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ON A STANDARD PRODUCT OF AN ARBITRARY FAMILY OF σ-FINITE BOREL MEASURES WITH DOMAINS IN POLISH SPACES

Let α be an infinite parameter set, and let $(\alpha_i)_{i\in I}$ be its any partition such that α_i is a non-empty finite subset for every $i \in I$. For $j \in \alpha$, let μ_j be a σ-finite Borel measure defined on a Polish metric space (E_j, ρ_j). We introduce a concept of a standard $(\alpha_i)_{i\in I}$-product of measures $(\mu_j)_{j\in\alpha}$ and investigate its some properties. As a consequence, we construct "a standard $(\alpha_i)_{i\in I}$-Lebesgue measure" on the Borel σ-algebra of subsets of $\mathbb{R}^\alpha$ for every infinite parameter set α which is invariant under a group generated by shifts. In addition, if $\mathrm{card}(\alpha_i) = 1$ for every $i \in I$, then "a standard $(\alpha_i)_{i\in I}$-Lebesgue measure" m^α is invariant under a group generated by shifts and canonical permutations of $\mathbb{R}^\alpha$. As a simple consequence, we get that a "standard Lebesgue measure" $m^{\mathbb{N}}$ on $\mathbb{R}^{\mathbb{N}}$ improves R. Baker's measure [2].

Let $(X_i, \mathbf{B}_i, \mu_i)$ $(i \in \mathbb{N})$ be a family of regular Borel measure spaces, where X_i is a Hausdorff topological space. In [4], it was proved that a Borel measure μ exists on $\prod_{i\in\mathbb{N}} X_i$ (with respect to the product topology) such that if $K_i \subseteq X_i$ is compact for all $i \in \mathbb{N}$ and $\prod_{i\in\mathbb{N}} \mu_i(K_i)$ converges, then $\mu(\prod_{i\in\mathbb{N}} K_i) = \prod_{i\in\mathbb{N}} \mu_i(K_i)$. Note that a special case of this result (in the case where $X_i = R$ and m_i is Lebesgue measure) has been proved only recently in [1]. Slightly later on, work [2] has improved the result in [4] as follows: there exists of a Borel measure λ on $\prod_{i\in\mathbb{N}} X_i$ such that if $R_i \subseteq X_i$ is measurable for $i \in \mathbb{N}$ and $\prod_{i\in\mathbb{N}} \mu_i(R_i)$ converges, then $\lambda(\prod_{i\in\mathbb{N}} R_i) = \prod_{i\in\mathbb{N}} \mu_i(R_i)$.

Note that both above-mentioned constructions in the case where multiplied measures coincide with a specific σ-finite Borel measure μ in a Hausdorff topological space X give a measure $\mu^{\mathbb{N}}$ which is not invariant under permutations of the $X^{\mathbb{N}}$. To eliminate this defect, we introduce a notion of a *standard product of measures* and prove its existence under some assumptions. Our approach, unlike [4], [1],[2], is based on the notion of a standard product of a family of real numbers. Main results of the article are the theorem about the existence of a *standard product of measures* and its invariance under action of some group of transformations. In the case where multiplied measures coincide with a Lebesgue measure on $\mathbb{R}$, our product occurs to be invariant under permutations of the $\mathbb{R}^{\mathbb{N}}$ (see [7]) unlike Baker's measures [1],[2]. In addition, our construction is essentially different from the points of view of [4] and [2], because it allows one to construct a *standard product of measures* for an arbitrary (not only for countable) family of σ-finite Borel measures with domains in Polish spaces.

Suppose that X is a topological space. The Borel sets $\mathcal{B}(X)$ are the σ-algebra generated by the open sets of a topological space X, and the Baire sets $\mathcal{B}_0(X)$ are the smallest σ-algebra making all real-valued continuous functions measurable. In 1957 (see [5]), Mařík proved that all normal countably paracompact spaces have the following property: Every Baire measure extends to a regular Borel measure. Spaces which have this property have come to be known as Mařík spaces.

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The present manuscript is devoted to the application of properties of some Mařík spaces to the definition of a product of an infinite family of σ-finite Borel measures with domains in Polish spaces.

In order to do it, we recall some important notions and well-known results from general topology and probability theory.

X is a Hausdorff space iff distinct points in X have disjoint neighbourhoods. X is a regular space if and only if, given any closed set F and any point x that does not belong to F, there exists a neighbourhood U of x and a neighbourhood V of F that are disjoint. X is a normal space if and only if, given any disjoint closed sets E and F, there are neighbourhoods U of E and V of F that are also disjoint. X is a regular Hausdorff space if and only if it is both regular and Hausdorff. X is a completely regular space if and only if, given any closed set F and any point x that does not belong to F, there is a continuous function f from X to the real line $\mathbb{R}$ such that $f(x)$ is 0 and $f(y)$ is 1 for every y in F. X is a Tychonoff space, if and only if it is both completely regular and Hausdorff.

Lemma 1 ([9], Theorem 4, p. 981) *The following statements about a product of nonempty metric spaces are equivalent:*
(i) The product is normal.
(ii) At most $\aleph_0$ of the factor spaces are noncompact.

Lemma 2 [10] *Every normal regular space is completely regular, and every normal Hausdorff space is Tychonoff.*

Recall that a Borel measure μ defined on a Hausdorff topological space (X,τ) is called Radon if

$$(\forall Y)(Y \in \mathcal{B}(X)\ \&\ 0 \le \mu(Y) < +\infty \to \mu(Y) = \sup_{\substack{K\subseteq Y \\ \text{K is compact in X}}} \mu(K))\ \diamondsuit$$

and called dense if the condition $\diamondsuit$ holds for $Y = X$.

A family $(U_i)_{i\in I}$ of open subsets in (X,τ) is called a generalized sequence if

$$(\forall i_1)(\forall i_2)(i_1 \in I\ \&\ i_2 \in I \to (\exists i_3)(i_3 \in I \to (U_{i_1} \subset U_{i_3}\ \&\ U_{i_2} \subset U_{i_3}))).$$

A Borel probability measure μ defined on X is called τ-smooth if, for an arbitrary generalized sequence $(U_i)_{i\in I}$, the condition

$$\mu(\bigcup_{i\in I} U_i) = \sup_{i\in I}\ \mu(U_i)$$

is valid.

A Baire probability measure μ on X is called τ_0-smooth if, for an arbitrary generalized sequence $(U_i)_{i\in I}$ of open Baire subsets in X, for which $\bigcup_{i\in I} U_i$ is also a Baire subset, the condition

$$\mu(\bigcup_{i\in I} U_i) = \sup_{i\in I}\mu(U_i)$$

is valid .

The following lemma plays a key role in our future investigations.

Lemma 3 ([12], Theorem 3.3, p. 42) *Let X be a completely regular topological space, and let μ be a Baire probability measure defined on the σ-algebra $\mathcal{B}_0(X)$. Then*
(a) if μ is τ_0-smooth, there exists a unique τ-smooth Borel extension on X.
(b) if the space X is Hausdorff and μ is dense on $\mathcal{B}_0(X)$, then μ admits a unique Radon extension on $\mathcal{B}(X)$.

Let $(E_j,\tau_j)_{j\in\alpha}$ be a family of Hausdorff topological spaces. By $(\prod_{j\in\alpha}E_j,\tau)$, we denote the Tychonoff product of the family of topological spaces $(E_j,\tau_j)_{j\in\alpha}$.

Lemma 4 *Let $(E_j,\rho_j)_{j\in\alpha}$ be a family of non-empty Polish metric spaces such that at most $\aleph_0$ of them are noncompact, and let μ_j be a Borel probability measure on E_j for $j\in\alpha$. Then the product measure $\prod_{j\in\alpha}\mu_j$ is τ_0-smooth and dense on $\prod_{j\in\alpha}E_j$.*

Proof. The product $\prod_{j\in\alpha}\mu_j$ is initially defined on the Baire σ-field $\mathcal{B}_0(\prod_{j\in\alpha}E_j)$ as in [12].

Let $(U_i)_{i\in I}$ be an arbitrary generalized sequence of open Baire subsets in $\prod_{j\in\alpha}E_j$, for which $\bigcup\limits_{i\in I}U_i$ is also a Baire subset.

The latter relation getting together with an assumption of Lemma 4 stated that at most $\aleph_0$ of the family $(E_j,\rho_j)_{j\in\alpha}$ are noncompact imply that there exist a countable subset $\alpha_0\subseteq\alpha$ and $U_{\alpha_0}\in\mathcal{B}(\prod_{j\in\alpha_0}E_j)$ such that

$$\bigcup_{i\in I}U_i=U_{\alpha_0}\times(\prod_{j\in\alpha\setminus\alpha_0}E_j)$$

and

$$(\forall j)(j\in\alpha\setminus\alpha_0\rightarrow(E_j,\rho_j)\text{ is compact}).$$

By the inner regularity of the Borel probability measure $\prod_{j\in\alpha_0}\mu_j$ with a domain in a Polish space, there exists an increasing family of compact sets $(F_k)_{k\in\mathbb{N}}$ such that $F_k\subseteq U_{\alpha_0}$ and

$$\lim_{n\to\infty}(\prod_{j\in\alpha_0}\mu_j)(F_n)=(\prod_{j\in\alpha_0}\mu_j)(U_{\alpha_0}).$$

We set $D_n=F_n\times\prod_{j\in(\alpha\setminus\alpha_0)}E_j$ for $n\in\mathbb{N}$. It is obvious that $(D_n)_{n\in\mathbb{N}}$ is an increasing family of compact subsets in $\prod_{j\in\alpha}E_j$ such that

$$\lim_{n\to\infty}(\prod_{j\in\alpha}\mu_j)(D_n)=(\prod_{j\in\alpha}\mu_j)(\cup_{i\in I}U_i).$$

It is obvious that $(U_i)_{i\in I}$ is covering D_n for every $n\in\mathbb{N}$. Hence, using the definition of a generalized sequence of open sets in a topological space, we can construct such a sequence $(i_n)_{n\in\mathbb{N}}$ of indices of I that the sequence $(U_{i_n})_{n\in\mathbb{N}}$ will be increasing and $D_n\subseteq U_{i_n}$ for $n\in\mathbb{N}$. We have

$$(\prod_{j\in\alpha}\mu_j)(D_n)\le(\prod_{j\in\alpha}\mu_j)(U_{i_n})$$

for every $n\in\mathbb{N}$. Hence,

$$(\prod_{j\in\alpha}\mu_j)(\cup_{i\in I}U_i)=\lim_{n\to\infty}(\prod_{j\in\alpha}\mu_j)(D_n)\le$$

$$\lim_{n\to\infty}(\prod_{j\in\alpha}\mu_j)(U_{i_n})\le(\prod_{j\in\alpha}\mu_j)(\cup_{i\in I}U_i).$$

The latter relation means that the condition

$$(\prod_{j\in\alpha}\mu_j)(\bigcup_{i\in I}U_i)=\sup_{i\in I}(\prod_{j\in\alpha}\mu_j)(U_i)$$

holds. Thus, the measure $\prod_{j\in\alpha}\mu_j$ is τ_0-smooth on $\prod_{j\in\alpha}E_j$.

Let us show that the measure $\prod_{j\in\alpha}\mu_j$ is dense.

We set

$\alpha_1=\{j:\ E_j\text{ is not compact}\}$.

It is clear that, for the Borel measure $\prod_{j\in\alpha_1}\mu_j$, there exists an increasing sequence of compact subsets $(F_k)_{k\in\mathbb{N}}$ in $\prod_{j\in\alpha_1}E_j$ that

$$\lim_{k\to+\infty}(\prod_{j\in\alpha_1}\mu_j)(F_k)=1.$$

Now it is easy to see that $(F_k\times\prod_{j\in\alpha\setminus\alpha_1}E_j)_{k\in\mathbb{N}}$ is an increasing sequence of compact subsets in $\prod_{j\in\alpha}E_j$ such that

$$\lim_{k\to+\infty}(\prod_{j\in\alpha}\mu_j)(F_k\times\prod_{j\in\alpha\setminus\alpha_1}E_j)=1. \quad \square$$

Lemma 5 *Let $(E_j,\rho_j)_{j\in\alpha}$ be a family of non-empty Polish metric spaces such that at most $\aleph_0$ of them are noncompact, and let μ_j be a Borel probability measure on E_j for $j\in\alpha$. Then there exists a unique τ-smooth Radon extension of the Baire measure $\prod_{j\in\alpha}\mu_j$ from the σ-algebra $\mathcal{B}_0(\prod_{j\in\alpha}E_j)$ to the σ-algebra $\mathcal{B}(\prod_{j\in\alpha}E_j)$.*

Proof. By Lemma 1, $\prod_{j\in\alpha}E_j$ is normal. Hence, in order to show that $\prod_{j\in\alpha}E_j$ is regular, it is sufficient to show that every point $(x_j)_{j\in\alpha}$ is closed in $\prod_{j\in\alpha}E_j$. But the latter relation follows from the Tychonoff well-known theorem because the point $(x_j)_{j\in\alpha}$ can be considered as a product of compact sets $(\{x_j\})_{j\in\alpha}$. Thus, it is normal and regular. By Lemma 2, we claim that the $(\prod_{j\in\alpha}E_j,\tau)$ is a completely regular topological space. Applications of Lemma 3 and Lemma 4 end the proof of Lemma 5. $\square$

We have the following lemma.

Lemma 6 ([6], Lemma 4.4, p. 67) *Let (E_1,τ_1) and (E_2,τ_2) be two topological spaces. By $B(E_1)$ and $B(E_2)$ (respectively, $B(E_1\times E_2)$), we denote the class of all Borel subsets generated by the topologies τ_1 and τ_2 (respectively, $\tau_1\times\tau_2$). If at least one of these topological spaces has a countable base, then the equality*

$$B(E_1)\times B(E_2)=B(E_1\times E_2)$$

holds.

Let us recall the definition of a standard product of non-negative real numbers

$$(\beta_j)_{j\in\alpha}\in[0,+\infty]^\alpha.$$

Definition 1 A standard product of the family of numbers $(\beta_j)_{j\in\alpha}$ is denoted by $(\mathbf{S})\prod_{j\in\alpha}\beta_j$ and defined as follows:

$(\mathbf{S})\prod_{j\in\alpha}\beta_j=0$ if $\sum_{i\in\alpha^-}\ln(\beta_j)=-\infty$, where $\alpha^-=\{j : ln(\beta_j)<0\}$ [1], and $(\mathbf{S})\prod_{j\in\alpha}\beta_j=e^{\sum_{j\in\alpha}\ln(\beta_j)}$ if $\sum_{j\in\alpha^-}\ln(\beta_j)\neq-\infty$.

Let (E,S) be a measurable space, and let $\mathcal{R}$ be any subclass of the σ-algebra S. Let $(\mu_B)_{B\in\mathcal{R}}$ be such a family of σ-finite measures that, for $B\in\mathcal{R}$, we have $\mathrm{dom}(\mu_B)=S\cap\mathcal{P}(B)$, where $\mathcal{P}(B)$ denotes the power set of the set B.

Definition 2 A family $(\mu_B)_{B\in\mathcal{R}}$ is called to be consistent if

$$(\forall X)(\forall B_1,B_2)(X\in S\ \&\ B_1,B_2\in\mathcal{R}\to\mu_{B_1}(X\cap B_1\cap B_2)=\mu_{B_2}(X\cap B_1\cap B_2)).$$

The following assertion plays a key role in our future investigation.

Lemma 7 ([7], Lemma 1) *Let $(\mu_B)_{B\in\mathcal{R}}$ be a consistent family of σ-finite measures. Then there exists a measure $\mu_{\mathcal{R}}$ on (E,S) such that*

(i) $\mu_{\mathcal{R}}(B)=\mu_B(B)$ for every $B\in\mathcal{R}$;

(ii) if there exists a non-countable family of pairwise disjoint sets $\{B_i : i\in I\}\subseteq\mathcal{R}$ such that $0<\mu_{B_i}(B_i)<\infty$, then the measure $\mu_{\mathcal{R}}$ is non-σ-finite;

[1] We set $ln(0)=-\infty$

(iii) if G is a group of measurable transformations of E such that $G(\mathcal{R}) = \mathcal{R}$ and

$$(\forall B)(\forall X)(\forall g)\big((B \in \mathcal{R} \ \& X \in S \cap \mathcal{P}(B) \ \& \ g \in G\big) \rightarrow \mu_{g(B)}(g(X)) = \mu_B(X)),$$

then the measure $\mu_{\mathcal{R}}$ is G-invariant.

Remark 1 Let $(E_j, \rho_j)_{j \in \alpha}$ be again a sequence of non-empty Polish metric spaces such that at most $\aleph_0$ of them are noncompact. Let $(\mu_j)_{j \in \alpha}$ be a sequence of Borel non-zero diffused finite measures with $\mathrm{dom}(\mu_j) = \mathcal{B}(E_j)$ for $j \in \alpha$ and

$$0 < (\mathbf{S}) \prod_{j \in \alpha} \mu_j(E_j) < +\infty.$$

By Lemma 5, we claim that there exists a unique τ-smooth and Radon Borel extension λ of the Baire probability measure $\prod_{j \in \alpha} \frac{\mu_j}{\mu_j(E_j)}$. A Borel measure

$$(\mathbf{S}) \prod_{j \in \alpha} \mu_j(E_j) \times \lambda$$

is called a standard product of the family of finite Borel measures $(\mu_j)_{j \in \alpha}$ and is denoted by $(\mathbf{S}) \prod_{j \in \alpha} \mu_j$.

We put

$$\tau_i = \prod_{j \in \alpha_i} \mu_j.$$

Lemma 8 *Let α be again an arbitrary infinite parameter set, and let $(\alpha_i)_{i \in I}$ be its any partition such that α_i is a non-empty finite subset of α for every $i \in I$. Let μ_j be a σ-finite diffused Borel measure defined on a Polish space (E_j, ρ_j) for $j \in \alpha$.*

By $\mathcal{R}_{(\alpha_i)_{i \in I}}$, we denote the family of all measurable rectangles $R \subseteq \prod_{j \in \alpha} E_j$ of the form $\prod_{i \in I} R_i$ with the property $0 \leq (\mathbf{S}) \prod_{i \in I} \tau_i(R_i) < \infty$ such that at most $\aleph_0$ of R_i's are noncompact(i.e., card$\{i : i \in I \ \& \ R_i$ is not compact in $\prod_{j \in \alpha_i} E_j\} \leq \aleph_0$.)

We suppose that there exists $R_0 = \prod_{i \in I} R_i^{(0)} \in \mathcal{R}_{(\alpha_i)_{i \in I}}$ such that

$$0 < (\mathbf{S}) \prod_{i \in I} \tau_i(R_i^{(0)}) < \infty.$$

For $X \in \mathcal{B}(R)$, we set $\mu_R(X) = 0$ if

$$(\mathbf{S}) \prod_{i \in I} \tau_i(R_i) = 0,$$

and

$$\mu_R(X) = (\mathbf{S}) \prod_{i \in I} \tau(R_i) \times \big(\prod_{i \in I} \frac{\tau_{i R_i}}{\tau_i(R_i)}\big)(X)$$

otherwise, where $\frac{\tau_{i R_i}}{\tau_i(R_i)}$ is a Borel probability measure defined on R_i as follows:

$$(\forall X)(X \in \mathcal{B}(R_i) \rightarrow \frac{\tau_{i R_i}}{\tau_i(R_i)}(X) = \frac{\tau_i(X)}{\tau_i(R_i)}).$$

Then the family of measures $(\mu_R)_{R \in \mathcal{R}}$ is consistent.

Proof. Let $R_1 = \prod_{i \in I} R_i^{(1)}$ and $R_2 = \prod_{i \in I} R_i^{(2)}$ be two elements of the class $\mathcal{R} = \mathcal{R}_{(\alpha_i)_{i \in I}}$.

Without loss of generality, it can be assumed that $0 < (\mathbf{S}) \prod_{i \in I} \tau_i(R_i^{(1)}) < \infty$ and $0 < (\mathbf{S}) \prod_{i \in I} \tau_i(R_i^{(2)}) < \infty$.

We will show that $\mu_{R_1}(X) = \mu_{R_2}(X)$ for every $X \in \mathcal{B}(R_1 \cap R_2)$. In this case, it is sufficient to show that $\mu_{R_1}(Y) = \mu_{R_2}(Y)$ for every elementary measurable rectangle $Y = \prod_{i \in I} Y_i$ in $R_1 \cap R_2$. Note here that, as an elementary measurable rectangle $Y = \prod_{i \in I} Y_i$

in $R_1 \cap R_2$, we assume a subset of $R_1 \cap R_2$ such that $Y_i \in \mathcal{B}(R_i^{(1)} \cap R_i^{(2)})$ for every $i \in \mathbb{N}$. Moreover, there exists a finite subset I_0 of I such that $Y_i = R_i^{(1)} \cap R_i^{(2)}$ for $i \in I \setminus I_0$.

For every $i \in I$ and for every $Y_i \in \mathcal{B}(R_i^{(1)} \cap R_i^{(2)})$, we have

$$\tau_i(Y_i \cap R_i^{(1)} \cap R_i^{(2)}) = \tau_i(Y_i \cap R_i^{(1)}) = \tau_i(Y_i \cap R_i^{(2)}).$$

The latter relation yields

$$(\mathbf{S}) \prod_{i \in I} \tau_i(Y_i \cap R_i^{(1)} \cap R_i^{(1)}) = (\mathbf{S}) \prod_{i \in I} \tau_i(Y_i \cap R_i^{(1)}) = (\mathbf{S}) \prod_{i \in I} \tau_i(Y_i \cap R_i^{(1)}).$$

Hence, we get

$$\mu_{R_1}(\prod_{i \in I} Y_i) = (\mathbf{S}) \prod_{i \in I} \tau_i(Y_i \cap R_i^{(1)}) = (\mathbf{S}) \prod_{i \in I} \tau_i(Y_i \cap R_i^{(1)} \cap R_i^{(2)})) =$$

$$(\mathbf{S}) \prod_{i \in I} \tau_i(Y_i \cap R_i^{(2)})) = \mu_{R_2}(\prod_{i \in I} Y_i).$$

Since a class $\mathcal{A}(R_1 \cap R_2)$ of all finite disjoint unions of elementary measurable rectangles in $R_1 \cap R_2$ is a ring, and since, by definition, the class $\mathcal{B}_0(R_1 \cap R_2)$ of Baire subsets of $R_1 \cap R_2$ is a minimal σ-ring generated by the ring $\mathcal{A}(R_1 \cap R_2)$, we claim (cf. [3], Theorem B, p. 27) that the class of all those sets of $R_1 \cap R_2$, for which this equality holds, coincides with the class $\mathcal{B}_0(R_1 \cap R_2)$.

Since restrictions of μ_{R_1} and μ_{R_2} to the class $\mathcal{B}_0(R_1 \cap R_2)$ coincide, and $R_1 \cap R_2$ is a product of non-empty Polish metric spaces such that at most $\aleph_0$ of them are noncompact, we claim by Lemma 5 that their Borel extensions coincide so that the extended Borel measure is unique, τ-smooth, and Radon. The latter relation means that the family of measures $(\mu_R)_{R \in \mathcal{R}}$ is consistent, and Lemma 8 is proved. □

Let α be again an arbitrary infinite parameter set, and let $(\alpha_i)_{i \in I}$ be its any partition such that α_i is a non-empty finite subset of the α for every $i \in I$. Let μ_j be a σ-finite continuous Borel measure defined on a Polish space (E_j, ρ_j) for $j \in \alpha$.

We denote, by $\mathcal{R}_{(\alpha_i)_{i \in I}}$, the family of all measurable rectangles $R \subseteq \prod_{j \in \alpha} E_j$ of the form $\prod_{i \in I} R_i$ with the property $0 \le (\mathbf{S}) \prod_{i \in I} \tau_i(R_i) < \infty$ such that at most $\aleph_0$ of R_i's are noncompact.

We suppose that there exists $R_0 = \prod_{i \in I} R_i^{(0)} \in \mathcal{R}_{(\alpha_i)_{i \in I}}$ such that

$$0 < (\mathbf{S}) \prod_{i \in I} \tau_i(R_i^{(0)}) < \infty.$$

We say that a Borel measure $\nu_{(\alpha_i)_{i \in I}}$ defined on $\mathcal{B}(\prod_{j \in \alpha} E_j)$ is called a standard $(\alpha_i)_{i \in I}$-product of the family of σ-finite continuous Borel measures $(\mu_j)_{j \in \alpha}$ if, for every

$$R = \prod_{i \in I} R_i \in \mathcal{R}_{(\alpha_i)_{i \in I}},$$

we have

$$\nu_{(\alpha_i)_{i \in I}}(R) = (\mathbf{S}) \prod_{i \in I} \tau_i(R_i),$$

where $\tau_i = \prod_{j \in \alpha_i} \mu_j$ for $i \in I$.

Theorem 1. *Let μ_j be a σ-finite diffused Borel measure defined on a Polish space (E_j, ρ_j) for $j \in \alpha$. Let α be again an arbitrary infinite parameter set, let $(\alpha_i)_{i \in I}$ be its* *any partition such that α_i is a non-empty finite subset of α for every $i \in I$, and let us suppose that there exists $R_0 = \prod_{i \in I} R_i^{(0)} \in \mathcal{R}_{(\alpha_i)_{i \in I}}$ such that*

$$0 < (\mathbf{S}) \prod_{i \in I} \tau_i(R_i^{(0)}) < \infty.$$

Then there exists a standard $(\alpha_i)_{i \in I}$-product of the family $(\mu_j)_{j \in \alpha}$.

Proof. For $X \in \mathcal{B}(R)$, we set $\mu_R(X) = 0$ if

$$(\mathbf{S}) \prod_{i \in I} \tau_i(R_i) = 0,$$

and

$$\mu_R(X) = (\mathbf{S}) \prod_{i \in I} \tau(R_i) \times \Big(\prod_{i \in I} \frac{\tau_{i R_i}}{\tau_i(R_i)} \Big)(X)$$

otherwise, where $\frac{\tau_{i R_i}}{\tau_i(R_i)}$ is a Borel probability measure defined on R_i as follows:

$$(\forall X)(X \in \mathcal{B}(R_i) \rightarrow \frac{\tau_{i R_i}}{\tau_i(R_i)}(X) = \frac{\tau_i(X)}{\tau_i(R_i)}).$$

By Lemma 8, the family of measures $(\mu_R)_{R \in \mathcal{R}}$ is consistent. We set

$$\nu_{(\alpha_i)_{i \in I}} = \mu_{\mathcal{R}_{(\alpha_i)_{i \in I}}},$$

where the measure $\mu_{\mathcal{R}_{(\alpha_i)_{i \in I}}}$ is defined by Lemma 7.

This completes the proof of Theorem 1. □

In the sequel, we denote a standard $(\alpha_i)_{i \in I}$-product of the family $(\mu_j)_{j \in \alpha}$ by

$$(\mathbf{S}, (\alpha_i)_{i \in I}) \prod_{j \in \alpha} \mu_j.$$

Here, we present a certain example of the family of σ-finite continuous Borel measures $(\mu_j)_{j \in \mathbb{N}}$ defined on the real axis $\mathbb{R}$ and of two different partitions $(\alpha_i)_{i \in \mathbb{N}}$ and $(\beta_i)_{i \in \mathbb{N}}$ of $\mathbb{N}$, for which

$$(\mathbf{S}, (\alpha_i)_{i \in \mathbb{N}}) \prod_{j \in \mathbb{N}} \mu_j \neq (\mathbf{S}, (\beta_i)_{i \in \mathbb{N}}) \prod_{j \in \mathbb{N}} \mu_j.$$

Example 1 We set $\alpha = \mathbb{N}$. For $j \in \mathbb{N}$, let l_j be a linear Lebesgue measure on $\mathbb{R}$. Let $\alpha_i = \{i\}$ and $\beta_i = \{2i+1, 2(i+1)\}$ for $i \in \mathbb{N}$.

We set

$$Y_i = [0, \frac{1}{2}] \times [0, 2].$$

It is obvious that

$$\big((\mathbf{S}, (\beta_i)_{i \in \mathbb{N}}) \prod_{j \in \mathbb{N}} l_j\big)(\prod_{i \in \mathbb{N}} Y_i) = 1$$

and

$$\big((\mathbf{S}, (\alpha_i)_{i \in \mathbb{N}}) \prod_{j \in \mathbb{N}} l_j\big)(\prod_{i \in \mathbb{N}} Y_i) = 0.$$

In view of Theorem 1 and Example 1, we state the following

Problem 1 Under assumptions of Theorem 1, describe all pairs of partitions $(\alpha_i)_{i \in I}$ and $(\beta_i)_{i \in I}$ of α, for which $(\mathbf{S}, (\alpha_i)_{i \in I}) \prod_{j \in \alpha} \mu_j = (\mathbf{S}, (\beta_i)_{i \in I}) \prod_{j \in \alpha} \mu_j$.

The next statement is an immediate consequence of Theorem 1.

Theorem 2. *Under assumptions of Theorem 1, if each measure μ_j is G_j-left-and-right-invariant, where G_j denotes a group of Borel transformations of the E_j for $j \in \alpha$, then the measure $(\mathbf{S},(\alpha_i)_{i\in I})\prod_{j\in\alpha}\mu_j$ is $\prod_{j\in\alpha}G_j$-left-and-right-invariant.*

Proof. We set $G=\prod_{j\in\alpha}G_j$. Let us show that the measure $(\mathbf{S},(\alpha_i)_{i\in I})\prod_{j\in\alpha}\mu_j$ is G-left-and-right-invariant. Indeed, let $g,f\in G$ and $X\in\mathcal{B}(\prod_{j\in\alpha}E_j)$.

If X is not covered by a countable family of elements of $\mathcal{R}_{(\alpha_i)_{i\in I}}$, then such will be gXf, because the class $\mathcal{R}_{(\alpha_i)_{i\in I}}$ is left-and-right-invariant, i.e., $g\mathcal{R}_{(\alpha_i)_{i\in I}}f=\mathcal{R}_{(\alpha_i)_{i\in I}}$ for every $g,f\in G$. Hence, by the definition of the measure $((\alpha_i)_{i\in I})\prod_{j\in\alpha}\mu_j$, we have

$$\big((\mathbf{S},(\alpha_i)_{i\in I})\prod_{j\in\alpha}\mu_j\big)(gXf)=+\infty.$$

Now let X be covered by the family $(A_k)_{k\in\mathbb{N}}$ of elements of $\mathcal{R}_{(\alpha_i)_{i\in I}}$ such that $A_0=\emptyset$. Then gXf will be covered by the family $(gA_kf)_{k\in\mathbb{N}}$ of elements of $\mathcal{R}_{(\alpha_i)_{i\in I}}$. Hence, we get

$$\big((\mathbf{S},(\alpha_i)_{i\in I})\prod_{j\in\alpha}\mu_j\big)(gXf)=\sum_{n=1}^{\infty}\lambda_{gA_nf}((gA_nf\setminus\cup_{k=0}^{n-1}gA_kf)\cap gXf)=$$

$$\sum_{n=1}^{\infty}\lambda_{gA_nf}(g((A_nf\setminus\cup_{k=0}^{n-1}A_kf)\cap Xf))=$$

$$\sum_{n=1}^{\infty}\lambda_{A_nf}((A_nf\setminus\cup_{k=0}^{n-1}A_kf)\cap Xf)=$$

$$\sum_{n=1}^{\infty}\lambda_{A_nf}((A_n\setminus\cup_{k=0}^{n-1}A_k)\cap X)f)=$$

$$\sum_{n=1}^{\infty}\lambda_{A_n}((A_n\setminus\cup_{k=0}^{n-1}A_k)\cap X)=\big((\mathbf{S},(\alpha_i)_{i\in I})\prod_{j\in\alpha}\mu_j\big)(X).\quad\square$$

By the scheme used in the proof of Theorem 2, one can prove the following assertion.

Theorem 3 *Under the assumptions of Theorem 1, if each measure μ_j is G_j-left-invariant, where G_j denotes a group of Borel transformations of the E_j for $j\in\alpha$, then the measure $(\mathbf{S},(\alpha_i)_{i\in I})\prod_{j\in\alpha}\mu_j$ is a $\prod_{j\in\alpha}G_j$-left-invariant.*

Observation 1. *Under the conditions of Theorem 1, the measure $(\mathbf{S},(\alpha_i)_{i\in I})\prod_{j\in\alpha}\mu_j$ is Radon.*

Proof. Let $0<\big((\mathbf{S},(\alpha_i)_{i\in I})\prod_{j\in\alpha}\mu_j\big)(X)<\infty$. This means that $X\in\mathcal{B}(\prod_{j\in\alpha}E_j)$ is covered by any countable family $(A_n)_{n\in\mathbb{N}}$ of elements of $\mathcal{R}_{(\alpha_i)_{i\in I}}$ such that $A_0=\emptyset$ and

$$\big((\mathbf{S},(\alpha_i)_{i\in I})\prod_{j\in\alpha}\mu_j\big)(X)=\sum_{n=1}^{\infty}\lambda_{A_n}((A_n\setminus\cup_{k=0}^{n-1}A_k)\cap X).$$

Since the measure λ_{A_n} is Radon, we can choose a compact set

$$F_n\subseteq(A_n\setminus\cup_{k=0}^{n-1}A_k)\cap X$$

such that

$$\lambda_{A_n}\big(((A_n\setminus\cup_{k=0}^{n-1}A_k)\cap X)\setminus F_n\big)<\frac{\epsilon}{2^{n+1}}$$

for $n\in\mathbb{N}$.

Moreover, we can choose a natural number n_ϵ such that

$$\sum_{n=n_\epsilon+1}^{\infty} \lambda_{A_n}((A_n \setminus \cup_{k=0}^{n-1} A_k) \cap X) < \frac{\epsilon}{2}.$$

Finally, we get

$$\big((\mathbf{S},(\alpha_i)_{i\in I})\prod_{j\in\alpha}\mu_j\big)(X \setminus \cup_{s=0}^{n_\epsilon} F_s) =$$

$$\sum_{n=1}^{\infty} \lambda_{A_n}((A_n \setminus \cup_{k=0}^{n-1} A_k) \cap (X \setminus \cup_{s=0}^{n_\epsilon} F_s)) =$$

$$\sum_{n=1}^{n_\epsilon} \lambda_{A_n}((A_n \setminus \cup_{k=0}^{n-1} A_k) \cap (X \setminus \cup_{s=0}^{n_\epsilon} F_s)) +$$

$$\sum_{n=n_\epsilon+1}^{\infty} \lambda_{A_n}((A_n \setminus \cup_{k=0}^{n-1} A_k) \cap (X \setminus \cup_{s=0}^{n_\epsilon} F_s)) \leq$$

$$\sum_{n=1}^{n_\epsilon} \lambda_{A_n}(((A_n \setminus \cup_{k=0}^{n-1} A_k) \cap X) \setminus F_n) +$$

$$\sum_{n=n_\epsilon+1}^{\infty} \lambda_{A_n}((A_n \setminus \cup_{k=0}^{n-1} A_k) \cap X) \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \quad \square$$

Remark 3 For $j \in \alpha$, we set $E_j = \mathbb{R}$ and $\mu_j = m$, where m denotes a linear Lebesgue measure on $\mathbb{R}$.

Let $(\alpha_i)_{i\in I}$ be any partition of α such that α_i is non-empty finite for every $i \in I$.

It is clear that $\prod_{j\in\alpha}[a_j, b_j] \in \mathcal{R}_{(\alpha_i)_{i\in I}}$ if $0 \leq (\mathbf{S})\prod_{i\in I} m^{\alpha_i}(\prod_{j\in\alpha_i}[a_j, b_j]) < \infty$, where m^{α_i} is a Lebesgue measure on $\mathbb{R}^{\alpha_i}$.

Then the measure $(\mathbf{S},(\alpha_i)_{i\in I})\prod_{j\in\alpha}\mu_j$ has the following property:

$$\big((\mathbf{S},(\alpha_i)_{i\in I})\prod_{j\in\alpha}\mu_j\big)(\prod_{j\in\alpha}[a_j, b_j]) = (\mathbf{S})\prod_{j\in\alpha}(b_i - a_i).$$

The measure $(\mathbf{S},(\alpha_i)_{i\in I})\prod_{j\in\alpha}\mu_i$ is called a standard "$(\alpha_i)_{i\in I}$-Lebesgue measure" on $\mathbb{R}^\alpha$.

When $\mathrm{card}(\alpha_i) = 1$ for every $i \in I$, then $(\mathbf{S},(\alpha_i)_{i\in I})\prod_{j\in\alpha}\mu_i$ is called a standard Lebesgue measure on $\mathbb{R}^\alpha$ and is denoted by m^α.

Let f be any permutation of α. A mapping $A_f : \mathbb{R}^\alpha \to \mathbb{R}^\alpha$ defined by $A_f((x_i)_{i\in\alpha}) = (x_{f(i)})_{i\in\alpha}$ for $(x_i)_{i\in\alpha} \in \mathbb{R}^\alpha$ is called a canonical permutation of $\mathbb{R}^\alpha$.

Note that, in our situation, $\mathcal{R}_{(\alpha_i)_{i\in I}}$ is the family of all measurable rectangles $R \subseteq \mathcal{B}(\mathbb{R}^\alpha)$ of the form $\prod_{i\in\alpha} Y_i$ with the property $0 \leq (\mathbf{S})\prod_{i\in\alpha} m(Y_i) < \infty$ such that at most $\aleph_0$ of them are noncompact(i.e., the card $\{i : i \in I \ \& \ Y_i$ is not compact in $\prod_{i\in\alpha_i} E_i\} \leq \aleph_0$.). It is obvious that a measure m^α is invariant under a group $\mathcal{P}(\mathbb{R}^\alpha)$ generated by shifts and canonical permutations of $\mathbb{R}^\alpha$ and

$$m^\alpha(\prod_{i\in\alpha} Y_i) = (\mathbf{S})\prod_{i\in\alpha} m(Y_i).$$

Remark 4 We can say that the main shortcoming of Baker's measures [1], [2] is that they are not invariant under the group of all canonical permutations of $\mathbb{R}^{\mathbb{N}}$.

Indeed, let us consider the following infinite-dimensional rectangular set

$$X = \prod_{k=1}^{\infty}[0, e^{\frac{(-1)^k}{k}}].$$

Then, for every non-zero real number a, there exists a canonical permutation f_a of $\mathbb{R}^\infty$ such that $\lambda(A_f(X)) = a$, where λ is any Baker's measure [1], [2].

Such a difference between our and Baker's measures is caused by the phenomenon that a standard (unlike an ordinary) product of the infinite family of real numbers is invariant under all permutations.

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