Theory of Stochastic Processes
Vol. 16 (32), no. 1, 2010, pp. 103–110

O. SUGAKOVA

# DENSITY ESTIMATION BY OBSERVATIONS WITH ADMIXTURE

We consider a two-component mixture model, in which the component of interest (the primary component) is assumed to be symmetrically distributed, and the admixture distribution has a known probability density function (pdf). The mixing probability and the mean of the primary component are unknown as well. A kernel estimate for the primary component's pdf is proposed. Under some assumptions, the asymptotic normality of this estimate is demonstrated.

## 1. Introduction

In this paper, we consider i.i.d. real valued observations $\{\xi_1, \xi_2, \dots, \xi_N\}$ taken from a mixture of two components: the primary component with an unknown distribution and the admixture, whose distribution is known. It will be assumed that both components have probability density functions and the pdf of the primary component is symmetric around its median. Thus, the pdf of $\xi_i$ is

$$\psi(x) = pf(x-a) + (1-p)f_0(x), \tag{1}$$

where $f_0(x)$ is the known admixture pdf, $a \in R$ is the median of the primary component, $f(x)$ is the pdf of the primary component centered by $a$ ($f$ is an even function: $f(-x) = f(x)$), and $p \in (0,1)$ is the mixing probability. The parameters $a$ and $p$ (which are called Euclidean parameters) are assumed to be unknown. The pdf $f$ is the nonparametric part of the model. There is a vast literature devoted to the estimation of parameters and distributions of two-component mixtures [1-3].

Model (1) was introduced in [1], where estimates for $a$ and $p$ are constructed, and their consistency is demonstrated. Another estimate for $a$ based on generalized estimating equations (GEE) approach was proposed in [3]. There, the asymptotic normality of this estimate is demonstrated.

An estimate $\hat{f}$ for $f$ was proposed in [1] on the basis of the estimates for $a$ and $p$ obtained in that article and a kernel estimate for $\psi$. The consistency of this estimate in the $L_1$- norm was demonstrated.

The aim of this paper is to investigate the asymptotic behavior of $\hat{f}$ and its modification in the case where the true pdf $f$ is twice continuously differentiable.

## 2. Statement of the Problem

To derive an estimate for $f$, we note that, by (1),

$$f(x) = \frac{1}{p}\left(\psi(x+a) - (1-p)f_0(x+a)\right). \tag{2}$$

We will replace the unknown parameters in this formula by their estimates. For the estimation of the pdf $\psi(x)$, we use the kernel estimate [4, p.57]

$$\hat{\psi}_N(x) = \frac{1}{Nh_N}\sum_{j=1}^{N} K\left(\frac{x-\xi_j}{h_N}\right), \tag{3}$$

2000 *Mathematics Subject Classification.* Primary 62G07; Secondary 62G20.

*Key words and phrases.* mixture, kernel estimate, Vapnik–Chervonenkis inequality.

where $K$ is a kernel that is pdf on $R$, and $\{h_N, N \geq 1\}$ is a bandwidth such that $h_N \to 0$ and $Nh_N \to \infty$ as $N \to \infty$.

A generalized estimating equation (GEE) for $a$ is proposed in [3]. Here, we consider unbiased estimating equations of the form

$$\hat{h}(\alpha) = 0, \tag{4}$$

where

$$\hat{h}(\alpha) = \frac{1}{N}\sum_{j=1}^{N}\left(g_1(x-\alpha)G_2(\alpha) - g_2(x-\alpha)G_1(\alpha)\right);$$

$g_1(x),\ g_2(x)$ are some fixed odd functions; $G_i(\alpha) = \int_{-\infty}^{\infty} g_i(x-\alpha)f_0(x)dx$.

Statistics $\hat{a}_N$ is called a GEE-estimate with the estimating pair $(g_1, g_2)$, if $\hat{h}(\hat{a}_N) = 0$ a.s. In [3], the asymptotic normality of $\hat{a}_N$ was demonstrated. A consistent GEE estimate $\hat{a}_N$ is constructed with the estimating pair $g_1(x) = x;\ g_2(x) = x^3$. This $\hat{a}_N$ is called a moment estimate.

To estimate the parameter $p$, we use the method of moments. We denote $m_0 = \int_{-\infty}^{\infty} xf_0(x)dx$. Equating the theoretical and empirical moments of observations, we obtain the equation for the estimate $\hat{p}_N$ :

$$\hat{p}(a - m_0) + m_0 = \frac{1}{N}\sum_{j=1}^{N}\xi_j.$$

Since $a$ is unknown, we replace it by a GEE estimate $\hat{a}_N$ from (4). So,

$$\hat{p}_N = \frac{1}{\hat{a}_N - m_0}\left(\frac{1}{N}\sum_{j=1}^{N}\xi_j - m_0\right). \tag{5}$$

The resulting estimate for $f$ is

$$\hat{f}_N(x) = \frac{1}{\hat{p}_N}\left(\hat{\psi}_N(x+\hat{a}_N) - (1-\hat{p}_N)f_0(x+\hat{a}_N)\right). \tag{6}$$

This estimate has the same form as the estimate proposed in [1], but we use here another estimates for $a$ and $p$. Additionally, we consider a symmetrized version of our estimate:

$$\tilde{f}_N(x) = \frac{\hat{f}_N(x) + \hat{f}_N(-x)}{2}. \tag{7}$$

## 3. Main results

Different numerical constants will be denoted by $c_1, c_2, \ldots, c_k$. The sign $\Rightarrow$ means the weak convergence of distributions.

To derive asymptotic results, we will use the following assumptions.

(i) Assumptions about the kernel:

$$K(x) - \text{ is a finitely supported function, i.e. } \exists\ [a,b] \in R, \text{ that } K(x) = 0 \text{ if } x \notin [a,b];$$

$$\mathrm{Var}_{[a,b]}K'(x) < c_1, \text{where } \mathrm{Var}_{[a,b]} \text{ means the functional variation on } [a,b];$$

$$d^2 = \int_{-\infty}^{\infty} K^2(z)dz < \infty; \quad \int_{-\infty}^{\infty} zK(z)dz = 0.$$

(ii) Assumptions about density's components:

$f(x)$ and $f_0(x)$ are twice continuously differentiable;

$$|f'(x)| < c_2; \quad |f_0'(x)| < c_3; \quad m_0 \neq a;$$

$$\int_{-\infty}^{\infty} x^2 f(x)dx < \infty; \qquad \int_{-\infty}^{\infty} x^2 f_0(x)dx < \infty.$$

(iii) Assumptions about the estimating functions for $a$:
$g_i'(x),\ G_i'(x)$ are continuous on $R$;

$$E(g_i(\xi_1 - a))^2 < \infty, i = 1, 2; \qquad E\frac{\partial}{\partial\alpha}h(\xi_1, \alpha)|_{\alpha=a} \neq 0;$$

for some $\varepsilon > 0, \delta > 0$ :

$$E \sup_{\alpha:|a-\alpha|<\varepsilon} |g_i'(\xi_1 - \alpha)|^{1+\delta} < \infty, i = 1, 2.$$

(iv) $\hat{a}_N$ is a consistent estimate for $a$;
(v) Assumption about the bandwidth $h_N$:

$$h_N = CN^{-1/5},\ C - \textit{some constant}.$$

*Remark* 3.1. In practice, the parameter $C$ of the bandwidths $h_N$ is usually taken to be dependent on the data $\xi_1, \ldots, \xi_N$. For example, in the so-called Silverman's rule-of-thumb, we have $C_N = \left(\frac{8\sqrt{\pi}D}{3d^2}\right)^{\frac{1}{5}} \hat{S}$, where $\hat{S}^2$ is the sample variance of the data. To simplify the presentation, we consider only a constant $C$, but the results can be generalized to the case where $C_N \to C$ as $n \to \infty$.

Denote $D = \int_{-\infty}^{\infty} z^2 K(z)dz$.

**Theorem 3.1.** *Assume that assumptions (i)-(v) hold.*
*Then*

$$N^{\frac{2}{5}}\left(\hat{f}_N(x) - f(x)\right) \Rightarrow \eta_1,$$

*where $\eta_1$ is a normal random variable distributed as*

$$N\left(\frac{D^2C^2}{2p}\psi''(x+a), \frac{d^2}{Cp^2}\psi(x+a)\right).$$

The next theorem describes the asymptotic behavior of the symmetrized estimate (7).

**Theorem 3.2.** *Assume that (i)-(v) hold. Then*

$$N^{\frac{2}{5}}\left(\widetilde{f}_N(x) - f(x)\right) \Rightarrow \eta_2,$$

*where $\eta_2$ is the normal random variable distributed as*

$$N\left(\frac{D^2C^2}{4p}\left(\psi''(x+a) + \psi''(-x+a)\right), \frac{d^2}{4Cp^2}\left(\psi(x+a) + \psi(-x+a)\right)\right).$$

*Remark* 3.2. For example, in the case where $f(x)$ and $f_0(x)$ are standard Gaussian densities and $\hat{a}_N$ is a moment estimate, assumptions (ii)-(iv) hold.

## 4. Proofs of results

*Proof. of Theorem 3.1.*
From

$$f(x) = f_0(x+a) + \frac{1}{p}(\psi(x+a) - f_0(x+a))$$

and (5), we get

$$\hat{f}_N(x) - f(x) = \varepsilon_1^N + \varepsilon_2^N + \varepsilon_3^N + \varepsilon_4^N,$$

where

$$\varepsilon_1^N = \left(1 - \frac{1}{\hat{p}_N}\right)(f_0(x+\hat{a}_N) - f_0(x+a));$$

$$\varepsilon_2^N = \frac{1}{\hat{p}_N}\left(\hat{\psi}_N(x+\hat{a}_N) - \psi(x+\hat{a}_N)\right); \quad \varepsilon_3^N = \frac{1}{\hat{p}_N}\left(\psi(x+\hat{a}_N) - \psi(x+a)\right);$$

$$\varepsilon_4^N = \frac{p-\hat{p}_N}{p\hat{p}_N}(\psi(x+a) - f_0(x+a)). \tag{8}$$

At first, we will consider the convergence rate of Euclidean parameters of the estimates $\hat{a}_N$ and $\hat{p}_N$. By Theorem 2.1 from [3]: if (iii) holds,

$$\hat{a}_N - a = O_p\left(\frac{1}{\sqrt{N}}\right). \tag{9}$$

*Remark* 4.1. For any random sequence $\varepsilon_N$, the notation $\varepsilon_N = O_p\left(\frac{1}{\sqrt{N}}\right)$ means that

$$\lim_{C\to\infty}\limsup_{N\to\infty}\mathbf{P}\left\{\sqrt{N}|\varepsilon_N| > C\right\} = 0.$$

We now consider an estimate of the parameter $p$. From (5)

$$\hat{p}_N - p = \frac{1}{\hat{a}_N - m_0}\left(\frac{1}{N}\sum_{j=1}^N \xi_j - m_0(1-p) - ap\right) - \frac{p}{\hat{a}_N - m_0}\left(\hat{a}_N - a\right). \tag{10}$$

By the central limit theorem [4, p.179],

$$\frac{1}{N}\sum_{j=1}^N \xi_j - m_0(1-p) - ap = O_p\left(\frac{1}{\sqrt{N}}\right). \tag{11}$$

Equations (9)-(11) imply that

$$\hat{p}_N - p = O_p\left(\frac{1}{\sqrt{N}}\right). \tag{12}$$

Due to (9), the boundedness of derivatives of $\psi(x)$, $f_0(x)$, and the Lagrangian theorem, we conclude that, for some intermediate points $\theta_1$ and $\theta_2$ between $\hat{a}_N$ and $a$,

$$\varepsilon_1^N = \left(1 - \frac{1}{\hat{p}_N}\right) f_0'(x+\theta_1)(\hat{a}_N - a) = O_p\left(\frac{1}{\sqrt{N}}\right),$$

$$\varepsilon_3^N = \frac{1}{\hat{p}_N}\psi'(x+\theta_2)(\hat{a}_N - a) = O_p\left(\frac{1}{\sqrt{N}}\right). \tag{13}$$

From (12) we derive the same convergence rate for $\varepsilon_4^N$.

$$\varepsilon_4^N = O_p\left(\frac{1}{\sqrt{N}}\right). \tag{14}$$

It remains to consider $\varepsilon_2^N$. Let us expand it as

$$\varepsilon_2^N = \delta_1^N + \delta_2^N + \delta_3^N; \quad \text{where} \tag{15}$$

$$\delta_1^N = \frac{1}{\hat{p}_N}\left(\hat{\psi}_N(x+\hat{a}_N) - \hat{\psi}_N(x+a)\right),$$

$$\delta_2^N = -\frac{1}{\hat{p}_N}\left(\psi(x+\hat{a}_N) - \psi(x+a)\right), \ \delta_3^N = \frac{1}{\hat{p}_N}\left(\hat{\psi}_N(x+a) - \psi(x+a)\right). \tag{16}$$

Obviously, $\delta_2^N$ is $O_p\left(\frac{1}{\sqrt{N}}\right)$ as $N\to\infty$. The asymptotic behavior of the third term is described by the following theorem.

**Theorem 4.1.** *[4, p.57] Assume that*

$$D=\int_{-\infty}^{\infty} z^2K(z)dz<\infty;\quad d^2=\int_{-\infty}^{\infty} K^2(z)dz<\infty;\quad \int_{-\infty}^{\infty} zK(z)dz=0;$$

*and the function $\psi(x)$ is doubly differentiable. The estimate $\hat{\psi}(x)$ defined by (3) can be represented as*

$$\hat{\psi}_N(x)=\psi_N(x)+\frac{\zeta_N(x)}{\sqrt{Nh_N}},$$

*where*

$$\psi_N(x)=E\hat{\psi}_N(x)=\int_{-\infty}^{\infty} K(z)\psi(x-zh_N)dz\to_{N\to\infty}\psi(x)\ ;$$

$$\zeta_N(x)=\frac{1}{\sqrt{Nh_N}}\sum_{j=1}^{N}\left(K\left(\frac{x-\xi_j}{h_N}\right)-EK\left(\frac{x-\xi_j}{h_N}\right)\right)\Rightarrow\zeta;$$

*where $\zeta$ is a zero mean Gaussian r.v. with variance $d^2\psi(x)$.*

By Taylor's formula,

$$\psi_N(x)=\int_{-\infty}^{\infty} K(z)\psi(x-zh_N)dz=$$

$$=\int_{-\infty}^{\infty} K(z)\left(\psi(x)-zh_N\psi'(x)+\frac{z^2{h_N}^2}{2}\psi''(\theta_{x,h_N})\right)dz=$$

$$=\psi(x)+\frac{{h_N}^2}{2}\int_{-\infty}^{\infty} z^2K(z)\psi''(\theta_{x,h_N})dz=\psi(x)+\frac{{h_N}^2D^2}{2}\psi''(x)+o({h_N}^2).$$

So, by Theorem 4.1,

$$\hat{\psi}_N(x+a)-\psi(x+a)=\frac{D^2\psi''(x+a)}{2}h_N^2+\frac{\zeta_N(x+a)}{\sqrt{Nh_N}}+o(h_N^2),$$

where $\zeta_N(x+a)$ converges weakly to $\zeta(x+a)$ distributed as $N(0,d^2\psi(x+a))$. By assumption (iiv), $h_N=CN^{-1/5}$, so

$$\delta_3^N=\frac{N^{-2/5}}{p}\left(\frac{C^2D^2\psi''(x+a)}{2}+\frac{\zeta_N(x+a)}{\sqrt{C}}\right)+$$

$$+\frac{N^{-2/5}}{p\hat{p}_N}(p-\hat{p}_N)\left(\frac{C^2D^2\psi''(x+a)}{2}+\frac{\zeta_N(x+a)}{\sqrt{C}}\right)+o(N^{-2/5})=$$

$$=\frac{N^{-2/5}}{p}\left(\frac{C^2D^2\psi''(x+a)}{2}+\frac{\zeta_N(x+a)}{\sqrt{C}}\right)+o(N^{-2/5}).\tag{17}$$

For the first term in (15), we have

$$\delta_1^N=\frac{\hat{\psi}'_N(x+\theta_3)}{\hat{p}_N}(\hat{a}_N-a),\tag{18}$$

where $\theta_3$ is an intermediate point between $\hat{a}_N$ and $a$.

Consider

$$\hat{\psi}'_N(x)=\frac{1}{Nh_N^2}\sum_{j=1}^{N}K'\left(\frac{x-\xi_j}{h_N}\right)=\frac{1}{h_N^2}\int_{-\infty}^{\infty}K'\left(\frac{x-y}{h_N}\right)d\mu_N(y),$$

where $\mu_N(y)=\frac{1}{N}\sum_{j=1}^{N}I\left\{\xi_j<y\right\}$ is the empirical measure ($I\{A\}$ is an indicator of a set $A$ ). Similarly, we denote

$$\tilde{\psi}_N(x)=\frac{1}{h_N^2}\int_{-\infty}^{\infty}K'\left(\frac{x-y}{h_N}\right)dP(y),$$

where $P(x) = \mathbf{P}\{\xi_1 < x\}$. Then

$$\hat{\psi}'_N(x) = \tilde{\psi}_N(x) + \frac{1}{h_N^2}\int_{-\infty}^{\infty} K'\left(\frac{x-y}{h_N}\right) d\left(\mu_N(y) - P(y)\right). \tag{19}$$

Changing variables and integrating by parts, we obtain

$$\tilde{\psi}_N(x) = \frac{1}{h_N^2}\int_{-\infty}^{\infty} K'\left(\frac{x-y}{h_N}\right)\psi(y)dy = \int_{-\infty}^{\infty} K(z)\psi'(x - zh_N)dz = \psi'(x) + O(h_N^2).$$

To estimate $\delta_1^N$, we need the following inequality ([5, p.138]): if $f(x)$ is a function continuous on $[a, b]$, and $g(x)$ is a function with bounded variation on $[a, b]$, then

$$\left|\int_a^b f(x)dg(x)\right| \le \max_{x\in[a,b]} |f(x)| \cdot \mathrm{Var}_{x\in[a,b]} g(x).$$

So,

$$\frac{1}{h_N^2}\left|\int_{-\infty}^{\infty} K'\left(\frac{x-y}{h_N}\right) d\left(\mu_N(y) - P(y)\right)\right| =$$
$$\frac{1}{h_N^2}\left|\int_{-\infty}^{\infty} \left(\mu_N(y) - P(y)\right) dK'\left(\frac{x-y}{h_N}\right)\right| \le$$
$$\le \frac{1}{h_N^2} var_{y\in R} K'\left(\frac{x-y}{h_N}\right) \cdot \sup_{y\in R} |\mu_N(y) - P(y)|.$$

By assumption (i), $\mathrm{Var}_{y\in R} K'\left(\frac{x-y}{h_N}\right)$ is finite. By the Vapnik–Chervonenkis inequality [6, p.231],

$$\mathbf{P}\left\{\sup_{y\in R} |\mu_N(y) - P(y)| > \varepsilon\right\} \le 6(2N+1)\exp\left(-\frac{\varepsilon^2(N-1)}{2}\right). \tag{20}$$

For $\varepsilon \ge \frac{\ln N}{\sqrt{N}}$, the right-hand side of (20) tends to zero as $N \to \infty$. So we conclude that

$$|\mu_N(y) - P(y)| = O_p\left(\frac{\ln N}{\sqrt{N}}\right).$$

Substituting the derived results on the asymptotics of both terms and $h_N = CN^{-1/5}$ to (19), we conclude that

$$\hat{\psi}'_N(x) = \psi'(x) + o(h_N^2) + O_p\left(\frac{1}{h_N^2}\frac{\log N}{\sqrt{N}}\right) = \psi'(x) + O_p(\ln N \cdot N^{-1/10}). \tag{21}$$

This yields $\delta_1^N = O_p\left(\frac{1}{\sqrt{N}}\right)$. So,

$$\varepsilon_2^N = \frac{N^{-2/5}}{p}\left(\frac{C^2D^2\psi''(x+a)}{2} + \frac{\zeta_N(x+a)}{\sqrt{C}}\right) + o_p(N^{-2/5}). \tag{22}$$

Substituting (13), (14), and (21) into (7), we derive the statement of Theorem 3.1 □

*Proof. of theorem 3.2.* Obviously,

$$\tilde{f}_N(x) - f(x) = \frac{\hat{f}_N(x) - f(x)}{2} + \frac{\hat{f}_N(-x) - f(-x)}{2}. \tag{23}$$

Let us apply Theorem 4.1 to the first and second terms of (23). We have

$$\hat{f}_N(x) - f(x) = \frac{N^{-2/5}}{p}\left(\frac{C^2D^2\psi''(x+a)}{2} + \frac{\zeta_N(x+a)}{\sqrt{C}}\right) + o_p(N^{-2/5});$$

$$\hat{f}_N(-x) - f(-x) = \frac{N^{-2/5}}{p}\left(\frac{C^2D^2\psi''(-x+a)}{2} + \frac{\zeta_N(-x+a)}{\sqrt{C}}\right) + o_p(N^{-2/5}), \tag{24}$$

where $\zeta_N(x+a)$ and $\zeta_N(-x+a)$ are defined as

$$\zeta_N(x+a) = \frac{1}{\sqrt{Nh_N}} \sum_{j=1}^{N} \left( K\left(\frac{x+a-\xi_j}{h_N}\right) - EK\left(\frac{x+a-\xi_j}{h_N}\right)\right);$$

$$\zeta_N(-x+a) = \frac{1}{\sqrt{Nh_N}} \sum_{j=1}^{N} \left( K\left(\frac{-x+a-\xi_j}{h_N}\right) - EK\left(\frac{-x+a-\xi_j}{h_N}\right)\right).$$

**Lemma 4.1.** *If (i)-(v) hold and* $x \neq a$,

$$\begin{pmatrix} \zeta_N(x+a) \\ \zeta_N(-x+a) \end{pmatrix} \Rightarrow \begin{pmatrix} \zeta^1 \\ \zeta^2 \end{pmatrix}$$

*, where* $\begin{pmatrix} \zeta^1 \\ \zeta^2 \end{pmatrix}$ *is a zero mean normal vector with covariance matrix*

$$B = d^2 \begin{pmatrix} \psi(x+a) & 0 \\ 0 & \psi(-x+a) \end{pmatrix}$$

*Proof. of lemma 4.1.* We will use the central limit theorem in the following form.

**Theorem 4.2.** *[7 , p. 201] Let* $\xi_{1,N}, \ldots, \xi_{N,N}$ *be random vectors independent for fixed* $N$ *with* $E\xi_{j,N} = 0$; $\zeta_N = \sum_{j=1}^N \xi_{j,N}$. *Denote*

$$\sigma^2_{j,N} = E\xi_{j,N}\xi^T_{j,N}; \quad \sigma^2_N = \sum_{j=1}^{N} \sigma^2_{j,N}.$$

*Assume that*
*1)*$\sigma^2_N \to \sigma^2$, *where* $\sigma^2$ *is some positive definite matrix;*
*2) The Lindeberg condition*

$$B_N = \sum_{j=1}^{N} E\left(\xi^2_{j,N};\ |\xi_{j,N}| > \tau\right) \to 0;\ N \to \infty$$

*holds for any constant* $\tau > 0$. *Then the random vector* $\zeta_N$ *converges weakly to a zero-mean normal vector with the covariance matrix* $\sigma^2$.

Let us show that the assumptions of Theorem 4.2 hold for the vector sequence $\begin{pmatrix} \zeta_N(x+a) \\ \zeta_N(-x+a) \end{pmatrix}$. By the theorem, 3.1

$$D\zeta_N(x+a) \to d^2\psi(x+a); \quad D\zeta_N(-x+a) \to d^2\psi(-x+a).$$

For the covariance of the entries, we have

$$Cov(\zeta_N(x+a), \zeta_N(-x+a)) =$$

$$= E\left(\sum_{j=1}^{N} \frac{1}{\sqrt{Nh_N}} \left(K\left(\frac{x+a-\xi_j}{h_N}\right) - EK\left(\frac{x+a-\xi_j}{h_N}\right)\right) \times \right.$$

$$\left. \times \sum_{j=1}^{N} \frac{1}{\sqrt{Nh_N}} \left(K\left(\frac{-x+a-\xi_j}{h_N}\right) - EK\left(\frac{-x+a-\xi_j}{h_N}\right)\right)\right) =$$

$$= \frac{1}{Nh_N} \sum_{j=1}^{N} E\left(K\left(\frac{x+a-\xi_j}{h_N}\right) - EK\left(\frac{x+a-\xi_j}{h_N}\right)\right)\left(K\left(\frac{-x+a-\xi_j}{h_N}\right) - \right.$$

$$\left. - EK\left(\frac{-x+a-\xi_j}{h_N}\right)\right) = \frac{1}{h_N}\left(EK\left(\frac{x+a-\xi_1}{h_N}\right) K\left(\frac{-x+a-\xi_1}{h_N}\right) - \right.$$

$$-EK\left(\frac{x+a-\xi_1}{h_N}\right)EK\left(\frac{-x+a-\xi_1}{h_N}\right)\Bigg) = \int_{-\infty}^{\infty} K(s)K\left(s-\frac{2x}{h_N}\right)\times$$
$$\times\psi(x+a-sh_N)ds - h_N\int_{-\infty}^{\infty} K(s)\psi(x+a-sh_N)ds\int_{-\infty}^{\infty} K(s)\psi(-x+a-sh_N)ds.$$

Since $K(x)$ is finitely supported, $\psi(x)$ is bounded, $x \neq a$, and $h_N \to 0$,

$$Cov(\zeta_N(x+a), \zeta_N(-x+a)) \to 0,\ N \to \infty.$$

Then, by (i), $K(x) < c_4$. So, by the Chebyshev inequality,

$$B_N = \frac{1}{Nh_N}\sum_{j=1}^{N} E\left(\left(K\left(\frac{x+a-\xi_j}{h_N}\right) - EK\left(\frac{x+a-\xi_j}{h_N}\right)\right)^2 + \right.$$
$$\left. + \left(K\left(\frac{-x+a-\xi_j}{h_N}\right) - EK\left(\frac{-x+a-\xi_j}{h_N}\right)\right)^2\right)\times$$
$$\times\chi\left\{\left(K\left(\frac{x+a-\xi_j}{h_N}\right) - EK\left(\frac{x+a-\xi_j}{h_N}\right)\right)^2 + \right.$$
$$\left. + \left(K\left(\frac{-x+a-\xi_j}{h_N}\right) - EK\left(\frac{-x+a-\xi_j}{h_N}\right)\right)^2 > \tau^2 Nh_N\right\} \leq$$
$$\leq \frac{8c_4^2}{h_N}\mathbf{P}\left\{\left(K\left(\frac{x+a-\xi_j}{h_N}\right) - EK\left(\frac{x+a-\xi_j}{h_N}\right)\right)^2 + \right.$$
$$\left. + \left(K\left(\frac{-x+a-\xi_j}{h_N}\right) - EK\left(\frac{-x+a-\xi_j}{h_N}\right)\right)^2 > \tau^2 Nh_N\right\} \leq \frac{8c_4^2}{h_N}\frac{8c_4^2}{\tau^2 Nh_N}$$

. Since $h_N = CN^{-1/5}$, the rhs of the inequality tends to zero. Lindeberg's condition is verified, and Lemma 4.1 is proved. □

To complete the proof of Theorem 3.2, it is sufficient to substitute (24) in (23) and take the asymptotics of normal components into account. □

## References


1. L. Bordes, C. Delmas and P. Vandekerkhove, *Semiparametric estimation of two-component mixture model where one component is known*, Scand. J. Statist. **33** (2006), 733–752.
2. L. Bordes, S. Mottelet, and P. Vandekerkhove, *Semiparametric estimation of a two-component mixture model*, Ann. Statist. **34** (2006), no. 3, 1204–1232.
3. O. V. Sugakova, *Estimate of center position for observations with admixture*, Theor. Probability and Math. Statist. **79**(2008) (in Ukrainian).
4. A. A. Borovkov, *Mathematical Statistics*, Nauka, Moscow, 1984 (in Russian).
5. G. M. Fikhtengolts, *Course of Differential and Integral Calculus*, Vol. 3, Gos. Izdat. Tekhn.-Theoret. Lit., Moscow, 1949 (in Russian).
6. V. N. Vapnik, A. Ya. Chervonenkis, *The Pattern Recognition Theory*, Nauka, Moscow, 1974 (in Russian).
7. A. A. Borovkov, *Theory of Probability*, Moscow, Nauka, 1986 (in Russian).



Department of Theory Probability and Mathematical Statistics, Faculty for Mechanics and Mathematics, Taras Shevchenko National University of Kyiv, Academician Glushkov Ave. 2, Build. 7, Kyiv 03127

*E-mail address*: sugak@univ.kiev.ua