$$

is a Brownian motion and informally $\dot{B} = x$ for $x \in \mathcal{S}'(\mathbb{R})$.

Let $(L^2) = L^2(\mathcal{S}'(\mathbb{R}), \mu)$ and let $(\mathcal{S})$ and $(\mathcal{S})^*$ denote the spaces of test functions and generalized functions on $\mathcal{S}'(\mathbb{R})$, respectively. Then we have a Gel'fand triple

$$(\mathcal{S}) \subset (L^2) \subset (\mathcal{S})^*.$$

Example 3.1. $\dot{B}(t) = \langle \cdot, \delta_t \rangle \in (\mathcal{S})^*$ for each $t \in \mathbb{R}$.

Example 3.2. $:e^{\dot{B}(t)}: = \sum_{n=0}^{\infty} \frac{1}{n!} \langle : \cdot^{\otimes n} :, \delta_t^{\otimes n} \rangle \in (\mathcal{S})^*$ for each $t \in \mathbb{R}$.

Example 3.3. $:e^{\langle \cdot, \eta \rangle}: = \sum_{n=0}^{\infty} \frac{1}{n!} \langle : \cdot^{\otimes n} :, \eta^{\otimes n} \rangle \in (\mathcal{S})$ if $\eta \in \mathcal{S}_c(\mathbb{R})$.

A basic tool in white noise theory is the S-transform defined below. It is used in characterizing generalized functions and test functions. Moreover, it is also used for computation on generalized functions.

Definition 3.1. The *S-transform* of $\Phi \in (\mathcal{S})^*$ is defined to be the function

$$(S\Phi)(\xi) = \langle\langle \Phi, :e^{\langle \cdot, \xi \rangle}: \rangle\rangle, \quad \xi \in \mathcal{S}_c(\mathbb{R}).$$

Example 3.4. $S(\dot{B}(t))(\xi) = \xi(t), \ \xi \in \mathcal{S}_c(\mathbb{R})$.

Example 3.5. $S(:e^{\dot{B}(t)}:)(\xi) = e^{\xi(t)}, \ \xi \in \mathcal{S}_c(\mathbb{R})$.

Example 3.6. $S(:e^{\langle \cdot, \eta \rangle}:)(\xi) = e^{\langle \xi, \eta \rangle}, \ \xi \in \mathcal{S}_c(\mathbb{R})$.

Definition 3.2. Let $\varphi \in (\mathcal{S})$ and $y \in \mathcal{S}'(\mathbb{R})$. Define the *directional derivative* $D_y \varphi$ by

$$(D_y \varphi)(x) = \lim_{\epsilon \to 0} \frac{\varphi(x + \epsilon y) - \varphi(x)}{\epsilon}$$

The operator D_y is a continuous linear operator from $(\mathcal{S})$ into itself. Its adjoint D_y^* is a continuous linear operator from $(\mathcal{S})^*$ into itself and for any $\Phi \in (\mathcal{S})^*$,

$$S(D_y^* \Phi)(\xi) = \langle y, \xi \rangle (S\Phi)(\xi), \quad \xi \in \mathcal{S}_c(\mathbb{R}).$$

The *white noise differentiation operator* ∂_t is defined to be the operator $\partial_t = D_{\delta_t}$. It is often called *Hida–Malliavin derivative* in white noise theory and Malliavin calculus. The white noise $\dot{B}(t)$ can be regarded as a continuous linear operator $\dot{B}(t) = \partial_t + \partial_t^*$ from $(\mathcal{S})$ into $(\mathcal{S})^*$.

Suppose $f(t)$ is a measurable function taking values in $(\mathcal{S})^*$. Then we can use the Pettis integral to define a *white noise integral* $\int_a^b \partial_t^* f(t)\,dt$, which in general is a generalized function in $(\mathcal{S})^*$.

Theorem 3.1. (Kubo–Takenaka [11]) *If $f(t)$ is adapted and $E\int_a^b |f(t)|^2\,dt < \infty$, then*

$$\int_a^b \partial_t^* f(t)\,dt = \int_a^b f(t)\,dB(t),$$

where the right-hand side is the Itô integral of $f(t)$.

Note that we do not have to assume that $f(t)$ is adapted for the white noise integral $\int_a^b \partial_t^* f(t)\,dt$. Thus in view of Theorem 3.1 the integral $\int_a^b \partial_t^* f(t)\,dt$ provides an extension of the Itô integral to anticipating stochastic integral. The only thing is that we need to require that $\int_a^b \partial_t^* f(t)\,dt$ is a random variable instead of just a generalized function in the space $(\mathcal{S})^*$.

A theorem due to N. Obata [20] (see also [12]) states that $\cup_{p>1} L^p(\mathcal{S}'(\mathbb{R}),\,\mu) \subset (\mathcal{S})^*$. Thus we can make the following definition.

Definition 3.3. A white noise integral $\int_a^b \partial_t^* f(t)\,dt$ is called a *Hitsuda-Skorokhod integral* if it belongs to $L^p(\mathcal{S}'(\mathbb{R},\,\mu)$ for some $p > 1$.

Example 3.7. $\int_0^1 \partial_t^* B(1)\,dt = B(1)^2 - 1$ is a Hitsuda-Skorokhod integral.

Example 3.8. In 1989 Buckdahn [4] solved the stochastic differential equation

$$dX_t = X_t\,dB(t), \quad 0 \le t \le 1, \quad X_0 = \operatorname{sgn}(B(1)).$$

The white noise formulation of this equation is given by

$$dX_t = \partial_t^* X_t\,dt, \quad 0 \le t \le 1, \quad X_0 = \operatorname{sgn}(B(1)).$$

By using the S-transform, we can derive the same solution

$$X_t = \operatorname{sgn}(B(1) - t)\,e^{B(t) - \frac{1}{2}t}.$$

See Example 13.30 in the book [12] for the derivation.

Example 3.9. In 1987 Buckdahn [3] solved the stochastic differential equation

$$dX_t = B(1)X_t\,dB(t), \quad 0 \le t \le 1, \quad X_0 = 1.$$

The white noise formulation of this equation is given by

$$dX_t = \partial_t^*\big(B(1)X_t\big)\,dt, \quad 0 \le t \le 1, \quad X_0 = 1.$$

By using the S-transform, we can derive the same solution

$$X_t = \exp\Big[B(1)\int_0^t e^{-(t-s)}\,dB(s) - \frac{1}{4}B(1)^2(1 - e^{-2t}) - t\Big].$$

See Example 13.35 in the book [12] for the derivation.

Example 3.10. In 1990 Kuo–Potthoff [14] solved the stochastic differential equation

$$dX_t = \partial_t^*\big(B(1) \diamond X_t\big)\,dt, \quad 0 \le t \le 1, \quad X_0 = 1,$$

where $\diamond$ denotes the Wick product. We can use the S-transform to derive the solution

$$X_t = \frac{1}{\sqrt{1 + t + t^2}}\,\exp\Big[-\frac{1}{2(1 + t + t^2)}\Big(tB(1)^2 - 2(1 + t)B(1)B(t) + B(t)^2\Big)\Big].$$

See Example 13.36 in the book [12] for the derivation, which is rather complicated.

Next we state an extension of Itô's formula from Theorem 13.19 in the book [12].

Example 3.11. Here is another example from Example 13.31 of the book [12]:

$$dX_t = \partial_t^* X_t\, dt + \operatorname{sgn}(B(1)-t)e^{B(t)-\frac{1}{2}t}\, dt, \quad 0 \le t \le 1, \quad X_0 = 1.$$

The solution is given by

$$X_t = e^{B(t)-\frac{1}{2}t}\big(1 + t\operatorname{sgn}(B(1)-t)\big).$$

Theorem 3.2. *Let $a \le t \le T$ and let $\theta(x,y)$ be a C^2-function such that*

$$\theta\big(B(\cdot), B(T)\big), \ \ \frac{\partial^2\theta}{\partial x^2}\big(B(\cdot), B(T)\big), \ \ \frac{\partial^2\theta}{\partial x \partial y}\big(B(\cdot), B(T)\big),$$

are all in $L^2\big([a,T]; (L^2)\big)$. Then the white noise integral

$$\int_a^t \partial_s^* \Big(\frac{\partial\theta}{\partial x}\big(B(s), B(T)\big)\Big)\, ds, \quad a \le t \le T,$$

is a Hitsuda-Skorokhod integral and the following Itô's formula

$$\theta\big(B(t), B(T)\big) = \theta\big(B(a), B(T)\big) + \int_a^t \partial_s^* \Big(\frac{\partial\theta}{\partial x}\big(B(s), B(T)\big)\Big)\, ds$$
$$+ \int_a^t \Big(\frac{1}{2}\frac{\partial^2\theta}{\partial x^2}\big(B(s), B(T)\big) + \frac{\partial^2\theta}{\partial x \partial y}\big(B(s), B(T)\big)\Big)\, ds.$$

holds in (L^2) for $a \le t \le T$.

Here we point out several drawbacks on the white noise approach to anticipating stochastic integration:

- It requires too much background and the associated Brownian motion $B(t,x)$ in Equation (2) is too restricted to be adopted to general problems.
- It is difficult to deal with pointwise multiplication of functions, although the Wick product can be used as a substitution.
- Computation involving the S-transform is usually very complicated, even for very simple examples.
- There is no available characterization theorem for generalized functions to be realized as random variables in $L^p(\mathcal{S}(\mathbb{R}), \mu)$ for some $p > 1$.
- It lacks probabilistic interpretation, e.g., it is unknown as how to deal with convergence in probability in terms of the S-transform.

4. A new viewpoint for stochastic integral

This section is the main part of this article. We will review the ideas from our previous paper [1] and present some new results.

4.1. **Counterpart of the Itô theory.** Recall the stochastic integral $\int_0^1 B(1)\, dB(t)$ in Equation (2). As explained in Subsection 3.1, Itô's ideas are as follows:

- Keep the integrand $B(1)$.
- Enlarge the filtration $\{\mathcal{F}_t\}$ and decompose the integrator $B(t)$.

Our new viewpoint in [1] comes from the simple observation that the anticipating integrand $B(1)$ has the following obvious decomposition

$$B(1) = \big(B(1) - B(t)\big) + B(t). \tag{3}$$

Note that the integral for the second term $B(t)$ is within the Itô theory. Thus we only need to define the stochastic integral $\int_0^1 \big(B(1)-B(t)\big)\, dB(t)$. This leads to the question: "*What is so special about the integrand* $B(1) - B(t)$?"

To find out the answer, consider another anticipating integrand $B(1)^2$. This integrand can be decomposed as

$$B(1)^2 = [B(1) - B(t)]^2 + 2B(t)[B(1) - B(t)] + B(t)^2.$$

Observe that the last term $B(t)^2$ and the factor $B(t)$ in the second term are adapted stochastic processes, while the first term $[B(1) - B(t)]^2$ and the factor $B(1) - B(t)$ in the second term have the same property (which is to be defined below) as that of the first term in Equation (3).

We can also try to decompose integrands such as $B(1)^n$ and $e^{B(1)}$ to discover the common property stated in the next definition.

Definition 4.1. A stochastic process $\varphi(t)$ is said to be *instantly independent* with respect to a filtration $\{\mathcal{F}_t\}$ if $\varphi(t)$ and $\mathcal{F}_t$ are independent for each t.

Clearly, $[B(1) - B(t)]^n$, $e^{B(1)-B(t)}$, and $\int_t^1 h(s)\, dB(s)$ are all instantly independent for $0 \le t \le 1$, where $h(s)$ is a deterministic function in $L^2([0, 1])$.

Lemma 4.1. *If a stochastic process $\varphi(t)$ is both adapted and instantly independent with respect to a filtration $\{\mathcal{F}_t\}$, then $\varphi(t)$ is a deterministic function.*

Proof. Since $\varphi(t)$ is adapted, we have $E(\varphi(t)|\mathcal{F}_t) = \varphi(t)$. On the other hand, since $\varphi(t)$ is instantly independent, we also have $E(\varphi(t)|\mathcal{F}_t) = E(\varphi(t))$. Hence $\varphi(t) = E(\varphi(t))$, which shows that $\varphi(t)$ is a deterministic function. □

In view of Lemma 4.1, we can regard the collection of instantly independent stochastic processes as a counterpart of the Itô theory. Namely, the Itô part consists of adapted stochastic processes and the counterpart consists of instantly independent stochastic processes. Moreover, observe from the above discussion that many anticipating stochastic processes can be decomposed into sums of the products of an Itô part and a counterpart. Thus our viewpoint in fact stems from Itô's ideas. We simply reverse the roles of the integrand and the integrator, i.e.,

- Keep the filtration $\{\mathcal{F}_t\}$ and the Brownian motion $B(t)$.
- Decompose an integrand as a sum of terms, each being the product of an adapted stochastic process and an instantly independent stochastic processes.

This leads to the question: "*How do we define a stochastic integral $\int_a^b f(t)\varphi(t)\, dB(t)$ for an adapted stochastic process $f(t)$ (in the Itô part) and an instantly independent stochastic process $\varphi(t)$ (in the counterpart)?*" The answer is in the next subsection.

4.2. **A new stochastic integral.** The second key idea in our approach to anticipating stochastic integration is the evaluation points for the integrand. Consider the instantly independent stochastic process $B(1) - B(t)$ in the right-hand side of Equation (3). How do we "define" the stochastic integral $\int_0^1 (B(1) - B(t))\, dB(t)$?

Let $\Delta = \{0 = t_0, t_1, t_2, \ldots, t_n = 1\}$ be a partition of the interval $[0, 1]$. On the subinterval $[t_{i-1}, t_i]$, we take the *"right endpoint"* t_i as the evaluation point for the integrand $B(1) - B(t)$ to form a Riemann like sum. Then we "define" the integral

$$\begin{aligned}
&\int_0^1 (B(1) - B(t))\, dB(t) \\
&\quad = \lim_{\|\Delta\|\to 0} \sum_{i=1}^n (B(1) - B(t_i))(B(t_i) - B(t_{i-1})) \\
&\quad = B(1)^2 - \lim_{\|\Delta\|\to 0} \sum_{i=1}^n B(t_i)(B(t_i) - B(t_{i-1})) \\
&\quad = B(1)^2 - \lim_{\|\Delta\|\to 0} \sum_{i=1}^n \big\{[B(t_i) - B(t_{i-1})] + B(t_{i-1})\big\}(B(t_i) - B(t_{i-1})) \\
&\quad = B(1)^2 - 1 - \int_0^1 B(t)\, dB(t), \qquad (4)
\end{aligned}$$

where the last integral is an Itô integral. It follows from Equations (3) and (4) that we have a new stochastic integral

$$\int_0^1 B(1)\, dB(t) = B(1)^2 - 1, \qquad (5)$$

which is different from the one in Equation (1) defined by K. Itô [9], but the same as the Hitsuda-Skorokhod integral in Example 3.7. Note that our new stochastic integral has expectation 0, a property that we want to keep for our new stochastic integral.

The above discussion leads to the following definition of a new stochastic integral of a stochastic process which is the product of an adapted stochastic process (in the Itô part) and an instantly independent stochastic process (in the counterpart).

Definition 4.2. For an adapted stochastic process $f(t)$ and an instantly independent stochastic process $\varphi(t)$, we define the *stochastic integral* of $f(t)\varphi(t)$ to be the limit

$$\int_a^b f(t)\varphi(t)\, dB(t) = \lim_{\|\Delta\|\to 0} \sum_{i=1}^n f(t_{i-1})\varphi(t_i)(B(t_i) - B(t_{i-1}))$$

provided that the limit in probability exists.

In general, for a stochastic process $F(t) = \sum_{n=1}^N f_n(t)\varphi_n(t)$ with $f_n(t)$'s being adapted and $\varphi_n(t)$'s instantly independent, we define

$$\int_a^b F(t)\, dB(t) = \sum_{n=1}^N \int_a^b f_n(t)\varphi_n(t)\, dB(t).$$

This stochastic integral is in fact well-defined. Obviously, there is a natural question: "*What is the class of stochastic processes for which the new stochastic integral is defined?*" Unfortunately, we do not have the answer yet.

Example 4.1. We mention two stochastic integrals from the paper [1].

$$\int_0^t B(1)B(s)\, dB(s) = \begin{cases} \frac{1}{2}B(1)(B(t)^2 - t) - \int_0^t B(s)\, ds, & 0 \le t \le 1, \\ \frac{1}{2}B(1)(B(t)^2 - t) - \int_0^1 B(s)\, ds, & t > 1. \end{cases}$$

In general, for a continuous function $f(x)$, we have

$$\int_0^t B(1)f(B(s))\, dB(s) = \begin{cases} B(1)\int_0^t f(B(s))\, dB(s) - \int_0^t f(B(s))\, ds, & 0 \le t \le 1, \\ B(1)\int_0^t f(B(s))\, dB(s) - \int_0^1 f(B(s))\, ds, & t > 1. \end{cases}$$

Example 4.2. Let $f(t)$ and $g(t)$ be two deterministic functions in $L^2([0,1])$. Then

$$\int_0^1 g(t)\Big(\int_0^1 f(s)\,dB(s)\Big)\,dB(t) = \int_{[0,1]^2} f(s)g(t)\,dB(s)dB(t), \tag{6}$$

where the right-hand side is a double Wiener–Itô integral (see Chapter 9 in [13]). To prove this equality, note that the Wiener integral of $f(s)$ in the left-hand side has the decomposition

$$\int_0^1 f(s)\,dB(s) = \int_0^t f(s)\,dB(s) + \int_t^1 f(s)\,dB(s),$$

where the first integral is in the Itô part and the second integral is in the counterpart. For convenience, let $\Delta B_i = B(t_i) - B(t_{i-1})$. By Definition 4.2, we have

$$\begin{aligned}
&\int_0^1 g(t)\Big(\int_0^1 f(s)\,dB(s)\Big)\,dB(t)\\
&= \lim_{\|\Delta\|\to 0}\sum_{i=1}^n g(t_{i-1})\Big(\int_0^{t_{i-1}} f(s)\,dB(s) + \int_{t_i}^1 f(s)\,dB(s)\Big)\Delta B_i\\
&= \lim_{\|\Delta\|\to 0}\sum_{i=1}^n g(t_{i-1})\Big(\int_0^1 f(s)\,dB(s) - \int_{t_i}^{t_{i-1}} f(s)\,dB(s)\Big)\Delta B_i\\
&= \int_0^1 f(s)\,dB(s)\int_0^1 g(t)\,dB(t) - \lim_{\|\Delta\|\to 0}\sum_{i=1}^n f(t_{i-1})g(t_{i-1})(\Delta B_i)^2\\
&= \int_0^1 f(s)\,dB(s)\int_0^1 g(t)\,dB(t) - \int_0^1 f(t)g(t)\,dt,
\end{aligned}$$

which is exactly the Wiener–Itô double integral in the right-hand side of Equation (6).

In [8] K. Itô proved the following well-known theorem on multiple Wiener–Itô integral (see also Theorem 9.6.7 in the book [13].)

Theorem 4.1. (K. Itô 1951) *Let $f \in L^2([a,b]^n)$ and $\widehat{f}$ its symmetrization. Then*

$$\begin{aligned}
&\int_{[a,b]^n} f(t_1,t_2,\ldots,t_n)\,dB(t_1)dB(t_1)\cdots dB(t_n)\\
&= n!\int_a^b\cdots\int_a^{t_{n-2}}\Big[\int_a^{t_{n-1}} \widehat{f}(t_1,\ldots,t_{n-1},t_n)\,dB(t_n)\Big]\,dB(t_{n-1})\cdots dB(t_1).
\end{aligned}$$

Note that the restriction to the region $a \le t_n \le t_{n-1} \le \cdots \le t_2 \le t_1 \le b$ for the iterated integrals is to ensure that in each step of the iteration the integrand is adapted so that the integral is defined as an Itô integral.

However, as seen from Example 4.2, there is no need to impose this restriction since in each step the integral is defined as a stochastic integral in Definition 4.2. By using the similar arguments as those in Example 4.2, we can prove the next theorem.

Theorem 4.2. *Let $f \in L^2([a,b]^n)$. Then*

$$\begin{aligned}
&\int_{[a,b]^n} f(t_1,t_2,\ldots,t_n)\,dB(t_1)dB(t_1)\cdots dB(t_n)\\
&= \int_a^b\cdots\int_a^b\Big[\int_a^b f(t_1,\ldots,t_{n-1},t_n)\,dB(t_n)\Big]\,dB(t_{n-1})\cdots dB(t_1).
\end{aligned} \tag{7}$$

Observe that we do not have to use the symmetrization $\widehat{f}$ in the right-hand side of Equation (7). If fact, it is obvious that the iterated new stochastic integrals for f and $\widehat{f}$ are equal. In view of this theorem a multiple Wiener–Itô integral can be evaluated as an iterated stochastic integral, just like multiple integrals and iterated integrals in the ordinary calculus.

5. Itô's formula and stochastic differential equations

In the paper [1] we have proved a special case of Itô's formula for the new stochastic integral. Here we will prove the formula for a more general case. An interval $[a,T]$ is fixed in this section.

Lemma 5.1. *Let $dX_t = h(t)\,dB(t) + g(t)\,dt$ with $h \in L^2([a,T])$ and $g \in L^1([a,T])$. Let $f(x)$ be a continuous function and $\varphi(x)$ a C^1-function. Let $\theta(x,y) = f(x)\varphi(y-x)$. Then for each $t \in [a,T]$, we have*

$$\sum_{i=1}^n \theta(X_{s_{i-1}}, X_T)(X_{s_i} - X_{s_{i-1}}) \longrightarrow \int_a^t \theta(X_s, X_T)\,dX_s + \int_a^t \frac{\partial\theta}{\partial y}(X_s, X_T)\,(dX_s)^2$$

in probability as $\|\Delta\| \to 0$. Here $\Delta = \{s_0, s_1, \ldots, s_n\}$ is a partition of $[a,t]$ with $s_0 = a$ and $s_n = t$. In the second integral, it is understood that $(dX_s)^2 = h(s)^2\,ds$.

Proof. Let $\Delta B_i = B(s_i) - B(s_{i-1})$, $\Delta s_i = s_i - s_{i-1}$, and $\Delta X_i = X_{s_i} - X_{s_{i-1}}$. Note that $\Delta X_i \approx h(s_{i-1})\Delta B_i + g(s_{i-1})\Delta t_i$. Then informally we have

$$\begin{aligned}\sum_{i=1}^n \theta(X_{s_{i-1}}, X_T)(X_{s_i} - X_{s_{i-1}}) &= \sum_{i=1}^n f(X_{s_{i-1}})\varphi(X_T - X_{s_{i-1}})\Delta X_i \\ &\approx \sum_{i=1}^n f(X_{s_{i-1}})\varphi(X_T - X_{s_{i-1}})h(s_{i-1})\Delta B_i \\ &\quad + \sum_{i=1}^n f(X_{s_{i-1}})\varphi(X_T - X_{s_{i-1}})g(s_{i-1})\Delta s_i. \end{aligned} \tag{8}$$

For the first summation, we have

$$\begin{aligned}&\sum_{i=1}^n f(X_{s_{i-1}})\varphi(X_T - X_{s_{i-1}})h(s_{i-1})\Delta B_i \\ &\quad\approx \sum_{i=1}^n f(X_{s_{i-1}})h(s_{i-1})\big\{\varphi(X_T - X_{s_i}) + \varphi'(X_T - X_{s_i})\Delta X_i\big\}\Delta B_i \\ &\quad\approx \sum_{i=1}^n f(X_{s_{i-1}})h(s_{i-1})\big\{\varphi(X_T - X_{s_i}) + \varphi'(X_T - X_{s_i})h(t_{s_{i-1}})\Delta B_i\big\}\Delta B_i \\ &\quad\to \int_a^t \theta(X_s, X_T)h(s)\,dB(s) + \int_a^t \frac{\partial\theta}{\partial y}(X_s, X_T)h(s)^2\,ds. \end{aligned} \tag{9}$$

For the second summation in Equation (8), we argue similarly to show that

$$\sum_{i=1}^n f(X_{s_{i-1}})\varphi(X_T - X_{s_{i-1}})g(s_{i-1})\Delta s_i \longrightarrow \int_a^t \theta(X_s, X_T)g(s)\,ds. \tag{10}$$

The assertion of the lemma follows from Equations (8), (9), and (10). □

Theorem 5.1. *Let $dX_t = h(t)\,dB(t) + g(t)\,dt$ with $h \in L^2([a,T])$ and $g \in L^1([a,T])$. Suppose $f(x)$ and $\varphi(x)$ are C^2-functions and let $\theta(x,y) = f(x)\varphi(y-x)$. Then the* *following equality holds for $a \le t \le T$,*

$$\theta(X_t, X_T) = \theta(X_a, X_T) + \int_a^t \frac{\partial\theta}{\partial x}(X_s, X_T)\, dX_s + \int_a^t \Big\{ \frac{1}{2}\frac{\partial^2\theta}{\partial x^2}(X_s, X_T) + \frac{\partial^2\theta}{\partial x \partial y}(X_s, X_T) \Big\} (dX_s)^2,$$

where $(dX_s)^2$ is understood to be given by $(dX_s)^2 = h(s)^2\, ds$.

Proof. We will derive the equality informally. Let $\Delta = \{s_0, s_1, \ldots, s_n\}$ be a partition of $[a, t]$ with $s_0 = a$ and $s_n = t$ and let $\Delta X_i = X_{s_i} - X_{s_{i-1}}$. Then

$$\begin{aligned} \theta(X_t, X_T) &= \theta(X_a, X_T) + \sum_{i=1}^{n} \big\{ \theta(X_{s_i}, X_T) - \theta(X_{s_{i-1}}, X_T) \big\} \\ &\approx \theta(X_a, X_T) + \sum_{i=1}^{n} \frac{\partial\theta}{\partial x}(X_{s_{i-1}}, X_T)\Delta X_i + \sum_{i=1}^{n} \frac{1}{2}\frac{\partial^2\theta}{\partial x^2}(X_{s_{i-1}}, X_T)(\Delta X_i)^2. \end{aligned}$$

Apply Lemma 5.1 to the function $\frac{\partial\theta}{\partial x}$ to show that the first summation converges in probability to

$$\int_a^t \frac{\partial\theta}{\partial x}(X_s, X_T)\, dX_s + \int_a^t \frac{\partial^2\theta}{\partial x \partial y}(X_s, X_T)\, (dX_s)^2.$$

On the other hand, the second summation converges in probability to

$$\int_a^t \frac{1}{2}\frac{\partial^2\theta}{\partial x^2}(X_s, X_T)\, (dX_s)^2.$$

Putting these two limits together, we obtain the formula in the theorem. □

Finally we give some examples of stochastic differential equations involving the new stochastic integral. At the moment we do not have a general theorem. But an interesting problem to start with is to consider a stochastic differential equation

$$dX_t = f(X_t)\, dB(t) + g(X_t)\, dt, \quad 0 \le t \le T,$$

with an initial condition that X_0 is anticipating, e.g., $X_0 = B(T)$. It is natural to ask the question, "*What is the relationship between the solution of this equation (assuming that there is a unique solution) and the solution of Itô's type equation with the initial condition $X_0 = x$?*"

We do not have an answer to this question yet. The next theorem is a special case relating to the exponential process in the Itô theory.

Theorem 5.2. *Let $h \in L^2([0, T])$ and let ξ be a random variable being independent of the filtration $\{\mathcal{F}_t\}$. Then the solution of the stochastic differential equation*

$$dX_t = h(t)X_t\, dB(t), \quad 0 \le t \le T, \quad X_0 = \xi + \int_0^T h(s)\, dB(s), \tag{11}$$

is given by

$$X_t = \Big(\xi + \int_0^T h(s)\, dB(s) - \int_0^t h(s)^2\, ds\Big) \exp\Big[\int_0^t h(s)\, dB(s) - \frac{1}{2}\int_0^t h(s)^2\, ds\Big]. \tag{12}$$

Proof. To check that the stochastic process X_t defined by Equation (12) is a solution of the stochastic differential equation (11), just apply the Itô formula in Theorem 5.1 to the function

$$\theta(t, x, y) = \Big(y - \int_0^t h(s)^2\, ds\Big) \exp\Big[x - \frac{1}{2}\int_0^t h(s)^2\, ds\Big].$$

Note that because of the variable t in $\theta(t,x,y)$ we need to add an extra term for $\partial\theta/\partial t$ in the Itô formula in Theorem 5.1. We omit the computation since it is somewhat tedious, but nevertheless straightforward.

To prove the uniqueness of a solution, suppose Y_t is another solution with the same initial condition $Y_0 = X_0$. Let $Z_t = X_t - Y_t$. Then Z_t satisfies the stochastic differential equation

$$dZ_t = h(t)Z_t\, dB(t), \quad 0 \le t \le T, \quad Z_0 = 0.$$

Observe that this equation is within the Itô theory and the solution is given by $Z_t = 0$. Hence $X_t = Y_t$ and the uniqueness of a solution is proved. □

There is another question: "*Suppose X_t^x, $a \le t \le T$, is the solution of a stochastic differential equation with $X_a^x = x$ in the Itô theory. Let η be a random variable which may not be measurable with respect to $\mathcal{F}_a$ (e.g., $B(T)$) and consider the stochastic process X_t^η. Then what is the stochastic differential equation that is satisfied by X_t^η?*"

Again we do not have an answer to this question yet. Here we only give an example from our previous paper [1].

Example 5.1. It is well known that the solution of the stochastic differential equation

$$dX_t = X_t\, dB(t), \quad t \ge 0, \quad X_0 = x,$$

is given by

$$X_t = xe^{B(t)-\frac{1}{2}t}.$$

It we change the initial condition to $B(1)$, then we have to get the stochastic process

$$Y_t = B(1)e^{B(t)-\frac{1}{2}t},$$

which is shown in [1] to be the solution of the stochastic differential equation

$$dY_t = Y_t\, dB(t) + \frac{1}{B(1)} Y_t\, dt, \quad 0 \le t \le 1, \quad Y_0 = B(1).$$

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