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STRONG INVARIANCE PRINCIPLE FOR A SUPERPOSITION OF RANDOM PROCESSES

The strong invariance principle (SIP) is proved for a superposition of random processes $S(N(t))$ under rather general assumptions on $S(t)$ and $N(t)$. As a consequence, a number of SIP-type results are obtained for random sums and used to investigate their rate of growth and fluctuation of increments.

1. Introduction

Starting from the pioneer works due to A. Skorokhod (1961) and W. Strassen (1964, 1965), the strong invariance principle (**SIP**, the other term – a.s. approximation) for sums of random variables (r.v.) was rather intensively investigated during more than four decades.

We use the notion "SIP" as an umbrella name for a wide class of limit theorems which provide sufficient (or necessary and sufficient) conditions for the possibility to construct i.r.v. $\{X_i, i \geq 1\}$ and a Lévy process $\{Y(t), t \geq 1\}$ in such a way that a.s.

$$|\sum_{i=1}^{[t]} X_i - m(t) - Y(t)| = o(r(t)) \vee O(r(t)), \tag{1}$$

where $m(t)$ is a non-random centering function, and the approximating error $r(t)$ is also a non-random function depending on assumptions posed on $\{X_i\}$. Such additional assumptions clear up the type of $Y(t)$ and the form of $r(.)$.

The first and most general results dealt with the case of sums of i.i.d.r.v. Numerous investigations in this area were carried out by a number of authors, among them Kiefer, M. Csörgő, Révész, Komlós, Major, Tusnady, Berkes, Horváth (quantile Hungarian method), Stout, Phillip, Berkes (reconstruction method based on a relationship between SIP and the convergence in the Prokhorov metrics), Horváth (inverse processes); Sakhanenko, Zaicev (non-identically distributed r.v.). For detailed references, see M. Csörgő, P. Révész (1981); M. Csörgő, L. Horváth (1993); N. Zinchenko (2000).

The further development was concerned with dependent r.v.: martingales, weakly dependent r.v. and mixing sequences. Last years, the interest in the SIP for dependent r.v. is remarkably increased. It is worth to mention the results for associated r.v. and fields due to Yu (1996), Balan (2005), Wu (2007) and recent fundamental monograph by Bulinski and Shaskin (2008).

Note that the complete solution of the problem of a.s. approximation depends not only on the distribution of $\{X_i, i \geq 1\}$, but also on a structure of the probability space, and (possibly) requires a "richer" probability space and equivalent r.v.$\{X_i^{'}, i \geq 1\}$. However, for brevity, we do not distinguish between r.v.$\{X_i\}$ and $\{X_i^{'}\}$, as well as between their sums.

We also use the concept of a.s. approximation in a **wider sense** and say that a random process $\xi(t)$ admits the a.s. approximation by the random process $\eta(t)$, if $\xi(t)$

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(or stochastically equivalent $\xi^{'}(t)$) can be constructed on the rich enough probability space together with $\eta(t)$ in such a way that a.s.

$$|\xi(t)-\eta(t)| = o(r_1(t)) \vee O(r_1(t)), \tag{2}$$

where $r_1(.)$ is again a non-random function, and we do not distinguish the notations $\xi(t)$ and $\xi^{'}(t)$.

Our investigation of the SIP for a superposition of random processes was stimulated by the rather general limit results for inverse and randomly stopped processes in the Skorokhod **D**-space due to Csörgő and Horváth (1993), Whitt (2002), Silvestrov (2004), as well as by various applications in risk and queuing theories and in storage models.

2. Main result

We start with some general result concerning a.s. approximation of the superposition of the random processes (not obligatory connected with the partial sums).

Let $Z^*(t)$, $S^*(t)$ be two real-valued random processes, and let N^* be the inverse of $Z^*(t)$ defined by

$$N^*(t) = \inf\{x>0 : Z^*(x)>t\}, \quad 0 \le t < \infty.$$

Theorem 2.1. *I. Suppose that, for some constants $m, \lambda > 0$, and $\tau > 0$, a.s.*

$$\sup_{0\le t\le T} \left|\tau^{-1}(Z^*(t)-t/\lambda) - W_1(t)\right| = O(r(T)), \tag{3}$$

where $W_1(t)$ is the standard Wiener process, $r(t)\uparrow\infty$, $r(t)/t \downarrow 0$ as $t\to\infty$, and

$$\sup_{0\le t\le T} \left|S^*(t) - mt - Y_{\alpha,\beta}(t)\right| = O(q(T)), \tag{4}$$

$Y_{\alpha,\beta}(t)$ being the α-stable process independent of $W_1(t)$, $|\beta|\le 1$, $0<\alpha<2$, $q(t)\uparrow\infty$, $q(t)/t\downarrow 0$ as $t\to\infty$. Then one can construct $S^*(t)$ and $N^*(t)$ on the same probability space in such a way that $\forall\varepsilon>0$ a.s.

$$\sup_{0\le t\le T} \left|S^*(N^*(t)) - (m\lambda)t - (Y_{\alpha,\beta}(t\lambda) - (m\lambda\tau)W_2(\lambda t))\right| =$$

$$= O\Big(q(T) + r(T) + \log T + (r(T) + (T\log\log T)^{1/2})^{1/(\alpha-\varepsilon)}\Big), \tag{5}$$

where $W_2(t)$ is a Wiener process independent of $Y_{\alpha,\beta}(t)$.

II. If $S^*(t)$ also admits a.s. approximation by a Wiener process (instead of the α-stable process with $\alpha<2$), i.e., for some constant $\sigma>0$, a.s.

$$\sup_{0\le t\le T} \left|\sigma^{-1}(S^*(t)-mt) - W_3(t)\right| = O(q(T)), \tag{6}$$

where $W_1(t)$ and $W_3(t)$ are independent Wiener processes, and $q(t)$ and $r(t)$ are as above, then a.s.

$$\sup_{0\le t\le T} \left|S^*(N^*(t)) - (m\lambda)t - \nu W(t)\right| = O\big(q(T) + r(T) + \log T\big), \tag{7}$$

where $W(t)$ is a standard Wiener process independent of $W_3(t)$, $\nu^2 = \lambda\sigma^2 + \lambda^3 m^2\tau^2$.

The version of this theorem was formulated in [19] with the sketch of the proof; here, we will establish the detailed proof. Such a proof, as well as various applications of Theorem 2.1, needs a number of auxiliary results connected with the known SIP for sums of i.i.d.r.v., renewal and related processes.

3. Auxiliary results

3.1. **SIP for sums of i.i.d.r.v. with finite variance.** Let $\{X_i, i \geq 1\}$ be i.i.d.r.v with a common distribution function (d.f.) $F(x)$, characteristic function (ch.f.) $f(u)$, and $EX_1 = m$, $VarX_1 = \sigma^2$,

$$S(n) = \sum_{i=1}^{n} X_i, \quad S(0) = 0, \quad S(x) = S([x]), [x] - \text{ entire of } x > 0.$$

Summarizing all known results in this area, we have, as in [5],

Theorem 3.1. *It is possible to define a partial sum process $\{S(t), t \geq 0\}$ and a standard Wiener process $\{W(t), t \geq 0\}$ in such a way that a.s.*

$$\sup_{0 \leq t \leq T} |S(t) - mt - \sigma W(t)| = o(r(T)), \tag{8}$$

where
$r(T) = T^{1/p}$ iff $E\,|X_1|^p < \infty, p > 2$; $r(T) = (T \ln\ln T)^{1/2}$ iff $E\,|X_1|^2 < \infty$;
right hand side of (8) is $O(\ln T)$ iff $E \exp(uX_1) < \infty$ for $u \in (0, u_0)$.

3.2. **SIP for sums of i.i.d.r.v. attracted to the stable law.** Suppose that i.i.d.r.v. $\{X_i, i \geq 1\}$ are in a domain of normal attraction of the stable law with $0 < \alpha < 2$, $|\beta| \leq 1$, the notation $\{X_i, i \geq 1\} \in DNA(G_{\alpha,\beta})$, i.e.

$$n^{-1/\alpha}\big(S(n) - a_n\big) \Rightarrow G_{\alpha,\beta},$$

where $a_n = nEX_1 = mn$ if $1 < \alpha < 2$, $a_n = 0$ if $0 < \alpha < 1$, and $a_n = (2/\pi)\beta \log n$ if $\alpha = 1$.

Here, $G_{\alpha,\beta}$ is d.f. of the stable law with $0 < \alpha < 2$, $|\beta| \leq 1$ and ch.f.

$$g_{\alpha,\beta}(u) = \exp(K_{\alpha,\beta}(u)), \quad K_{\alpha,\beta}(u) = -|u|\big(1 - i\beta(u/|u|)\varpi(u,\alpha)\big), \tag{9}$$

where $\varpi(u,\alpha) = \tan(\pi\alpha/2)$ if $0 < \alpha < 2, \alpha \neq 1$, and $\varpi(u,\alpha) = -(2/\pi)\log|u|$ if $\alpha = 1$.

Now the approximating process $Y(t) = Y_\alpha(t) = Y_{\alpha,\beta}(t)$, $t \geq 0$, is a stable Lévy process with ch.f. $g_{\alpha,\beta}(t;u) = \exp(tK_{\alpha,\beta}(u))$. In this case, SIP was studied by Zinchenko [15]; Berkes, Dehling, Dobrovski, and Philipp [1] with additional
Assumption (C) : *there are $a_1 > 0, a_2 > 0$ and $l > \alpha$ such that, for $|u| < a_1$,*

$$|f(u) - g_{\alpha,\beta}(u)| < a_2|u|^l, \tag{10}$$

where $f(u)$ is a ch.f. of $(X_1 - EX_1)$ if $1 < \alpha < 2$, and ch.f. of X_1 if $0 < \alpha \leq 1$.

Theorem 3.2. *([15]) Put $m = EX_1$ for $1 < \alpha < 2$ and $m = 0$ for $0 < \alpha \leq 1$. Under assumption (C), a.s.*

$$\sup_{0 \leq t \leq T} \big|S([t]) - mt - Y_{\alpha,\beta}(t)\big| = O(T^{1/\alpha - \rho_0}), \quad \rho_0 = \min\left(\frac{l-\alpha}{80\alpha}, \frac{2-\alpha}{2\alpha}\right). \tag{11}$$

3.3. **SIP for counting (renewal) processes.** Suppose that $\{Z_i, i \geq 1\}$ is another sequence of i.i.d.r.v. independent of $\{X_i, i \geq 1\}$ with d.f. $F_1(x)$, ch.f. $f_1(u)$ and $EZ_1 = 1/\lambda > 0$,

$$Z(n) = \sum_{i=1}^{n} Z_i, \quad Z(0) = 0, \quad Z(x) = Z([x]),$$

and define the **renewal (counting) process** $N(t)$ associated with partial sums $Z(n)$ as

$$N(t) = \inf\{x \geq 0 : Z(x) > t\}.$$

In the case $\tau^2 = varZ_1 < \infty$, $EZ_1 = 1/\lambda > 0$ Csörgő, Horvách, Steinebach, Aalex, Deheuvels, Mason, and van Zwet (see [5]) studied an a.s. approximation of the type

$$|\lambda t - N(t) - \tau\lambda^{3/2}W(t)| = o(r(t)) \vee O(r(t)) \tag{12}$$

and proved that the conditions, which provide (12) and corresponding optimal errors, are the same as those for $S(n)$, see Theorem 3.1.

The case $\{Z_i, i \geq 1\} \in DNA(G_{\alpha,\beta})$ is covered by the following:

Theorem 3.3. *If $\{Z_i\}$ satisfy (C) with $1 < \alpha < 2$, then a.s.*

$$\sup_{0 \leq t \leq T} \left|\lambda t - N(t) - \lambda^{1+1/\alpha} Y_{\alpha,\beta}(t)\right| = o(r_1(T)), \tag{13}$$

where $r_1(T)$ is any upper function for the α-stable Levy process.

The proof of this theorem is analogous to the proof of Theorem 1 in [20], where $r_1(T) = T^{1/\alpha+\delta}$ for any $\delta > 0$ was considered.

3.4. **SIP for a general inverse process.** Let $\{Z^*(t), t \geq 0\}$ be a real-valued stochastic process. We define its inverse $N^*(t)$ by

$$N^*(t) = \inf\{x \geq 0 : Z^*(x) > t\}, \quad 0 \leq t < \infty.$$

The next theorem [5] established that if $\{Z^*(t), t \geq 0\}$ can be a.s. approximated by a Wiener process, then $N^*(t)$ can be also approximated by another Wiener process.

Theorem 3.4. *Assume that, with some positive constants λ and τ,*

$$\sup_{0 \leq t \leq T} \left|\tau^{-1}(Z^*(t) - t/\lambda) - W(t)\right| = O(r(T)), \tag{14}$$

where $W(t)$ is a Wiener process, $r(t) \uparrow \infty$, $r(t)/t \downarrow 0$ as $t \to \infty$. Then we can define a Wiener process $\{W^*(t), t \geq 0\}$ such that a.s.

$$\sup_{0 \leq t \leq T} \left|(N^*(t) - \lambda t) - (\tau \lambda^{3/2}) W(t)\right| = O(r(T) + \log t). \tag{15}$$

3.5. **SIP for dependent r.v.** Below, we present two examples of SIP-type results for dependent r.v.; much more facts can be found in [2],[8],[14].

Proposition 3.1. *([7]) Let $\{X_i, i \geq 1\}$ be a stationary Gaussian sequence centered at expectations. Suppose that, for some $\varepsilon > 0$,*

$$E\{X_1 X_n\} = O(n^{-1-\varepsilon}), \quad EX_1^2 + 2\sum_{i \geq 1} EX_1 X_i = \sigma_1^2 < \infty. \tag{16}$$

Then there exists a Wiener process $\{W(t), t \geq 0\}$ such that a.s.

$$\sup_{0 \leq t \leq T} |S(t) - \sigma_1 W(t)| = (T^{1/2-\vartheta}), \quad \vartheta = \min(1, \varepsilon)/500. \tag{17}$$

Definition 3.1. R.v. $X_1, \ldots, X_n$ are associated, if, for any two coordinate-wise nondecreasing functions $f, g : R^n \to R^1$,

$$Cov\big(f(X_1, \ldots, X_n), g(X_1, \ldots, X_n)\big) \geq 0$$

whenever the covariance is defined. A sequence $\{X, X_i, i \geq 1\}$ is associated, if every finite subcollection is associated.

Proposition 3.2. *([20, 2]). Let $\{X_i, i \geq 1\}$ be a strictly stationary associated sequence centered at expectations. Suppose that $E|X_1|^{2+\delta} < \infty$ for some $\delta > 0$ and the Cox–Grimmett coefficient*

$$u(n) = \sup_{k \geq 1} \sum_{j:|j-k| \geq n} Cov(X_j X_k) = O(e^{-\theta n}) \tag{18}$$

for some $\theta > 0$. Then there exists a Wiener process $\{W(t), t \geq 0\}$ such that, for some $\vartheta > 0$, a.s.

$$\sup_{0 \leq t \leq T} |S(t) - \sigma_1 W(t)| = O(T^{1/2-\vartheta}). \tag{19}$$

4. Proof of Theorem 2.1.

We carry out the proof of the first part of Theorem 2.1, while the proof of the second part was presented in [5]. Our approach is based on the ideas of Csörgő and Horvách who studied the case of the approximation with the help of a Wiener process.

Obviously, we can express

$$S^*(N^*(t)) - m\lambda t = S^*(N^*(t)) - mN^*(t) + m(N^*(t) - \lambda t).$$

Thus, we have the inequality

$$\Delta(T) = \sup_{0\le t\le T} \big|S^*(N^*(t)) - m\lambda t - (Y_\alpha(\lambda t) - (m\lambda\tau)W_2(\lambda t))\big| \le$$
$$\le \sup_{0\le t\le T} \big|S^*(N^*(t)) - mN^*(t) - Y_\alpha(N^*(t))\big| +$$
$$+ \sup_{0\le t\le T} m\big|(N^*(t) - \lambda t) + \lambda\tau W_2(\lambda t)\big| + \sup_{0\le t\le T} \big|Y_\alpha(N^*(t)) - Y_\alpha(\lambda t)\big|$$
$$\le \Delta_1(T) + m\Delta_2(T) + \Delta_3(T). \tag{20}$$

On the next steps, we estimate each $\Delta_i(T)$ using conditions (3) and(4), SIP for $N^*(t)$ from Theorem 3.4, and the growth rate for the stable and Wiener processes.

By (3) and LLN for $W(t)$, we have a.s. $\limsup_{t\to\infty} Z(t)/T = 1/\lambda$, which implies $\limsup_{t\to\infty} N(t)/T = \lambda$, i.e. $\forall\varepsilon > 0$ for large T a.s.

$$N(T) \le (1+\varepsilon)\lambda T. \tag{21}$$

Hence, from (4) and (21) $\forall\varepsilon > 0$, a.s.

$$\sup_{0\le t\le T} \big|S^*(N^*(t)) - mN^*(t) - Y_\alpha(N^*(t))\big| = O\big(q\big(\lambda(1+\varepsilon)T\big)\big),$$

so

$$\Delta_1(T) = O\big(q\big(\lambda(1+\varepsilon)T\big)\big).$$

Now we will demonstrate that $q\big(\lambda(1+\varepsilon)T\big) = O\big(q(T)\big)$. Really, if $\lambda(1+\varepsilon) \ge 1$, then $q(T)/T \ge ((1+\varepsilon)\lambda T)^{-1} q\big(\lambda(1+\varepsilon)T\big)$, and if $\lambda(1+\varepsilon) < 1$, then $q\big(\lambda(1+\varepsilon)T\big) \le q(T)$. Hence,

$$\Delta_1(T) = O\big(q(T)\big). \tag{22}$$

Condition (2) and SIP for the inverse process (Theorem 3.4) ensure the possibility to construct $N^*(t)$ together with the Wiener process $W_2(t)$ in such a way that a.s.

$$\sup_{0\le t\le T} \big|(N^*(t) - \lambda t) + \lambda\tau W_2(\lambda t)\big| = O\big(r(T) + \log T\big). \tag{23}$$

Thus,

$$\Delta_2(T) = O\big(r(T) + \log T\big). \tag{24}$$

The LIL for Wiener process and (23) also yield the existence of such $C, C_1 > 0$ that, for large T, a.s.

$$\sup_{0\le t\le T} \big|\lambda t - N^*(t))\big| \le \sup_{0\le t\le T} \big|(N^*(t) - \lambda t) + \lambda\tau W_2(\lambda t)\big| + \sup_{0\le t\le T} |\lambda\tau W_2(\lambda t)| \le$$
$$\le C_1\big(r(T) + \log t + (T\log\log T)^{1/2}\big) \le C\big(r(T) + (T\log\log T)^{1/2}\big). \tag{25}$$

Thus, from the fact that $\forall\varepsilon > 0$, a.s.

$$\sup_{0\le t\le T} |Y_\alpha(t)| = o(T^{1/\alpha+\varepsilon}),$$

which follows from the integral test for upper/lower functions for the Lévy process [6, Ch. 4], we obtain a.s.

$$\sup_{0\le t\le T} |Y_\alpha(N^*(t) - Y_\alpha(\lambda t)| = \sup_{0\le t\le T} |Y_\alpha(N^*(t) - \lambda t)| =$$

$$= \sup_{0\le t\le C(r(T)+(T\log\log T)^{1/2})} |Y_\alpha(t)| = o\Big((r(T) + (T\log\log T)^{1/2})^{1/(\alpha-\varepsilon)} \Big),$$

i.e.

$$\Delta_3 = o\Big(\big(r(T) + \sqrt{T\log\log T}\big)^{1/(\alpha-\varepsilon)} \Big). \tag{26}$$

Combining together (22), (24), and (26), we complete the proof.

The form of Theorem 2 is convenient for the investigation of the random sums

$$S(N(t)) = \sum_{i=1}^{N(t)} X_i,$$

where $N(t)$ is a renewal process. So, this theorem can serve as a source of numerous SIP-type limit theorems for the random sums under various assumptions on the dependence and moment conditions of terms and inter-occurrence intervals.

5. Strong invariance principle for random sums

Let the partial sum processes $S(t)$ and $Z(t)$ be as those in Section 3, let the renewal process $N(t)$ be the inverse one of $Z(t)$, and $D(t) = S(N(t)) = \sum_{i=1}^{N(t)} X_i$, $EZ_1 = 1/\lambda > 0$, $EX_1 = m$.

We start with the case where both i.i.d.r.v. $\{X_i, i\ge 1\}$ and $\{Z_i, i \ge 1\}$ have finite moments of order greater than 2. The following theorem is a straightforward consequence of Theorem 2.1; the corresponding $r(t)$ and $q(t)$ are obtained from Theorem 3.1 and (12).

Theorem 5.1. *([5]) Let $varX_1 = \sigma^2 < \infty$, $\tau^2 = varZ_1 < \infty$. The mentioned r.v. can be constructed on the same probability space together with the Wiener process $W(t)$ in such a way that a.s.*

$$\sup_{0\le t\le T} |S(N(t)) - \lambda mt - \nu W(t)| = o(r_3(T)), \nu^2 = \lambda\sigma^2 + \lambda^3 m^2\tau^2, \tag{27}$$

where
(i) *$r_3(T) = T^{1/p}$, $p = \min\{p_1, p_2\}$, if $E|X_1|^{p_1} < \infty$, $E|Z_1|^{p_2} < \infty$, $p_1 > 2$, $p_2 > 2$;* ***(ii)*** *$r_3(T) = (T\ln\ln T)^{1/2}$, if $p = 2$;* ***(iii)*** *the right-hand side of (13) is $O(\ln T)$, if $E\exp(uX_1) < \infty$ and $E\exp(uZ_1) < \infty$ for all $u \in (0, u_o)$.*

In the case $\{X_i, i \ge 1\} \in DNG_{\alpha,\beta}$ and satisfy **(C)**, by Theorem 3.2, $q(t) = t^{1/\alpha-\varrho_1}$, $\varrho_1 > 0$, and the worst estimate for $r(t)$ is $(t\log\log t)^{1/2}$. These facts lead to

Theorem 5.2. *([20]) Let $\{X_i, i\ge 1\}$ satisfy (C) with $1 < \alpha < 2$, and $EZ_1^2 < \infty$. Then a.s.*

$$\sup_{0\le t\le T} \big|S(N(t)) - m\lambda t - Y_{\alpha,\beta}(\lambda t)\big| = o(T^{1/\alpha-\varrho_1}), \varrho_1 = \varrho_1(\alpha, l) > 0. \tag{28}$$

Corollary 5.1. *Theorems 5.1 and 5.2 hold if $N(t)$ is a homogeneous Poisson process with intensity λ, corresponding $\nu^2 = \lambda EX_1^2$.*

For Poisson sums with dependent terms, it is easy to derive

Corollary 5.2. *Statement (27) holds with $r_3(t) = (T^{1/2-\vartheta})$, $\vartheta > 0$, if the terms $\{X_i, i \ge 1\}$ constitute a stationary Gaussian sequence, whose covariance satisfies (16), or a strictly stationary associated sequence with covariance satisfying Proposition 3.2.*

The approach analogous to the proof of Theorem 2.1. provides

Theorem 5.3. *Let $\{X_i, i\ge 1\}$ satisfy (C) with $1 < \alpha_1 < 2$, $|\beta_1| \le 1$ and $\{Z_i\}$ satisfy (C) with $1 < \alpha_2 < 2$, $\alpha_1 < \alpha_2$, then a.s.*

$$\sup_{0\le t\le T} \big|D(t) - m\lambda t - Y_{\alpha_1,\beta_1}(\lambda t)\big| = o(T^{1/\alpha_1-\varrho_2}) \text{ for some } \varrho_2 = \varrho_2(\alpha_1, l) > 0.$$

6. The rate of growth of random sums and their increments

In this section, we demonstrate the possible way of application of the SIP: using the SIP with an appropriate error term, one can transfer, almost without the proof, the results on the asymptotic behavior of the Wiener or stable processes on the rate of growth of random sums $D(t) = S(N(t))$ and their increments.

Theorem 6.1. *(Classical LIL for random sums). Let* $\{X_i, i \geq 1\}$ *and* $\{Z_i, i \geq 1\}$ *be independent sequences of i.i.d.r.v. with* $EX_1 = m < \infty$, $0 < EZ_1 = 1/\lambda < \infty$, $\sigma^2 = VarX_1 < \infty$, $\tau^2 = VarZ_1 < \infty$. *Then a.s.*

$$\limsup_{t\to\infty} \frac{|D(t) - m\lambda t|}{\sqrt{2t\ln\ln t}} = \nu, \quad \nu^2 = \lambda\sigma^2 + \lambda^3 m^2\tau^2.$$

Remark 6.1. Analogous LIL holds also for stationary sequences satisfying Corollary 5.2.

When the terms $\{X_i, i \geq 1\}$ are attracted to the asymmetric stable law $G_{\alpha,-1}$, we have

Theorem 6.2. *Let* $\{X_i, i \geq 1\}$ *satisfy* ***(C)*** *with* $1 < \alpha_1 < 2$, $\beta_1 = -1$, *and* $\{Z_i, i \geq 1\}$ *satisfy* ***(C)*** *with* $\alpha_2 \in (\alpha_1, 2)$, $|\beta_2| \leq 1$, *or* $EZ_1^2 < \infty$, $EX_1 = m < \infty$, $0 < EZ_1 = 1/\lambda < \infty$. *Then a.s.*

$$\limsup_{t\to\infty} \frac{D(t) - m\lambda t}{t^{1/\alpha_1}(B_1^{-1}\ln\ln t)^{1/\theta_1}} = \lambda^{1/\alpha_1}, \tag{29}$$

$$B_1 = (\alpha_1 - 1)\alpha_1^{-\theta_1}|\cos(\pi\alpha_1/2)|^{1/(\alpha_1-1)}, \quad \theta_1 = \alpha_1/(\alpha_1 - 1). \tag{30}$$

The proof follows from Theorems 5.2 and 5.3 and LIL for an asymmetric stable process [5], [17].

Corollary 6.1. *Theorems 6.1 and 6.2 are true when* $N(t)$ *is a homogeneous Poisson process.*

Next, we study the magnitude of increments $D(T+a_T) - D(T)$. When both $\{X_i\}$ and $\{Z_i\}$ have finite variance, the centered process $D(t) - m\lambda t$ can be a.s. approximated by a Wiener process $W(t)$ with an appropriate error term, whose form depends on additional moment conditions. This gives a possibility to extend the Erdös–Rényi–Csörgő–Révész results [4, 5, 7] for $W(T+a_T) - W(T)$ on the asymptotics of $D(T+a_T) - D(T)$. Notice that the additional assumptions which determine the form of the approximation term have impact on the length of intervals a_T which appear in the next theorems.

Theorem 6.3. *Let* $\{X_i, i \geq 1\}$ *and* $\{Z_i, i \geq 1\}$ *be the independent sequences of i.i.d.r.v.,* $EX_1 = m$, $varX_1 = \sigma^2$, $EZ_1 = 1/\lambda > 0$, $varZ_1 = \tau^2$,

$$E\exp(uX_1) < \infty, \quad E\exp(uZ_1) < \infty, \tag{31}$$

as $|u| < u_0$, $u_0 > 0$, *the function* $a_T, T \geq 0$, *satisfies the following conditions:* $0 < a_T < T$, *and* T/a_T *does not decrease in* T. *In addition,*

$$a_T/\ln T \to \infty \text{ as } T \to \infty. \tag{32}$$

Then a.s.

$$\limsup_{T\to\infty} \frac{|D(T+a_T) - D(T) - m\lambda a_T|}{\gamma(T)} = \nu, \tag{33}$$

where

$$\nu^2 = \lambda\sigma^2 + \lambda^3 m^2\tau^2, \quad \gamma(T) = \{2a_T(\ln\ln T + \ln T/a_T)\}^{1/2}.$$

Theorem 6.4. *Let* $\{X_i, i \geq 1\}$, $\{Z_i, i \geq 1\}$ *and* a_T *satisfy all conditions of the previous Theorem with the following assumption used instead of (31):*

$$EX_1^{p_1} < \infty, \quad p_1 > 2, \quad EZ_1^{p_2} < \infty, \quad p_2 > 2.$$

Then (33) is true if $a_T > c_1 T^{2/p}/\ln T$ *for some* $c_1 > 0$, $p = \min\{p_1, p_2\}$.

When $\{X_i, i \geq 1\}$ are attracted to an asymmetric stable law, Theorems 5.2 and 5.3 and a variant of the Erdös–Rényi–Csörgő–Révész-type law for the α-stable Lévy process without positive jumps [17] yield

Theorem 6.5. *Suppose that $\{X_i, i \geq 1\}$ and $\{Z_i, i \geq 1\}$ satisfy the conditions of Theorem 6.2. The function a_T is non-decreasing, $0 < a_T < T$, T/a_T is also non-decreasing and provides ${d_T}^{-1}T^{1/\alpha - \varrho_2} \to 0$ for certain $\varrho_2 > 0$ determined by the error term in SIP. Then a.s.*

$$\limsup_{T\to\infty} \frac{D(T+a_T) - D(T) - m\lambda a_T}{d_T} = \lambda^{1/\alpha_1}, \tag{34}$$

where the normalizing function

$$d_T = a_T^{1/\alpha_1}\{B_1^{-1}(\ln\ln T + \ln T/a_T)\}^{1/\theta_1}$$

and B_1, θ_1 are defined by (30).

Now consider the classical risk process $U(t) = u + ct - \sum_{i=1}^{N(t)} X_i$, where $u \geq 0$ denotes the initial capital; $c > 0$ stands for the premium income rate; i.i.d.r.v $\{X_i, i \geq 1\}$ are interpreted as claim sizes; the Poisson process $N(t)$ stands for the number of claims until time t. In such a model, $D(t) = S(N(t))$ is interpreted as the total claim amount process, and the above results can be used to investigate its growth rate [19].

In the case of small claims (with finite exponential moments) or large claims (but with finite moments of order $p > 2$) for large t, we can a.s. indicate upper/lower bounds for the growth of total claim amounts $D(t)$ as $m\lambda t \pm \nu\sqrt{2t\ln\ln t}$ and for the reserve capital $U(t)$ as $u + t\rho m\lambda \pm \nu\sqrt{2t\ln\ln t}$, where $\rho = (c - \lambda m)/\lambda m > 0$ is a safety loading.

For very large claims in a domain of normal attraction of the asymmetric stable law $G_{\alpha,1}$ with $1 < \alpha < 2$, $\beta = 1$ (for instance, a Pareto-type r.v. with corresponding $1 < \alpha < 2$), Theorem 6.2 for large t yields the a.s. upper bound for the risk process

$$U(t) \leq u + \rho m\lambda t + (1+\epsilon)\lambda^{1/\alpha} t^{1/\alpha}(B^{-1}\ln\ln t)^{1/\theta}, \quad \forall \epsilon > 0.$$

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