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ASYMPTOTIC BEHAVIOR OF THE CONDITIONAL PROBABILITY OF THE NONLINEAR BOUNDARY CROSSING BY A RANDOM WALK

We study the asymptotic behavior of the conditional probability of the boundary crossing by a random walk with distribution belonging to the attraction domain of a stable distribution with parameter α.

1. Introduction

Let ξ_n, $n \geq 1$ be a sequence of independent identically distributed random variables determined on some probability space $(\Omega, \mathcal{F}, P)$, and let $f_a(t)$, $t > 0$, be some family of nonlinear (non-random) functions (boundaries) with respect to the parameter $a > 0$.

Consider the first passage time

$$\tau_a = \inf\{n \geq 1 : S_n \geq f_a(n)\} \quad (\inf\{\varnothing\} = \infty)$$

in the case where the random walk $S_n = \xi_1 + \cdots + \xi_n$, $n \geq 1$ crosses a nonlinear boundary $f_a(t)$, $t > 0$.

In the present paper, we study the asymptotic behavior $P\left(\tau_a \geq n / \overline{S}_n = x\right)$, $\overline{S}_n = \frac{S_n}{n}$, of the conditional probability of the nonlinear boundary crossing by a random walk with infinite variance.

This problem was studied in [1],[2],[4] in the case of a finite variance $D\xi_1 < \infty$ for some class of nonlinear boundaries $f_a(t)$ and random walks.

As was noted in [1],[2] (see also [4]), such conditional probabilities play an important role in studying the local probabilities arising in the boundary problems with random walks.

2. Conditions and Notations

We assume that $\mu = E\xi_1 > 0$ and the distribution of a random variable ξ_1 belongs to the attraction domain of a stable distribution $G_\alpha(x)$ with the characteristic index $\alpha \in (1, 2]$ and the density $g_\alpha(x)$. That is, the convergence

$$P\left(\frac{S_n - n\mu}{A(n)} \leq x\right) \to G_\alpha(x) \ \text{ as } n \to \infty, \ x \in R,$$

holds. Here, $A(n) = n^{1/\alpha} L(n)$, $L(x)$, and $x > 0$ is a showily varying function at infinity [7].

For the boundary $f_a(t)$, we assume that it satisfies the following regularity conditions (see [5]):

1) For each a, the function $f_a(t)$ monotonically increases and is continuously differentiable for $t > 0$,

2) $n(a) \to \infty$ and $a \to \infty$ so that $\frac{1}{n} f_a(t) \to \mu$ and $f'_a(n) \to \theta \in [0, \mu)$,

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3) For each a, the function $f_a'(t)$ oscillates weakly at infinity, i.e. $\frac{f_a'(m)}{f_a'(n)} \to 1$ as $n \to \infty$ and $\frac{n}{m} \to 1$.

By W, we denote a class of functions satisfying conditions 1)-3).

The class W is described in [5] in detail.

By $l_a(n,x) = P\left(\tau_a \geq n / \overline{S}_n = x\right)$, we denote the conditional probability of the crossing of a nonlinear boundary $f_a(t)$ by the sums S_n, $n \geq 1$ at $\overline{S}_n = x$.

We have

$$\begin{gathered} l_a(n,x) = P\left(S_k \leq f_a(k), 1 \leq k \leq n-1 / \overline{S}_n = x\right) = \\ = P\left(S_n - S_k \geq S_n - f_a(k) : 1 \leq k \leq n-1 / \overline{S}_n = x\right) = \\ = P\left(S_{n-k} \geq nx - f_a(k) : 1 \leq k \leq n-1 / \overline{S}_n = x\right) = \\ = P\left(S_k \geq f_a(n) - f_a(n-k) + nx - f_a(n) : 1 \leq k \leq n-1 / \overline{S}_n = x\right) \end{gathered}$$

or

$$l_a(n,x) = P\left(S_k \geq k f_a'(m) + \delta_a(n,x) : 1 \leq k \leq n-1 / \overline{S}_n = x\right), \tag{1}$$

where $\delta_a(n,x) = nx - f_a(n)$, and $m = m(n,k)$ is some intermediate point from the segment $[n-k, n]$.

Equality (1) is a starting point for the approximation of the conditional probability of crossing, $l_a(n,x)$, as $a \to \infty$ and $n = n(a) \to \infty$.

Introduce the notation

$$\begin{gathered} L_a'(n,x,u) = P\left(S_k - k f_a'(m) \geq u,\ 1 \leq k \leq n-1 / \overline{S}_n = x\right), \\ L_a(n,x,u) = P\left(S_k - k\theta \geq u,\ 1 \leq k \leq n-1 / \overline{S}_n = x\right), \\ \psi(u) = P(T \geq u),\ T = \inf_{k \geq 1}(S_k - k\theta),\ u \in R. \end{gathered}$$

It is easy to see that

$$l_a(n,x) = L_a'(n,x,\delta_a(n,x)). \tag{2}$$

Remark 2.1. Note that the function $L_a(n,x,u)$ of u is non-increasing and continuous from the left-hand side, and the function $\psi(u)$ possesses these properties and has a set of continuity points consisting of all those $u \geq 0$, for which

$$P(\min(S_1 - \theta, S_2 - 2\theta, \ldots, S_k - k\theta) = u) = 0 \quad \text{for all } k \geq 1.$$

3. Formulation and Proof of the Basic Result

Theorem 3.1. *Let $f_a(t) \in W$, and let the above-mentioned conditions for the distribution of the random variable ξ_1 be satisfied. Furthermore, for the characteristic function $h(t)$ of the random variable ξ_1, let the condition*

$$\int_{-\infty}^{\infty} |h(t)|^m \, dt < \infty \tag{3}$$

hold for some $m \geq 1$.

If $x = x(a) \to \mu$ and $n = n(a) \to \infty$ as $a \to \infty$ so that

$$x - \mu = O(A(n)/n) \quad \text{and} \quad \delta(n,x) = O(1),$$

then

$$L_a(n,x,u) \to \psi(u) \quad (L_a'(n,x,u) \to \psi(u))$$

for all $u \geq 0$.

Corollary 3.1. *Let the conditions of Theorem 3.1 be satisfied. If $\delta(n,x) \to u \geq 0$, then*

$$l_a(n,x) \to \psi(u) \quad \text{as } a \to \infty.$$

Corollary 3.2. *While fulfilling the conditions of Corollary 3.1, we have*

$$l_a(n,x) \to (\mu-\theta)\, r(u), \ a \to 0,$$

where

$$r(u) = \frac{1}{ES'_\tau} P(S'_\tau > u), \ \tau = \inf\{n \geq 1 : S'_n = S_n - \theta n > 0\}.$$

Note that the function $r(u)$, $u \geq 0$ is the density of a limit distribution of overshoots of the random walk $S'_n = S_n - \theta n$, $n \geq 1$ (see [2]).

Remark 3.1. Note that condition (3) implies, in particular, that the function $\psi(u)$ is continuous for $u \geq 0$, since the sum S_n has a bounded continuous density for all $n \geq m$, while satisfying condition (3). Note also that the condition $\delta(n,x) = O(1)$ can be replaced by the condition $\frac{f_a(n)-n\mu}{A(n)} = O(1)$.

In order to prove the theorem, we need the following auxiliary lemmas of independent interest.

By $p_n(x)$, we denote the distribution density of the sum for $n \geq m$ and assume

$$q_{nk} = q_{nk}(x_1,\ldots,x_k/x) = \begin{cases} \frac{p_{n-k}(nx-y_k)}{p_n(nx)}, & \text{if } p_n(nx) > 0, \\ 1, & \text{if } \quad p_n(nx) = 0 \end{cases}$$

and

$$Q_{nk} = Q_{nk}(B/x) = \int \ldots \int_B q_{nk} F(dx_1)\ldots F(dx_k),$$

where

$$y_k = \sum_{i=1}^{k} x_i, \ 1 \leq k \leq n-1, \quad n \geq m, \quad B \in \beta\left(R^k\right) \quad \text{and } F(x) = P(\xi_1 \leq x).$$

It is clear that Q_{nk} is the conditional distribution of the vector $(\xi_1,\ldots,\xi_k)$ given that $\overline{S}_n = x$.

Lemma 3.1. *Let the distribution of the random variable ξ_1 belong to an attraction domain of the stable distribution $G_a(x)$ with characteristic index $\alpha \in (1,2]$ and density $g_\alpha(x)$, and let the condition* (3) *be satisfied. Then*

1) For each k, the conditional distribution Q_{nk} weakly converges as $n \to \infty$ to an unconditional distribution of the vector $(\xi_1,\ldots,\xi_k)$, and the convergence is uniform with respect to $x : x - \mu = O(A(n)/n)$, $A(n) = n^{1/\alpha} L(n)$.

2) For any $\delta \in (0,1)$, there exists a constant $M = M(\delta)$ such that $q_{nk} \leq M$ for all $x_1,\ldots,x_k$, $k \leq (1-\delta)n$, $n \geq m$ and x, $x - \mu = O(A(n)/n)$.

Proof. Under the conditions of the proved lemma, a local limit theorem for the sum S_n, $n \geq m$, holds uniformly with respect to $x \in R$ (see [6],[7]):

$$P_n(x) = \frac{1}{A(n)} g_\alpha\left(\frac{x-n\mu}{A(n)}\right) + o(1/A(n)) \qquad (4).$$

From relation (4) and properties of the density $g_\alpha(x)$ for the fixed y and k, we have

$$\lim_{n\to\infty} \frac{P_{n-k}(nx-y)}{g_n(nx)} = \lim_{n\to\infty} \frac{g_\alpha\left(\frac{n(x-\mu)+k\mu-y}{A(n)}\right)}{g_n\left(\frac{n(x-\mu)}{A(n)}\right)} = 1. \qquad (5)$$

By (4), statement 1) of Lemma 3.1 follows from the definition of the conditional distribution of Q_{nk} and from relation (5).

Statement 2) of Lemma 3.1 follows from relation (4) by the upper bound

$$\sup_{x\in R} A(n)\, p_n(x) < c < \infty$$

and the lower bound $\inf\limits_{x\left|x-\mu\right|\leq\frac{cA(n)}{n}} A(n)\,p_n(nx) > 0$, which follows from the boundedness of the stable distribution.

Lemma 3.2. *For any number* $\delta\in(0,1)$,

$$J_1 = P\left(S_1 < y; \exists i\in(n\delta, n-1]\,/\overline{S}_n = x\right)\to 0$$

as $a\to\infty$ *uniformly with respect to* $y\in R$ *and* x, $x-\mu = O\left(\frac{A(n)}{n}\right)$. *Moreover,* $J_2 = P\left(S_1 < y; \exists i\in(k, n\delta]\,/\overline{S}_n = x\right)\to 0$ *as* $k\to\infty$ *uniformly with respect to* a *and* $x, y\in R$.

Proof. At first, we estimate J_1. Assuming $j = n-i$ and considering that the difference $S_n - S_{n-i}$ for each i is distributed as the sum S_i, we have

$$J_1 = P\left(S_n - S_{n-i} < y, \exists i\in(n\delta, n-1]\,/\overline{S}_n = x\right) = \\ = P\left(S_i > nx - y, \exists j\in[1, n(1-\delta))\,/\overline{S}_n = x\right).$$

By statement 2) of Lemma 3.1 and from the last equality, we get

$$J_1 \leq MP\left(S_j > nx - y, \exists j\in[1, n(1-\delta))\right) = MP\left(\tau_c'\leq n(1-\delta)\right), \tag{6}$$

where

$$\tau_c' = \inf\{i\geq 1 : S_i > c\},\quad c = nx - y.$$

It is well known that [2]

$$\frac{\tau_c'}{c}\overset{p.n}{\to}\frac{1}{\mu}\quad\text{as}\quad c\to\infty,\quad(a\to\infty).$$

Considering that

$$\frac{c}{n}\to\mu\quad\text{as } n\to\infty,\quad(a\to\infty),$$

we have

$$\frac{\tau_c'}{c}\overset{p.n}{\to}1\quad\text{as}\quad a\to\infty.$$

Therefore,

$$P\left(\tau_c'\leq n(1-\delta)\right)\to 0\quad\text{as}\quad a\to\infty.$$

Then, it follows from (6) that $J_1\to 0$ as $a\to\infty$.

We now prove that $J_2\to 0$ as $k\to\infty$.

From statement 2) of Lemma 3.1, we have

$$J_2\leq MP\left(S_i < y, \exists i\in(k, n\delta]\right)\leq MP\left(S_i < y, \exists i > k\right).$$

The right-hand side of the last inequality is independent of a and, by the strong law of large numbers, tends to zero as $k\to\infty$.

Proof of Theorem 3.1. Let

$$L_{a,k}'(n,x,u) = P\left(S_i - if_a'(m)\geq u, 1\leq i\leq k/\overline{S}_n = x\right),$$
$$L_{a,k}(n,x,u) = P\left(S_i - i\theta\geq u, 1\leq i\leq k/\overline{S}_n = x\right),$$
$$\psi_k(u) = P\left(T_k\geq u\right),\ T_k = \inf_{1\leq i\leq k}\left(S_i - i\theta\right)$$

It follows from statement 1) of Lemma 3.1 that, for each k,

$$L_{a,k}(n,x,u)\to\psi_k(u)\quad\text{as}\quad a\to\infty. \tag{7}$$

It follows from the condition $f_a'(m)\to\theta\in[0,\mu]$ as $a\to\infty$ that

$$L_a'(n,x,u) - L_a(n,x,u)\to 0 \tag{8}$$

and

$$L_{a,k}'(n,x,u) - L_{a,k}(n,x,u)\to 0\quad\text{as } a\to\infty.$$

It is obvious that, for each fixed u,

$$\psi_k(u) \to \psi(u) \quad \text{as} \quad k \to \infty. \tag{9}$$

We now prove that

$$\Delta_k(a) = L'_{a,k}(n,x,u) - L_{a,k}(n,x,u) \to 0 \tag{10}$$

as $a \to \infty$ and $k \to \infty$. For $\delta \in (0,1)$, we have

$$0 \le \Delta_k(a) \le P\left(S_i < u, \exists i \in (k,n) / \overline{S}_n = x\right) \le P\left(S_i < u, \exists i \in (k, n\delta] / \overline{S}_n = x\right) +$$
$$+P\left(S_i < u, \exists i \in (n\delta, n] / \overline{S}_n = x\right) = J_2 + J_1.$$

By Lemma 2, $J_1 \to 0$ as $a \to \infty$ and $J_2 \to 0$ as $k \to \infty$. Therefore, (9) holds.

Now, the statement of the theorem follows from (7),(9), and (10).

The statement of Corollary 3.1 of the theorem follows from equality (2) and relation (8). The statement of Corollary 3.2 follows from the statement of Corollary 3.1 and from work [2, Ch. 2, Theorem 2,7].

References

1. A. A. Borovkov and A. A. Mogulskii, *Limit theorems in the problem of the boundary crossing by a multivariate walk*, Sib. Mat. Zh., **42** (2001), no. 2, 284-318.
2. M. Woodroofe, *Nonlinear Renewal Theory in Sequential Analysis*, SIAM, Philadelphia, 1982.
3. D. Siegmund, *Sequential Analysis. Tests and Confidence Intervals*, Springer, New York, 1985.
4. F. H. Ragimov, *On the limit behavior of conditional probabilities of the crossing of nonlinear boundaries by a perturbed random walk*, Transactions of the NAS of Azerbaijan, (2006), no. 1, 163-168.
5. F. H. Ragimov, *Asymptotic expansion of the distribution of the crossing time for nonlinear boundaries*, Teor. Veroyat. Primen., **37** (1992), 580-583.
6. I. A. Ibrahimov and Yu. V. Linnik, *Independent and Stationary Sequences of Random Variables*, Wolter-Noordhoff, Groningen, 1971.
7. W. Feller, *An Introduction to Probability Theory and Its Applications*, Wiley, New York, 1970.

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