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CONVERGENCE OF INDEPENDENT RANDOM VARIABLE SUM DISTRIBUTIONS TO SIGNED MEASURES AND APPLICATIONS TO THE LARGE DEVIATIONS PROBLEM

We study properties of symmetric stable measures with index $\alpha > 2$, $\alpha \neq 2k$, $k \in \mathbb{N}$. Such measures are signed ones and hence they are not probability measures. We show that, in some sense, these signed measures are limit measures for sums of independent random variables.

1. Introduction

Let $\{\xi_i\}_{i=1}^{\infty}$ be a sequence of independent identically distributed symmetric random variables (for simplicity, we consider only a symmetric case). We suppose that the distribution $\mathcal{P}_1$ of the random variable ξ_1 satisfies the following condition:

$$\mathcal{P}_1((x,+\infty)) = \mathbf{P}(\xi_1 > x) = \frac{b}{x^{\alpha}}(1+o(1)), \quad x \to +\infty. \tag{1}$$

It is well known (see [1]) that, for $\alpha \in (0,2)$, the distribution $\mathcal{P}_1$ belongs to a domain of attraction of a symmetric stable law with index α. This means that the distributions of normalized sums

$$\frac{1}{n^{1/\alpha}} \sum_{i=1}^{n} \xi_i$$

weakly converge, as $n \to \infty$, to a symmetric stable distribution $\mathcal{Q}^{\alpha}$ with index α. The characteristic function $\widehat{q^{\alpha}}(p)$ of $\mathcal{Q}^{\alpha}$ is equal to $\exp(-c|p|^{\alpha})$, $c > 0$.

It is worth to note (see [1]) that the asymptotic behavior of the tail probability $\mathbf{P}(\xi_1 > x)$ coincides with the asymptotic behavior of the tail probability of the limit stable distribution. The corresponding statement is not true for the Central Limit Theorem.

In our work, we suppose that the random variable ξ_1 satisfies (1) with $\alpha > 2$, $\alpha \neq 2k$, $k \in \mathbb{N}$. As above, we consider only a symmetric case. It follows from (1) that the distribution $\mathcal{P}_1$ belongs to the domain of attraction of the normal law. This means that the distributions of the normalized sums $\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \xi_i$ weakly converge to a normal distribution.

In our paper, we consider the distributions of normalized sums

$$\frac{1}{n^{1/\alpha}} \sum_{i=1}^{n} \xi_i \tag{2}$$

with a weaker normalization $n^{1/\alpha}$ (exactly the same normalization as in the case $\alpha \in (0,2)$). By $\mathcal{P}_n$ and p_n, we denote, respectively, the distribution of the random variable (2) and the corresponding density. Additionally, we suppose that $p_n \in L_2(\mathbb{R})$ for all n large enough.

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It is clear that the sequence $\mathcal{P}_n$ has no weak limit. Nevertheless, we construct a sequence of corrections $\mathcal{B}_n^\alpha$ (each correction is an operator $L_2(\mathbb{R}) \to L_2(\mathbb{R})$) such that the sequence $\mathcal{B}_n^\alpha p_n$ converges in $L_2(\mathbb{R})$ to a function q^α, i.e. $q^\alpha = (L_2)\lim_{n\to\infty} \mathcal{B}_n^\alpha p_n$.

Each operator $\mathcal{B}_n^\alpha$ acts as follows:

$$\mathcal{B}_n^\alpha p_n = p_n + p_n * \zeta_n.$$

Here, for every n, ζ_n is an inverse Fourier transform of some smooth even function with bounded support, and $\int_{-\infty}^{\infty} \zeta_n(x)dx = 0$.

For $\alpha \in \cup_{k=1}^{\infty}(4k, 4k+2)$, the Fourier transform $\widehat{q^\alpha}$ of the limit function q^α is

$$\widehat{q^\alpha}(p) = \exp(-c|p|^\alpha), \quad c > 0 \tag{3}$$

(exactly as in the case $\alpha \in (0,2)$). But, for $\alpha \in \cup_{k=1}^{\infty}(4k-2, 4k)$, the function $\widehat{q^\alpha}$ is of another form, namely

$$\widehat{q^\alpha}(p) = \exp(c_0|p|^\alpha - c_1 p^{4k}), \; c_0, c_1 > 0. \tag{4}$$

By $\mathcal{Q}^\alpha$, we denote a signed measure in $\mathbb{R}$ with the density q^α. It follows from (3) and (4) that, for every $\alpha > 2$, $\mathcal{Q}^\alpha(\mathbb{R}) = \int_{-\infty}^{\infty} q^\alpha(x)dx = 1$, but the limit measure $\mathcal{Q}^\alpha$ is not a probability measure, because it is a signed one.

It is very important to note that, for every $\alpha > 2, \alpha \neq 2k, \; k \in \mathbb{N}$, the support of the negative part of the signed measure $\mathcal{Q}^\alpha$ belongs to some finite interval. Outside this interval, the measure $\mathcal{Q}^\alpha$ is positive. Moreover, the asymptotic behavior of its density at infinity is of the form $\frac{c}{|x|^{1+\alpha}}, \; c > 0$ (exactly as in the classical case $\alpha \in (0,2)$). This fact partly explains the difference between the form of the Fourier transform of q^α for $\alpha \in \cup_{k=1}^{\infty}(4k-2, 4k)$ and for $\alpha \in \cup_{k=1}^{\infty}(4k, 4k+2)$. On the contrary, the inverse Fourier transform of the function $f(p) = \exp(-c|p|^\alpha)$ for $\alpha \in \cup_{k=1}^{\infty}(4k-2, 4k)$ is negative outside some finite interval.

Although the limit measure $\mathcal{Q}^\alpha$ is not a probability one, nevertheless in the last section of the paper, we prove a theorem about large deviations of sums of independent random variables using the positive part (actually, the positive tail) of the limit measure. This result is close to the result of S.V. Nagaev ([3, 2]).

The results of the present paper extend those of our previous paper [5], where only the case $\alpha \in (2,4) \cup (4,6)$ was considered.

2. Necessary notations

We introduce necessary notations. Let $\{\xi_i\}_{i=1}^{\infty}$ be a sequence of independent identically distributed symmetric random variables. We suppose that the distribution of the random variable ξ_1 satisfies (1). Put $a = \alpha b$.

We suppose that $\alpha > 2$ and $\alpha \neq 2k, \; k \in \mathbb{N}$. By $f = f(p)$, we denote the characteristic function of ξ_1. For $k < \alpha$, we denote the $k-$th order moment of the random variable ξ_1 by $\mu_k = \mathbb{E}\xi_1^k$ and the corresponding semiinvariant by s_k. Since $\mathcal{P}_1$ is symmetric, $f(p)$ is a real-valued function, and $\mu_k \neq 0$ and $s_k \neq 0$ only for even numbers k.

In addition, we suppose that

$$|f(p)| \leq \frac{K}{|p|^\delta} \tag{5}$$

for some $K > 0, \; \delta > 0$, so that, for $n > \frac{2}{\delta}$, the distribution of the sum $\sum_{i=1}^n \xi_i$ has a square-integrable density.

3. Limit theorems. The case $\alpha \in \cup_{k=1}^{\infty}(4k, 4k+2)$

Let $\alpha \in (4k, 4k+2)$ for some $k \in \mathbb{N}$. We suppose that the characteristic function $f(p)$ of the random variable ξ_1 satisfies (5) and consider the representation

$$f(p) = \mathbb{E}\cos p\xi_1 = 1 - \frac{\mu_2 p^2}{2!} + \frac{\mu_4 p^4}{4!} - \cdots + \frac{\mu_{4k} p^{4k}}{(4k)!} +$$

$$\mathbb{E}\Big(\cos p\xi_1 - 1 + \frac{p^2\xi_1^2}{2!} - \frac{p^4\xi_1^4}{4!} + \dots - \frac{p^{4k}\xi_1^{4k}}{(4k)!}\Big) =$$
$$1 - \frac{\mu_2 p^2}{2!} + \frac{\mu_4 p^4}{4!} - \dots + \frac{\mu_{4k}p^{4k}}{(4k)!} - T(p), \tag{6}$$

where

$$T(p) = -2\int_0^\infty \Big(\cos px - 1 + \frac{p^2x^2}{2!} - \frac{p^4x^4}{4!} + \dots - \frac{p^{4k}x^{4k}}{(4k)!}\Big) d\mathcal{P}_1(x) > 0, \tag{7}$$

Lemma 3.1. *$T(p) \sim c_2|p|^\alpha$, as $p \to 0$, where*

$$c_2 = -2a\int_0^\infty (\cos y - 1 + \frac{y^2}{2!} - \frac{y^4}{4!} + \dots - \frac{y^{4k}}{(4k)!})\frac{dy}{y^{1+\alpha}} = \frac{\pi a}{\Gamma(1+\alpha)\sin\frac{\pi\alpha}{2}} > 0.$$

Proof. We present $-T(p)$ in the form

$$-T(p) = 2\int_0^\infty \Big(\cos px - 1 + \frac{p^2x^2}{2!} - \dots - \frac{p^{4k}x^{4k}}{(4k)!}\Big) d\mathcal{P}_1(x) =$$
$$2\int_0^1 (\dots) d\mathcal{P}_1 + 2\int_1^\infty (\dots) d\mathcal{P}_1.$$

It is clear that $\int_0^1 (\dots) d\mathcal{P}_1 = o(|p|^\alpha)$. Taking (1) into account, we obtain

$$\int_1^\infty (\dots) d\mathcal{P}_1 = a|p|^\alpha \int_0^\infty (\cos y - 1 + \frac{y^2}{2!} - \dots - \frac{y^{4k}}{(4k)!})\frac{dy}{y^{1+\alpha}} + o(|p|^\alpha). \quad \square$$

Further, it follows from (6) and Lemma 1 that, for the logarithm of the characteristic function, we have the representation

$$\log f(p) = v(p) - T(p) + o(|p|^\alpha) = v(p) - c_2|p|^\alpha + o(|p|^\alpha), \tag{8}$$

where

$$v(p) = -\frac{s_2p^2}{2!} + \frac{s_4p^4}{4!} - \dots + \frac{s_{4k}p^{4k}}{(4k)!},$$

and $s_2, \dots, s_{4m}$ are semiinvariants of the random variable ξ_1.

Lemma 3.2. *There exist $\varepsilon_0, d_0 > 0$ such that, for $|p| \le \varepsilon_0$,*

$$0 \le f(p)e^{-v(p)} \le e^{-d_0|p|^\alpha}$$

and

$$e^{-v(p)} \ge 1.$$

Proof. It is trivial that the second inequality is true in a small neighborhood of the origin. The first one follows from (8) and Lemma 1. $\square$

Choose a decreasing function $\chi \in C^\infty(\mathbb{R})$ such that $\chi(x) = 1$ for $x \le 1$ and $\chi(x) = 0$ for $x \ge 2$. We define a sequence of smooth functions $\varphi_n = \varphi_n(p)$ by

$$\varphi_n(p) = \exp\big(-v(p)\chi(n^{1/(4k+2)}|p|)\big).$$

Further, by $\mathcal{P}_n$, we denote the distribution of the random variable $\frac{1}{n^{1/\alpha}}\sum_{i=1}^n \xi_i$, and f_n stands for the characteristic function of $\mathcal{P}_n$. We have

$$f_n(p) = (f(\frac{p}{n^{1/\alpha}}))^n = \Big(1 - \frac{\mu_2p^2}{2!n^{2/\alpha}} + \frac{\mu_4p^4}{4!n^{4/\alpha}} - \dots + \frac{\mu_{4k}p^{4k}}{(4k)!n^{4k/\alpha}} - T(\frac{p}{n^{1/\alpha}})\Big)^n.$$

It is clear that such a sequence of functions has no limit. In order to obtain a convergent sequence, we multiply the function f_n by

$$\psi_n(p) = \big(\varphi_n(\frac{p}{n^{1/\alpha}})\big)^n. \tag{9}$$

Theorem 3.1. *The sequence $f_n\psi_n$ converges in $L_2(\mathbb{R})$ as $n \to \infty$ to a function $\widehat{q^\alpha}$, where $\widehat{q^\alpha}(p) = \exp(-c_2|p|^\alpha)$.*

Proof. First, we show that $f_n\psi_n$ converges to $\widehat{q^\alpha}$ pointwise. Note that, for every fixed p and n large enough, $|p|n^{1/(4k+2)-1/\alpha} < 1$. So, using (8), we have

$$\Big(f(\frac{p}{n^{1/\alpha}})\varphi_n(\frac{p}{n^{1/\alpha}})\Big)^n = \exp\Big[n\Big(\log f(\frac{p}{n^{1/\alpha}}) - v(\frac{p}{n^{1/\alpha}})\Big)\Big] =$$
$$\exp\Big[n\Big(v(\frac{p}{n^{1/\alpha}}) - c_2\frac{|p|^\alpha}{n} + o(\frac{1}{n}) - v(\frac{p}{n^{1/\alpha}})\Big)\Big] =$$
$$\exp\big[-c_2|p|^\alpha + o(1)\big] \underset{n\to\infty}{\longrightarrow} \exp(-c_2|p|^\alpha).$$

Next, we show that $f_n\psi_n \to \widehat{q^\alpha}$ not only pointwise but in L_2−sense as well.

We have

$$f_n(p)\psi_n(p) = A_n(p) + B_n(p), \tag{10}$$

where

$$A_n(p) = f_n(p)\psi_n(p)\mathbf{1}_{[0,2n^{1/\alpha}n^{-1/(4k+2)}]}(|p|)$$

and

$$B_n(p) = f_n(p)\psi_n(p)\mathbf{1}_{(2n^{1/\alpha}n^{-1/(4k+2)},\infty)}(|p|).$$

Note that, for every $p \in \mathbb{R}$, $A_n(p) \to \widehat{q^\alpha}(p)$ as $n \to \infty$, and, by Lemma 2, the functions A_n^2 are majorized by the function $e^{-2d_0|p|^\alpha}$. Using the Lebesgue majorized convergence theorem, we get that $A_n \underset{n\to\infty}{\longrightarrow} \widehat{q^\alpha}$ in $L_2(\mathbb{R})$.

It remains to check that $\|B_n\|_{L_2} \to 0$ as $n \to \infty$. First, we note that $M = \sup|f(p)| <$ 1 on the interval $[2n^{1/\alpha-1/(4k+2)}, \infty)$, and, hence,

$$|f(p)| \le \min(M, \frac{K}{p^\delta}) = M\min(1, \frac{K}{Mp^\delta}). \tag{11}$$

Using (11), we have

$$\|B_n\|^2_{L_2} = \int_{2n^{1/\alpha}n^{-1/(4k+2)}}^\infty \big(f_n(p)\psi_n(p)\big)^2 dp =$$
$$\int_{2n^{1/\alpha}n^{-1/\gamma}}^\infty \big(f(\frac{p}{n^{1/\alpha}})\big)^{2n} dp = n^{1/\alpha}\int_{2n^{-1/\gamma}}^\infty (f(u))^{2n} du \le$$
$$n^{1/\alpha}M^{2n}\Big(\int_{2n^{-1/\gamma}}^{(K/M)^{1/\delta}} du + \int_{(K/M)^{1/\delta}}^\infty (\frac{K}{Mu^\delta})^{2n} du\Big) \le$$
$$n^{1/\alpha}M^{2n}\Big(\Big(\frac{K}{M}\Big)^{1/\delta} + \frac{1}{2n\delta - 1}\Big(\frac{K}{M}\Big)^{1/\delta}\Big).$$

Hence, $\|B_n\|_{L_2} \to 0$ as $n \to \infty$. □

4. Limit theorems. The case $\alpha \in \cup_{k=1}^\infty (4k-2, 4k)$

Let $\alpha \in (4k-2, 4k)$ for some $k \in \mathbb{N}$. Then, for the characteristic function $f(p)$ of the random variable ξ_1, we have the representation

$$f(p) = \mathbb{E}\cos(p\xi_1) =$$
$$1 - \frac{\mu_2 p^2}{2!} + \cdots - \frac{\mu_{4k-2}p^{4k-2}}{(4k-2)!} + \mathbb{E}\Big(\cos(p\xi_1) - 1 + \frac{p^2\xi_1^2}{2!} - \cdots + \frac{p^{4k-2}\xi_1^{4k-2}}{(4k-2)!}\Big) =$$
$$1 - \frac{\mu_2 p^2}{2!} + \cdots - \frac{\mu_{4k-2}p^{4k-2}}{(4k-2)!} + T(p), \tag{12}$$

where

$$T(p) = 2\int_0^\infty \Big(\cos px - 1 + \frac{p^2x^2}{2!} - \cdots + \frac{p^{2k-2}x^{2k-2}}{(2k-2)!}\Big) d\mathcal{P}_1(x) > 0. \tag{13}$$

Lemma 4.1. *$T(p) \sim c_0|p|^\alpha$ as $p \to 0$, where*

$$c_0 = 2a \int_0^\infty \big(\cos y - 1 + \frac{y^2}{2!} - \dots + \frac{y^{4k-2}}{(4k-2)!}\big)\frac{dy}{y^{1+\alpha}} = \frac{-\pi a}{\Gamma(1+\alpha)\sin\frac{\pi\alpha}{2}} > 0. \tag{14}$$

Proof. It can be proved by the same arguments as those in Lemma 1. □

Further, it follows from (12) and Lemma 3 that

$$\log f(p) = v(p) + c_0|p|^\alpha + o(|p|^\alpha), \tag{15}$$

where

$$v(p) = -\frac{s_2p^2}{2!} + \frac{s_4p^4}{4!} - \dots - \frac{s_{4k-2}p^{4k-2}}{(4k-2)!},$$

(we recall that $s_2, \dots, s_{4k-2}$ are semiinvariants of the random variable ξ_1.)

Lemma 4.2. *There exist $\varepsilon_0, d_0 > 0$ such that, for $|p| \le \varepsilon_0$, the following inequality is true:*

$$0 \le f(p)e^{-v(p)} \le e^{d_0|p|^\alpha}. \tag{16}$$

Proof. The statement of this lemma follows from (15) and Lemma 3. □

We choose a decreasing function $\chi \in C^\infty(\mathbb{R})$, such that $\chi(x) = 1$ for $x \le 1$ $\chi(x) = 0$ for $x \ge 2$. Further, for $\alpha \in (4k-2, 4k)$, we choose $\gamma = \gamma(\alpha)$, so that $\gamma = 4k$ for $k > 1$. For $k = 1$, we put $\gamma = 4$ for $\alpha \in [3, 4)$, and $\gamma \in (\alpha, \min(4, \frac{2\alpha}{4-\alpha}))$ for $\alpha \in (2, 3)$.

Now we fix a positive constant c_1 and define a sequence of functions ψ_n, $n \in \mathbb{N}$ by

$$\psi_n(p) = \exp\bigg(-n\Big(v\big(\frac{p}{n^{1/\alpha}}\big) + \frac{c_1p^{4k}}{n}\Big)\chi\Big(\frac{|p|}{n^{1/\alpha-1/\gamma}}\Big)\bigg). \tag{17}$$

Lemma 4.3. *There exists n_0 such that, for all $n \ge n_0$,*

$$\inf_p \psi_n(p) = 1.$$

Proof. We prove the lemma only for $k > 1$, when $\gamma = 4k$ (the case $k = 1$ can be treated by the same arguments).

The function $\psi_n(p)$ is not equal to 1 only if

$$|p| \le 2\frac{n^{1/\alpha}}{n^{1/4k}}. \tag{18}$$

We now show that, for such p and n large enough,

$$\frac{c_1p^{4k}}{n} \le -v\big(\frac{p}{n^{1/\alpha}}\big). \tag{19}$$

First, we note that $-v(p) \sim \frac{s_2p^2}{2}$, as $p \to 0$ and, hence, for p small enough, we have

$$-v(p) > \frac{s_2p^2}{4}. \tag{20}$$

It follows from (20) that (19) is true if p satisfies the condition

$$|p| \le Mn^{(1-\frac{2}{\alpha})\frac{1}{4k-2}}, \tag{21}$$

where $M = (\frac{s_2}{4c_1})^{\frac{1}{4k-2}}$.

To prove the statement of the lemma, it is sufficient to prove that inequality (18) stronger than (21) or

$$0 < \frac{1}{\alpha} - \frac{1}{4k} < \frac{\alpha-2}{\alpha(4k-2)}. \tag{22}$$

It is easy to check that (22) is true for $k > 1$, $\alpha \in (4k-2, 4k)$. □

By $\mathcal{P}_n$ and f_n, we denote, respectively, the distribution of the random variable $\frac{1}{n^{1/\alpha}} \sum_{i=1}^n \xi_i$ and the characteristic function of $\mathcal{P}_n$.

We have

$$f_n(p) = (f(\frac{p}{n^{1/\alpha}}))^n = \left(1 - \frac{\mu_2 p^2}{2!n^{2/\alpha}} + \cdots - \frac{\mu_{4k-2}p^{4k-2}}{(4k-2)!n^{(4k-2)/\alpha}} + T(\frac{p}{n^{1/\alpha}})\right)^n. \quad (23)$$

It is clear that this sequence has no limit. In order to construct a convergent sequence, we multiply f_n by ψ_n given by (17).

Theorem 4.1. *The sequence $f_n\psi_n$ converges in $L_2(\mathbb{R})$ as $n \to \infty$ to the function $\widehat{q^\alpha}$, where $\widehat{q^\alpha}(p) = \exp\left(c_0|p|^\alpha - c_1 p^{4k}\right)$ with a constant $c_0 > 0$ defined by (14).*

Proof. First, we show that, for every $p \in \mathbb{R}$, $f_n(p)\psi_n(p) \to \widehat{q^\alpha}(p)$.

Note that, for every p fixed and n large enough, $|p|n^{1/\gamma-1/\alpha} < 1$. So, using (15) and Lemma 3, we obtain

$$f_n(p)\psi_n(p) = \exp\left(n\left(\log f(\frac{p}{n^{1/\alpha}}) - v(\frac{p}{n^{1/\alpha}})\right)\right)\exp(-c_1p^{4k}) =$$

$$\exp\left(n\left(v(\frac{p}{n^{1/\alpha}}) + c_0\frac{|p|^\alpha}{n} + o(\frac{1}{n}) - v(\frac{p}{n^{1/\alpha}})\right)\right)\exp(-c_1p^{4k}) \underset{n\to\infty}{\longrightarrow} e^{c_0|p|^\alpha - c_1p^{4k}}.$$

To prove the theorem, it is sufficient to check that the convergence $f_n\psi_n \to \widehat{q^\alpha}$ takes place not only pointwise, but in L_2 sense as well. This can be proved by the same arguments as those in Theorem 1. □

5. The application to the large deviation problem

As above, by $\mathcal{P}_n$ and p_n, we denote the distribution of $\frac{1}{n^{1/\alpha}}\sum_{i=1}^n \xi_i$ and the corresponding density, respectively. Further, by ζ_n, we denote the inverse Fourier transform of the function $\psi_n - 1$. We recall that the functions ψ_n are defined by (9) for $\alpha \in \cup_{k=1}^\infty(4k, 4k+2)$ and by (17) for $\alpha \in \cup_{k=1}^\infty(4k-2, 4k)$.

It is clear that

$$\int_{-\infty}^{\infty} \zeta_n(x)dx = \psi_n(0) - 1 = 0. \quad (24)$$

It follows from Theorems 1 and 2 that the sequence $p_n + p_n * \zeta_n$ converges in $L_2(\mathbb{R})$ to the function q^α. The asymptotic behavior of q^α is described by the following statement.

Lemma 5.1. *We have*

$$q^\alpha(x) = \frac{a}{x^{1+\alpha}} + O(\frac{1}{x^{1+2\alpha}}) \quad (25)$$

as $x \to \infty$, where $a = \frac{-c_0}{\pi}\Gamma(1+\alpha)\sin\frac{\pi\alpha}{2}$ for $\alpha \in (4k-2, 4k)$ and $a = \frac{c_2}{\pi}\Gamma(1+\alpha)\sin\frac{\pi\alpha}{2}$ for $\alpha \in (4k, 4k+2)$.

Proof. The proof can be found in [4]. □

The main aim of this section is to show that the asymptotic behavior of p_n is the same as the asymptotic behavior of q^α.

We need some additional assumptions. Namely, we suppose that, for

$$\alpha \in \bigcup_{k=1}^{\infty}(4k-2, 4k),$$

the characteristic function f of the random variable ξ_1 has the representation

$$f(p) = 1 - \frac{\mu_2 p^2}{2!} + \cdots - \frac{\mu_{4k-2}p^{4k-2}}{(4k-2)!} + c_0|p|^\alpha - R(p), \quad (26)$$

where the function $R(p)$ is of the form

$$R(p) = p^{[\alpha]+1}R_0(p), \quad (27)$$

and the function R_0 is $[\alpha]+2$ times continuously differentiable in a neighborhood of the origin (as usual, by $[\alpha]$, we denote the integer part of α).

Similarly, we suppose for $\alpha \in \cup_{k=1}^{\infty}(4k, 4k+2)$ that the characteristic function f has the representation

$$f(p) = 1 - \frac{\mu_2 p^2}{2!} + \cdots + \frac{\mu_{4k} p^{4k}}{(4k)!} - c_2|p|^\alpha + R(p), \tag{28}$$

where the function $R(p)$ satisfies (27). We also suppose that, outside of a neighborhood of the origin, the function f is $[\alpha]+2$ times continuously differentiable, and all derivatives $f^{(i)}(p)$, $i = 1, 2 \ldots, [\alpha]+2$, are bounded.

For example, for every $A > 0$, the distribution $\mathcal{P}_1$ with density V of the form

$$V(x) = \frac{\alpha A}{2}\frac{1}{|x|^{1+\alpha}}\mathbf{1}_{[A,\infty)}(|x|)$$

satisfies the last condition. Moreover, its characteristic function $f(p)$ satisfies (26) for $\alpha \in \cup_{k=1}^{\infty}(4k-2, 4k)$ and (28) for $\alpha \in \cup_{k=1}^{\infty}(4k, 4k+2)$.

Below, we need the following trivial statement.

Lemma 5.2. *Let $\widehat{g}$ be the Fourier transform of a function g. Suppose that, for some $l \in \mathbb{N}$, $\widehat{g}^{(l)} \in L_1(\mathbb{R})$. Then, for every $x \in \mathbb{R}$, we have*

$$|g(x)| \le \frac{1}{2\pi}\frac{1}{|x|^l}\int_{\mathbb{R}} |\widehat{g}^{(l)}(p)|dp. \tag{29}$$

By h_n, we denote the function

$$h_n(p) = f_n(p)\psi_n(p) - \widehat{q^\alpha}(p).$$

It follows from Theorems 1 and 2 that $\|h_n\|_{L_2} \to 0$ as $n \to \infty$.

In what follows, by C, we denote a positive constant. The same letter C may denote different constants.

We now consider separately the case $\alpha \in (4k-2, 4k)$ and $\alpha \in (4k, 4k+2)$. First, we consider the case $\alpha \in (4k, 4k+2)$.

Theorem 5.1. *For $\alpha \in (4k, 4k+2)$, we have*

$$\int_{\mathbb{R}} |h_n^{([\alpha]+2)}(p)|dp \le \frac{C}{n^{\frac{4m+2}{\alpha}-1}}.$$

Proof. Let $\alpha \in [4k+1, 4k+2)$ (the case $\alpha \in (4k, 4k+1)$ can be considered by the same arguments). First, we estimate

$$\int_{|p| \le n^{1/\alpha - \frac{1}{4k+2}}} |h_n^{(4k+3)}(p)|dp.$$

It follows from (28) that

$$\log f(p) = v(p) - c_2|p|^\alpha + p^{4k+2}T_0(p), \tag{30}$$

and the function T_0 is $4k+3$ times continuously differentiable in a neighborhood of the origin.

For $|p| \le n^{1/\alpha - \frac{1}{4k+2}}$, using (30), we have

$$\begin{aligned} h_n(p) = f_n(p)\psi_n(p) - \exp(-c_2|p|^\alpha) &= \exp\left(n\left(\log f(\frac{p}{n^{1/\alpha}}) - v(p)\right)\right) - \exp(-c_2|p|^\alpha) \\ &= \exp(-c_2|p|^\alpha)\left[\exp\big(-n(\frac{p}{n^{1/\alpha}})^{4k+2}T_0(\frac{p}{n^{1/\alpha}})\big) - 1\right]. \end{aligned} \tag{31}$$

Note that function (31) is $4k+3$ times differentiable at the origin.

From (31), we get the following estimate:

$$\int_{|p| \le n^{1/\alpha - \frac{1}{4k+2}}} |h_n^{(4k+3)}(p)|dp \le \frac{C}{n^{\frac{4k+2}{\alpha}-1}}. \tag{32}$$

Further, arguing as in the proof of Theorem 1, we can show that, for every $N \in \mathbb{N}$, there exists $C > 0$ such that

$$\int_{|p|>n^{1/\alpha - \frac{1}{4k+2}}} |h_n^{(4k+3)}(p)|dp \le Cn^{-N}. \quad \square$$

Theorem 5.2. *For $\alpha \in (4k-2, 4k)$, we have*

$$\int_{\mathbb{R}} |h_n^{([\alpha]+2)}(p)|dp \le \frac{C}{n^{\frac{4k}{\alpha}-1}}.$$

Proof. The statement can be proved by the same arguments as those in Theorem 3. $\square$

It follows from Lemma 7 and Theorems 3 and 4 that, for every $x \in \mathbb{R}$, $\alpha \in (4k-2, 4k)$, we have

$$|p_n(x) + p_n * \zeta_n(x) - q^\alpha(x)| \le \frac{C}{n^{\frac{4k}{\alpha}-1}|x|^{[\alpha]+2}}. \tag{33}$$

For $\alpha \in (4k, 4k+2)$, we have

$$|p_n + p_n * \zeta_n(x) - q^\alpha(x)| \le \frac{C}{n^{\frac{4m+2}{\alpha}-1}|x|^{[\alpha]+2}}. \tag{34}$$

Combining (33),(34), and Lemma 6, we get

$$p_n + p_n * \zeta_n(x) \sim \frac{a}{|x|^{1+\alpha}}, \quad \text{as } x \to \infty.$$

Our next aim is to compare the asymptotic behaviors of $p_n + p_n * \zeta_n$ and p_n. To this end, we use Lemma 7 once again.

We put $d_n(p) = f_n(p)\psi_n(p) - f_n(p) = f_n(p)\big(\psi_n(p) - 1\big)$.

Theorem 5.3. *For every $\alpha \in \cup_{k=1}^{\infty}\Big((4k-2, 4k) \cup (4k, 4k+2)\Big)$, we have*

$$\int_{\mathbb{R}} |d_n^{([\alpha]+2)}(p)|dp \le C\rho_n^{[\alpha+1]-\alpha},$$

where $\rho_n = n^{1/2-1/\alpha} \to \infty$ as $n \to \infty$.

Proof. Assume that $\alpha \in [4k+1, 4k+2)$. In this case, we have to prove that

$$\int_{\mathbb{R}} |d_n^{(4k+3)}(p)|dp \le C\rho_n^{4k+2-\alpha}. \tag{35}$$

As above, we derive firstly an estimate for $\int_{|p|\le n^{1/\alpha - \frac{1}{4k+2}}} |d_n^{(4k+3)}(p)|dp$. When $|p| \le n^{1/\alpha - \frac{1}{4k+2}}$, by (9) and (31), we have

$$d_n(p) = f_n(p)\psi_n(p) - f_n(p) = f_n(p)\psi_n(p)[1 - \psi_n^{-1}(p)] =$$
$$\exp(-c_2|p|^\alpha)\exp\left(n\big(\frac{p}{n^{1/\alpha}}\big)^{4k+2}T_0\big(\frac{p}{n^{1/\alpha}}\big)\right)\left[1 - \exp\left(nv\big(\frac{p}{n^{1/\alpha}}\big)\right)\right]. \tag{36}$$

We get

$$\int_{|p|\le n^{1/\alpha - \frac{1}{4k+2}}} |d_n^{(4k+3)}(p)|dp =$$
$$\int_{|p|\le\rho_n^{-1}} |d_n^{(4k+3)}(p)|dp + \int_{\rho_n^{-1}<|p|\le n^{1/\alpha - \frac{1}{4k+2}}} |d_n^{(4k+3)}(p)|dp = A_n + B_n. \tag{37}$$

It is not difficult to prove that there exists a constant $C > 0$ such that

$$|A_n| \le C\frac{n}{n^{2/\alpha}}\int_0^{\rho_n^{-1}} p^{\alpha-4k-1}dp = C\rho_n^2\rho_n^{4k-\alpha} = C\rho_n^{4k+2-\alpha}$$

and

$$|B_n| \le C\int_{\rho_n^{-1}<|p|\le n^{1/\alpha - \frac{1}{4k+2}}} p^{\alpha-4k-3}e^{-c_2p^\alpha}dp \le \int_{\rho_n^{-1}}^{\infty} p^{\alpha-4k-3}e^{-c_2p^\alpha}dp \le C\rho_n^{4k+2-\alpha}.$$

Our next step is to estimate

$$\int_{n^{1/\alpha-\frac{1}{4k+2}}\le|p|\le 2n^{1/\alpha-\frac{1}{4k+2}}} |d_n^{(4k+3)}(p)|dp.$$

Note that, for every $n \in \mathbb{N}$, $p \in \mathbb{R}$, we have $\psi_n^{-1}(p) \le 1$. Arguing as above, we get that, for every $N \in \mathbb{N}$, there exists $C > 0$ such that

$$\int_{n^{1/\alpha-\frac{1}{4k+2}}\le|p|\le 2n^{1/\alpha-\frac{1}{4k+2}}} |d_n^{(4k+3)}(p)|dp \le Cn^{-N}.$$

To complete the proof, we note that $d_n(p) = 0$ for $|p| > 2n^{1/\alpha-\frac{1}{4k+2}}$.

The estimate for the case $\alpha \in (4k-2, 4m) \cup (4k, 4k+1)$ can be proved by the same arguments. □

Corollary 5.1. *Let us discuss the connection between our results and the local limit theorems for large deviations (see* [3],[2]*). To be definite, we assume that* $\alpha \in [4k+1, 4k+2)$*. It follows from Theorems 3, 4, and 5 that*

$$p_n(x) = \frac{a}{|x|^{1+\alpha}} + O(\frac{1}{|x|^{4k+3}}) + g_n(x), \ \text{as } x \to \infty,$$

where

$$|g_n(x)| \le \frac{\left(n^{1/2-1/\alpha}\right)^{4k+2-\alpha}}{|x|^{4k+3}}.$$

It is clear now that the asymptotic behavior $p_n(x) \sim \frac{a}{|x|^{1+\alpha}}$, $x \to \infty$ *is true if*

$$\frac{\left(n^{1/2-1/\alpha}\right)^{4k+2-\alpha}}{|x|^{4k+3}} = o(\frac{1}{|x|^{1+\alpha}}),$$

or, what is the same,

$$\sqrt{n}\frac{1}{xn^{1/\alpha}} \underset{n\to\infty}{\longrightarrow} 0.$$

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